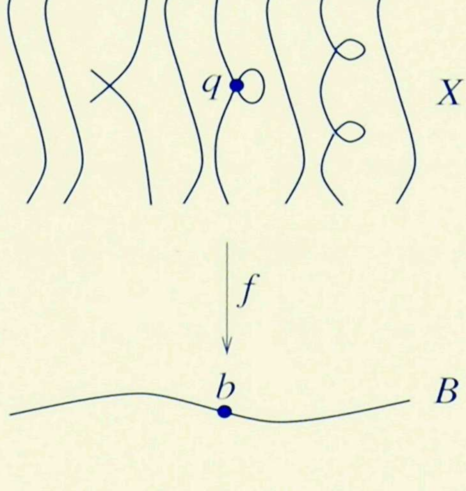
The birational geometry of the moduli space of curves



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The birational geometry of the moduli space of curves

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To my wife Lia

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Amsterdam, May 2000 Gavril Farkas

Contents

Introduction

0.1 Algebraic curves and their moduli

The subject of this thesis is the geometry of the moduli space $\mathcal{M}_g$ of algebraic curves of genus g. This is the universal parameter space for curves (Riemann surfaces) of genus g, in the sense that its points correspond one to one to isomorphism classes of curves.

Algebraic curves, the objects classified by $\mathcal{M}_g$, started to appear systematically in mathematics around the middle of the 19th century, although interest in algebraic curves can be traced back to Euler's study of *abelian integrals* in the 18th century. For most of the 19th century, the curves people were looking at were plane curves, that is, subsets of $\mathbb{P}^2$ satisfying an equation $F(x, y, z) = 0$, where F is a homogeneous polynomial in three variables. Two plane curves were said to belong to the same class, if there was a birational transformation of $\mathbb{P}^2$ carrying one curve into another. The advantage of such an approach (which surely appears quite cumbersome and ineffective nowadays) is that given a plane curve of degree d, by varying the coefficients of the polynomial equation one immediately obtains the family of all plane curves of degree d, which is itself an algebraic variety, the projective space $\mathbb{P}^{d(d+3)/2}$.

It was Riemann in his famous papers on function theory from 1857 who started the process of lifting the curves from $\mathbb{P}^2$ and began to view them as abstract objects. By realizing curves as branched covers of $\mathbb{P}^1$, he even managed to show that for $g \geq 2$ curves of genus g depend on $3g-3$ minimal parameters, which he called *moduli*.

As Brill and Noether pointed out, Riemann himself did not think of a space whose points would correspond to classes of curves. However, in the late 19th century, the concept of a moduli space of curves was floating around and the existence of a variety parametrizing genus g curves was widely assumed. At that time people were already actively studying properties of $\mathcal{M}_g$. For instance, in 1882, Klein using topological arguments due to Clebsch showed that the space of n-sheeted coverings of the Riemann sphere with $b = 2g + 2n - 2$ branch points is irreducible, implicitly proving the irreducibility of $\mathcal{M}_g$. Although Severi and B. Segre among others continued the investigation of $\mathcal{M}_g$ in the first decades of the 20th century, the first rigorous construction of $\mathcal{M}_g$ (as an analytic variety) was due to Teichmüller in 1940. Work by Baily in 1962 showed that $\mathcal{M}_g$ was an algebraic (quasi-projective) variety and the first purely algebraic construction of $\mathcal{M}_g$ was carried out by Mumford in 1965 using "Geometric Invariant Theory" (cf. [Mu2]).

At this point we want to make more precise what we understand by $\mathcal{M}_g$. The moduli space of curves $\mathcal{M}_g$ is an algebraic variety satisfying the following properties:

- For an algebraically closed field k, the points in $\mathcal{M}_g(k)$ correspond 1:1 to isomorphism classes of curves of genus g over k. If C is a complex smooth curve of genus g, we denote by $[C] \in \mathcal{M}_g$ its moduli point.
- For any flat family $\pi : \mathcal{C} \to B$ of smooth curves of genus g, the moduli map $m : B \to \mathcal{M}_g$ given by $m(b) := [C_b]$ for $b \in B$, is holomorphic. Moreover, $\mathcal{M}_g$ is in some sense minimal with respect to this universal property.

