BBH 8

Aufgabe 1

1) $\mathbb{R}^3$: $g(x,y) = x_1y_2 + x_2y_1 + x_1y_3 + x_3y_1 + x_2y_3 + x_3y_2$

bilineare Form

a) $f(x) = x_1x_2 + x_2x_1 + x_1x_3 + x_3x_1 + x_2x_3 + x_3x_2$

$= 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = 2(x_1x_2 + x_1x_3 + x_2x_3)$

b)

$$(x_1 \quad x_2 \quad x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{i,j} a_{ij}x_ix_j$$

$\Rightarrow a_{11} = a_{22} = a_{33} = 0$

$\Rightarrow a_{ij} = 1 \quad i \neq j$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = M_K(f)$$

c) $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}, \begin{pmatrix} 1 \\ 8 \\ 27 \end{pmatrix} \right\}$

$$\Rightarrow S = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{pmatrix}$$

$$\Rightarrow M_B(f) = (S)^t A S = \begin{pmatrix} 6 & 28 & 78 \\ 28 & 112 & 288 \\ 78 & 288 & 702 \end{pmatrix}$$

2) $f : K^2 \to K$

1. $f(x,y) = x+y$

$f(\lambda x + \mu z, y) = \lambda x + \mu z + y$ ~~[illegible]~~ $\neq$

$\lambda x + y + \mu z + y = \lambda f(x,y) + \mu f(z,y)$

$f(x, \lambda y + \mu z) = x + \lambda y + \mu z = \lambda f(x,y) + \mu f(x,z)$ ✓

2. $f(x,y) = x^2 + y^2$

$f(\lambda x, y) = \lambda^2 x^2 + y^2 = \lambda^2 f(x,y)$ $\neq$

3. $f(x,y) = xy$

$f(\lambda x + \mu z, y) = (\lambda x + \mu z) y = \lambda x y + \mu z y$

$= \lambda f(x,y) + \mu f(z,y)$ ✓

$f(x, \lambda z + \mu y) = x(\lambda z + \mu y) = \lambda x z + \mu x y$

$= \lambda f(x,z) + \mu f(x,y)$ ✓

4. $f(x,y) = xy^2$

$f(x, \lambda y) = x \lambda^2 y^2 = \lambda^2 f(x,y)$ $\neq$

Aufgabe 2

1) $(V, \langle , \rangle)$ Euklidischer Vektorraum mit Norm

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Parallelogrammgleichung: $\forall\ v, w \in V$:

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

$$\langle v+w, v+w \rangle + \langle v-w, v-w \rangle = \langle v, v+w \rangle + \langle w, v+w \rangle + \langle v, v-w \rangle - \langle w, v-w \rangle$$

$$= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle + \langle v, v \rangle - \langle v, w \rangle - \langle w, v \rangle + \langle w, w \rangle$$

$$= 2\langle v, v \rangle + 2\langle w, w \rangle.$$

2) V $\mathbb{C}$-Vektorraum $s: V \times V \to \mathbb{C}$ Sesquilinearform mit $q: V \to \mathbb{C}$ zugehörige quadratische Form

z.z. $$s(v, w) = \frac{1}{4}\Big(q(v+w) - q(v-w) + i\,q(v+iw) - i\,q(v-iw)\Big)$$

~~$s(v,w) = \overline{s(w,v)}$~~ $s(v, w) = \overline{s(w, v)}$

$$q(v+w) = s(v+w, v+w) = s(v, v) + s(v, w) + s(w, v) + s(w, w)$$
$$= q(v) + q(w) + 2\,\mathrm{Re}(s(v, w))$$

$$q(v-w) = s(v-w, v-w) = s(v, v) - s(v, w) - s(w, v) + s(w, w)$$
$$= q(v) + q(w) - 2\,\mathrm{Re}(s(v, w))$$

$$\Rightarrow 4\,\mathrm{Re}(s(v, w)) = q(v+w) - q(v-w) \Rightarrow \mathrm{Re}(s(v, w)) = \frac{1}{4}\big(q(v+w) - q(v-w)\big)$$

$$\mathrm{Im}(s(v,w)) = \mathrm{Re}(-i s(v,w)) = \mathrm{Re}(s(v,iw))$$

Aber $\mathrm{Re}(s(v,iw)) = \frac{1}{4}\left(q(v+iw) - q(v-iw)\right)$

$$\Rightarrow \quad s(v,w) = \frac{1}{4}\left(q(v+w) - q(v-w) + i\, q(v+iw) - i\, q(v-iw)\right)$$

3) K Körper $\quad V$ K-Vektorraum

$\forall\ f : V \times V \to K$ Bilinearform

$f = f_1 + f_2$ wobei

$f_1 : V \times V \to K$ symmetrische Bf.

$f_2 : V \times V \to K$ schiefsymmetrische Bf

Nehmen wir an, dass $f = f_1 + f_2$

f_1 symmetrisch

f_2 antisymmetrisch

$$\Rightarrow \quad f(v,w) = f_1(v,w) + f_2(v,w)$$

$$f(w,v) = f_1(w,v) + f_2(w,v)$$

$$= f_1(v,w) - f_2(v,w)$$

$$\Rightarrow \quad f_1(v,w) = \frac{f(v,w) + f(w,v)}{2} \quad \Rightarrow \text{Existenz und Eindeutigkeit}$$

$$f_2(v,w) = \frac{f(v,w) - f(w,v)}{2}$$