BGH 6

Aufgabe 1

a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 6 \end{pmatrix} \Rightarrow p_A(\lambda) = -\lambda(\lambda-9)(\lambda+1)$

$\Rightarrow m_A(\lambda) = \lambda(\lambda-9)(\lambda+1)$

$B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix} \Rightarrow B' = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ mit

$p_B(\lambda) = (\lambda-3)(\lambda-2)$

$\Rightarrow B'$ diagonalisierbar

$\Rightarrow m_B(\lambda) = (\lambda-2)^2(\lambda-3).$

b) i) vom Analysis, Taylor:

$$g(x) = g(\lambda) + g'(\lambda)(x-\lambda) + \frac{g''(\lambda)}{2!}(x-\lambda)^2 + \frac{g'''(\lambda)}{3!} * (x-\lambda)^3 \ldots$$

$$\Rightarrow g(T) = g(\lambda)I + g'(\lambda)(T-\lambda I) + \frac{g''(\lambda)}{2!}(T-\lambda I)^2 + \frac{g'''(\lambda)}{3!}(x-\lambda)^3$$

aber $m_T(x) = (x-\lambda)^2 \Rightarrow (T-\lambda I)^2 = 0$ und $(T-\lambda I)^q = 0 \quad \forall q > 2$

$\Rightarrow g(T) = g(\lambda)I + g'(\lambda)(T-\lambda I).$

ii) $H = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad p_H(t) = \begin{vmatrix} 2-t & 1 \\ -1 & -t \end{vmatrix} = t(t-2)+1 = (t-1)^2$

$$H - I_2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \Rightarrow (H-I)^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \Rightarrow (t-1)^2 = m_H(t)$$

Sei $g(H) = \sum_{k=0}^{199} H^k$.

$$\Rightarrow g(H) = g(1)I + g'(1)(H-I)$$

$$g(1) = 0 + 1 + \dots + 199 = \frac{199 \cdot 200}{2} = 19900$$

$$g(x) = 1 + x + x^2 + \dots + x^{199}$$

$$g'(x) = 1 + 2x + 3x^2 + \dots + 199x^{198}$$

$$g'(1) = 1 + 2 + 3 + \dots + 199 = 19900$$

$$\Rightarrow g(H) = 19900(I + H - I) = 19900H.$$

c) $m_T = g \cdot h$, $\quad ggT(g,h) = 1 \qquad T \in End(V)$

i) z.z. $V = \ker g(T) \oplus \ker h(T)$

$ggT(g,h) = 1 \Rightarrow \exists$ Polynome $y_1(t), y_2(t) \in k[t]$ so dass
$$g(t)y_1(t) + h(t)y_2(t) = 1.$$

$$\Rightarrow g(T)y_1(T) + h(T)y_2(T) = I \Rightarrow \forall v \in V,$$
$$(g(T)y_1(T) + h(T)y_2(T))(v) = v$$

$v_1 := g(T)y_1(T)(v)$; $\quad v_2 := h(T)y_2(T)(v)$

$\Rightarrow \quad h(T)(v_1) = h(T)g(T)y_1(T)(v) = m_T(T)y_1(T)(v) = 0$

$\Rightarrow \quad v_1 \in \ker h(T) =: U_2$

und $\quad v_2 \in \ker g(T) =: U_1$.

$\Rightarrow \quad \forall\, v \in V, \quad v = v_1 + v_2 \quad$ mit $\quad v_1 \in \ker h(T)$
$\qquad v_2 \in \ker g(T)$

Eindeutigkeit: $\quad v = v_1 + v_2 = w_1 + w_2 \quad$ mit $\quad v_1, w_1 \in \ker h(T)$
$\qquad v_2, w_2 \in \ker g(T)$

$v_1 = g(T)y_1(T)(v) = g(T)y_1(T)(w_1 + w_2) = g(T)y_1(T)(w_1) = w_1$

und $\quad v_2 = w_2 \Rightarrow \quad V = \ker g(T) \oplus \ker h(T)$.

(ii) $\quad U_1$ T-invariant $\Leftrightarrow (v \in U_1 \Rightarrow T(v) \in U_1) \Leftrightarrow g(T)T(v) = 0$

$g(T)T(v) = T g(T)(v) = T(0) = 0$.

$\Rightarrow \quad U_1, U_2$ T-invariant

$r(T)$ beliebiges Polynom: $\quad g(T)r(T)(v) = r(T)g(T)(v) = 0$

$\Rightarrow \quad U_1, U_2 \quad r(T)$-invariant.

Aufgabe 2

b) $P(T) = (T-I)^3 (T+I)^2$

$$\begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& -1 & \\ &&&& -1 \end{pmatrix}$$

$m(T) = (T-I)(T+I)$

oder

$$\begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& -1 & 1 \\ &&&& -1 \end{pmatrix}$$

$m(T) = (T-I)(T+I)^2$

oder

$$\begin{pmatrix} 1 & 1 &&& \\ & 1 &&& \\ && 1 && \\ &&& -1 & \\ &&&& -1 \end{pmatrix}$$

$(T-I)^2(T+I)$

oder

$$\begin{pmatrix} 1 & 1 &&& \\ & 1 & 1 && \\ && 1 && \\ &&& -1 & \\ &&&& -1 \end{pmatrix}$$