It turns out that there exists a unique variety $\mathcal{M}_g$ satisfying these properties and one says that $\mathcal{M}_g$ *coarsely represents* the moduli functor of curves (we refer to [Mu2] for precise definitions of these terms).

The space $\mathcal{M}_g$ is an irreducible quasi-projective variety of dimension $3g-3$ for $g \geq 2$. We have that $\mathcal{M}_0$ is a point and $\mathcal{M}_1$ is the affine line $\mathbb{A}^1$. Since smooth curves can degenerate to singular ones, $\mathcal{M}_g$ is not a compact variety. One can compactify $\mathcal{M}_g$ by enlarging the class of curves we parametrize and allowing certain singular curves, called *stable curves*. These are connected, nodal curves, such that any smooth rational component meets the curve in at least 3 points. We get in this way the Deligne-Mumford moduli space $\overline{\mathcal{M}}_g$ of stable curves (cf. [DM]), which is an irreducible projective variety with only mild singularities ($\mathbb{Q}$-factorial).

The boundary $\overline{\mathcal{M}}_g - \mathcal{M}_g$ corresponding to singular curves is a union of irreducible divisors Δ_i for $0 \leq i \leq [g/2]$. The general point of Δ_0 corresponds to an irreducible curve with one node, whereas for $1 \leq i \leq [g/2]$ the general point of Δ_i corresponds to a curve $C_1 \cup_q C_2$, where C_1 and C_2 are smooth curves of genus i and $g-i$ meeting transversally at q.

0.2 How rational is $\mathcal{M}_g$?

For low genus there are explicit descriptions of the variety $\mathcal{M}_g$. Any smooth curve of genus 2 can be realized through the equation

$$y^2 = (x-\alpha_1)\cdots(x-\alpha_6), \quad \text{where } \alpha_1, \ldots, \alpha_6 \in \mathbb{C}.$$

If we consider the quotient of $\mathrm{Sym}^6(\mathbb{P}^1)$ under the action of $PGL(2)$, we realize $\mathcal{M}_2$ as a quotient of an open subset of $\mathbb{C}^3$ by the symmetric group S_6.

In the case of $\mathcal{M}_3$, a non-hyperelliptic curve of genus 3 can be uniquely embedded as a smooth plane quartic. We thus have a dominant rational map from the $\mathbb{P}^{14}$ of plane quartics to $\mathcal{M}_3$. Almost every curve of genus 3 can be realized by varying the coefficients in the equation

$$\sum_{i,j,k \geq 0, i+j+k=4} a_{ijk} x^i y^j z^k = 0.$$

What is essential here, is that the coefficients $a_{ijk} \in \mathbb{C}$ can vary freely, they do not have to satisfy any equations, only a few polynomial inequalities, so that we have a way of getting our hands on (almost) every curve of genus 3. We say that the variety $\mathcal{M}_3$ is *unirational.*

More generally, a variety X is called *unirational* if there exists a rational dominant map from a projective space $\mathbb{P}^n$ to X. Unirationality is a very desirable property as it gives a parametrization of the variety. One says that X is *uniruled* if there exists a rational dominant map from a product variety $Y \times \mathbb{P}^1$ to X and which is not constant on fibres $\{y\} \times \mathbb{P}^1$, where $y \in Y$. Equivalently, X is uniruled if through a general point $x \in X$ there passes a rational curve. Clearly unirationality implies uniruledness.

