Blatt 9

Aufgabe 1

1. $\left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}} \right\}$

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \qquad w_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$\|v_1\| = \sqrt{1+4+4} = 3$$

$$v_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \qquad w_2 : \quad \frac{\langle v_1, v_2 \rangle v_1}{\langle v_1, v_1 \rangle} = \langle w_1, v_2 \rangle w_1$$

$$= \frac{1}{9} \cdot (3+4+2) v_1 = v_1$$

$$v_2 - v_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$w_2 = \frac{1}{\left\|\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}\right\|} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} \qquad w_3 : \quad \frac{\langle v_1, v_3 \rangle v_1}{\langle v_1, v_1 \rangle} = \frac{1}{9} (2+2+14) v_1 = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

$$\frac{\langle v_2, v_3 \rangle}{\langle v_2, v_2 \rangle} v_2 = \frac{1}{9+4+1} \cdot (6+2+7) v_2 = \frac{15}{14} v_2$$

$$v_3 - \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} - \frac{15}{14} \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} \Rightarrow w_3 = \begin{pmatrix} 2/3\sqrt{5} \\ -\sqrt{5}/3 \\ 4/3\sqrt{5} \end{pmatrix}$$

2. $V = \mathbb{R}^4$

$$U = \left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ -13 \end{pmatrix} \right\rangle$$

$$U^\perp = \ker \begin{pmatrix} 2 & 1 & 0 & 3 \\ 4 & 2 & 1 & -1 \\ 1 & 0 & 2 & -13 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 & 3 \\ 4 & 2 & 1 & -1 \\ 1 & 0 & 2 & -13 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -7 \end{pmatrix}$$

$$\Rightarrow \left\langle \begin{pmatrix} 1 \\ 1 \\ -7 \\ -1 \end{pmatrix} \right\rangle = U^\perp$$

Aufgabe 2

1. V endlich-dimensionaler Vektorraum

$v_1 \dots v_r$ orthonormale Familie

i) $v_1 \dots v_r$ Basis

ii) $\forall\ v, w \in V, \quad \langle v, w \rangle = \sum_{j=1}^{r} \langle v, v_j \rangle \langle v_j, w \rangle$

iii) $\forall\ v \in V, \quad \|v\|^2 = \sum_{j=1}^{r} |\langle v, v_j \rangle|^2$

iv) $\forall\ v \in V : \langle v, v_j \rangle = 0 \quad \forall j \Rightarrow v = 0$

i) ⟹ ii) $\quad v = a_1 v_1 + \dots + a_r v_r \qquad \langle v, v_j \rangle = a_j$

$\qquad w = b_1 v_1 + \dots + b_r v_r \qquad \langle v_j, w \rangle = \overline{b_j}$

$$\langle v, w \rangle = \langle a_1 v_1 + \dots + a_r v_r ; b_1 v_1 + \dots + b_r v_r \rangle$$
$$= a_1 \overline{b_1} + \dots + a_r \overline{b_r}$$
$$= \sum_{j=1}^{r} \langle v, v_j \rangle \langle v_j, w \rangle$$

ii) ⟹ iii)

$$\|v\|^2 = \langle v, v \rangle = \sum_{j=1}^{r} \langle v, v_j \rangle \langle v_j, v \rangle$$
$$= \sum_{j=1}^{r} \langle v, v_j \rangle \overline{\langle v, v_j \rangle}$$
$$= \sum_{j=1}^{r} |\langle v, v_j \rangle|^2.$$

iii) ⟹ iv) $\quad \langle v, v_j \rangle = 0 \Rightarrow \|v\|^2 = 0 \Rightarrow v = 0.$

iv) ⟹ i) $\quad v_1 \dots v_r$ orthogonal $\Rightarrow v_1 \dots v_r$ lin. unab.

noch zu zeigen dass $v_1 \dots v_r$ erzeugend System für V ist.

$v \in V, \quad v \neq 0 \Rightarrow \exists j$ so dass $\langle v, v_j \rangle \neq 0$

$v - \langle v, v_j \rangle v_j = 0 \Rightarrow v = \langle v, v_j \rangle v_j$

$v - \langle v, v_j \rangle v_j \neq 0 \Rightarrow \exists i$ so dass $\langle v, v_i \rangle \neq 0$

$\Rightarrow v - \langle v, v_j \rangle v_j - \langle v, v_i \rangle v_i \begin{cases} = 0 \\ \neq 0 \end{cases}$

$\Rightarrow \dots \Rightarrow w = v - \sum_{j=1}^{r} \langle v, v_j \rangle v_j$ und $\langle w, v_j \rangle = 0 \ \forall j$

$\Rightarrow w = 0 \Rightarrow v = \sum_{j=1}^{r} \langle v, v_j \rangle v_j$

2. $\mathcal{B}$ kanonische Basis von $\mathbb{R}^3$, $A : \mathbb{R}^3 \to \mathbb{R}^3$ lin. Abb

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} . \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ -2 & -1 & -1 \end{pmatrix} = G$$

$$q : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \qquad q(v,w) = g(A(v), w)$$

$$q(v,w) = g(A(v), w) = (Av)^t G w = v^t (A^t G) w$$

$$= v^t \begin{pmatrix} 0 & 0 & -1 \\ -2 & -3 & 0 \\ -6 & -2 & -2 \end{pmatrix} w$$

$$C = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \right\}; \qquad S = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ -1 & 1 & -2 \end{pmatrix}$$

$$S^t = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & -3 & -2 \end{pmatrix}$$

$$M_C(q) = S^t \begin{pmatrix} 0 & 0 & -1 \\ -2 & -3 & 0 \\ -6 & -2 & -2 \end{pmatrix} S = \begin{pmatrix} -3 & 10 & 3 \\ 2 & -16 & 8 \\ 10 & 39 & -27 \end{pmatrix}$$