$(T-I)^3(T+I)$

oder

$$\begin{pmatrix} 1 & 1 &&& \\ & 1 & 1 && \\ && 1 && \\ &&& -1 & 1 \\ &&&& -1 \end{pmatrix}$$

$(T-I^3)(T+I)$

c) Für jedes minimal Polynom $m(T)$ gibt es nur eine eindeutige Parditionmatrix (siehe oben) $\Rightarrow m(T) = m(T')$
$\Rightarrow T \sim T'$.

a) $A = \begin{pmatrix} 0 & -4 & -8 & 0 \\ -3 & 4 & 0 & -2 \\ 2 & 0 & 4 & 1 \\ 0 & 8 & 12 & 0 \end{pmatrix}$ $\quad P_A(\lambda) = (\lambda - 2)^4$

$\tilde{A} = A - 2I = \begin{pmatrix} -2 & -4 & -8 & 0 \\ -3 & 4 & 0 & -2 \\ 2 & 0 & 2 & 1 \\ 0 & 8 & 12 & -2 \end{pmatrix}$ und $\tilde{A}^2 = 0$

$\ker \tilde{A}$: $\left(\begin{array}{cccc|c} -2 & -4 & -8 & 0 & 0 \\ -3 & 4 & 0 & 2 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & 8 & 12 & -2 & 0 \end{array}\right) \xrightarrow{Z_1 + Z_3} \left(\begin{array}{cccc|c} 0 & -4 & -6 & 1 & 0 \\ -3 & 2 & 0 & -2 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & 8 & 12 & -2 & 0 \end{array}\right)$

$$\xrightarrow{\frac{1}{2}z_4+z_1} \left(\begin{array}{cccc|c} 0 & -4 & -6 & 1 & 0 \\ -3 & 2 & 0 & -2 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{z_2+\frac{3}{2}z_3+\frac{1}{2}z_1} \left(\begin{array}{cccc|c} 0 & -4 & -6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \dim \ker \tilde{A} = 2$ und

$$\ker \tilde{A} = \left\langle \begin{pmatrix} 2 \\ 3 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 4 \end{pmatrix} \right\rangle$$

$\Rightarrow \dim \mathbb{R}^4 \setminus \ker \tilde{A} = 2$

$\Rightarrow v_1, v_2 \subset \mathbb{R}^4 \setminus \ker \tilde{A}$ lin. unab.

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{A} v_1 = \begin{pmatrix} -2 \\ -3 \\ 2 \\ 0 \end{pmatrix} \quad \Rightarrow S^{-1} = \begin{pmatrix} -2 & 1 & -4 & 0 \\ -3 & 0 & 2 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{A} v_2 = \begin{pmatrix} -4 \\ 2 \\ 0 \\ 8 \end{pmatrix}$$

$$\Rightarrow S = \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 1 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1/8 \\ 0 & 1 & 3/2 & -1/4 \end{pmatrix} \Rightarrow S\tilde{A}S^{-1} + 2I = \begin{pmatrix} 2 & 1 & & \\ 0 & 2 & & \\ & & 2 & 1 \\ & & 0 & 2 \end{pmatrix}$$

Aufgabe 3 $\quad g(t) = (t-1)^2 (t-3)^4 (t-5)$

$g_1(t) = (t-1)^2 \qquad g_2(t) = (t-3)^4 \qquad g_3(t) = t-5$

$\Rightarrow z_1 = g_2 g_3 = (t-3)^4 (t-5) \qquad z_2 = g_1 g_3 = (t-1)^2 (t-5)$

$z_3 = g_1 g_2 = (t-1)^2 (t-3)^4$ und $ggT(z_1, z_2, z_3) = 1$

$\Rightarrow \exists\ y_1, y_2, y_3$ Polynome sodass

$y_1(t) z_1(t) + y_2(t) z_2(t) + y_3(t) z_3(t) = 1$

$$\mathrm{ggT}(z_1, z_2, z_3) = \mathrm{ggT}(z_1, \mathrm{ggT}(z_2, z_3)) = \mathrm{ggT}(z_1, (t-1)^2)$$

$\mathrm{ggT}(z_2, z_3) = (t-1)^2 \Rightarrow \exists\ a_1, a_2$ Polynome so dass

$$a_1(t)\, z_2(t) + a_2(t)\, z_3(t) = (t-1)^2$$

z.B. $a_1(t) = -\frac{1}{16}(t-1)(t^2 - 6t + 13)$

$a_2(t) = \frac{1}{16}$

$\Rightarrow \exists\ b_1, b_2$ Polynome so dass

$$b_1(t)\, z_1(t) + b_2(t)(t-1)^2 = 1$$

z.B. $b_1(t) = \frac{1}{256}(5 - 9t)$

$b_2(t) = \frac{1}{256}(9t^4 - 140t^3 - 822t^2 - 2180t + 2281)$

$\Rightarrow \quad b_1(t)\, z_1(t) + b_2(t)(a_1 z_2(t) + a_2 z_3(t)) = 1$

$\Rightarrow \quad y_1 = b_1, \quad y_2 = a_1 b_2, \quad y_3 = a_2 b_2.$

$\Rightarrow \forall\ v \in V \qquad v = v_1 + v_2 + v_3$

$v_1 = z_1(f)\, y_1(f)(v)$

$v_2 = z_2(f)\, y_2(f)(v)$

$v_3 = z_3(f)\, y_3(f)(v)$