An important birational invariant of an algebraic variety is its *Kodaira dimension*. For a smooth, projective variety X, the Kodaira dimension $\kappa(X)$ is defined as follows: let us consider $\omega_X = \mathcal{O}(K_X)$ the canonical sheaf on X and for $m \geq 1$ such that $|mK_X| \neq \emptyset$, we take the rational map $\phi_{mK_X} : X --\to \mathbb{P}^{h^0(mK_X)-1}$. We define

$$\kappa(X) := \max\{\dim \phi_{mK_X}(X) : m \in \mathbb{Z}_{\geq 1} \text{ such that } |mK_X| \neq \emptyset\}.$$

Clearly $\kappa(X) \in \{-\infty, 0, \ldots, \dim(X)\}$. If $\kappa(X) = \dim(X)$, we say that X is of *general type*. If X is uniruled, then $|mK_X| = \emptyset$ for all $m \geq 1$, hence $\kappa(X) = -\infty$. For an arbitrary projective variety X, we define $\kappa(X) := \kappa(\tilde{X})$, where $\tilde{X}$ is a desingularization of X.

A famous conjecture in the classification theory of higher dimensional algebraic varieties predicts that X is uniruled if and only if $\kappa(X) = -\infty$. This is known to be true when $\dim(X) \leq 3$.

Severi was the first to study the rationality of $\mathcal{M}_g$. The existence of a rational parametrization of $\mathcal{M}_g$ would mean that we can describe most curves of genus g by equations depending on free parameters. In other words, we can write down more or less explicitly the general curve of genus g and to paraphrase Mumford (see [Mu]) we would be able to boast: "We have seen every curve once".

On the other hand, the non-uniruledness of $\mathcal{M}_g$ would have pretty spectacular consequences for the geometry of curves of genus g. For instance, it would imply that if C is a general curve of genus g and S is a surface containing C such that $\dim |C| \geq 1$, then S must be birational to $C \times \mathbb{P}^1$. To rephrase it, the general curve of genus g does not appear in a non-trivial linear system on any non-ruled surface (cf. [HM]). We refer to the beginning of Chapter 1 for a more detailed history of the problem of unirationality of $\mathcal{M}_g$.

0.3 The Brill-Noether Theorem

We view curves as (abstract) 1-dimensional smooth, complete, algebraic varieties. Understanding the various embeddings of curves in projective spaces will add a great deal to our knowledge of the geometry of algebraic curves.

Let C be an algebraic curve. A nondegenerate map $f : C \to \mathbb{P}^r$ is given by a linear series of dimension r on C, that is, a pair $l = (\mathcal{L}, V)$, where $\mathcal{L}$ is a line bundle on C and $V \subseteq H^0(C, \mathcal{L})$ is a subspace of dimension $r + 1$. Assuming that l is *base-point-free* (i.e. for every $p \in C$ there is $s \in V$ such that $s(p) \neq 0$), by choosing a base $(s_0, \ldots, s_r)$ in V, we obtain a map $f : C \to \mathbb{P}^r$, given by $f(p) := [s_0(p), \dots, s_r(p)]$. To $l = (\mathcal{L}, V)$ we also associate the r-dimensional system of divisors

$$V' = \{\mathrm{div}(s) : s \in V\} \subseteq \mathbb{P}(H^0(C, \mathcal{L})).$$

If $\deg(\mathcal{L}) = d$, following classical terminology, we say that $l = (\mathcal{L}, V)$ is a $\mathfrak{g}^r_d$, which means "a group of d points moving with r degrees of freedom". We can ask the following question: What kind of linear series does a curve of genus g have? The answer is the following famous result (cf. [BN]):

Theorem 0.1 (Brill-Noether, 1874) *A general curve of genus g has a $\mathfrak{g}^r_d$ if and only if $\rho(g, r, d) = g - (r+1)(g - d + r) \geq 0$.*

To mention a few things about the history of this theorem, we note that in the 1920's after the result had been taken for granted for 40 years, Severi realized that Brill and Noether had only proved that any component of the variety of $\mathfrak{g}^r_d$'s on C has dimension $\geq \rho(g, r, d)$ but they failed to prove the existence of such a component (or the nonexistence when $\rho(g, r, d) < 0$). The Brill-Noether Theorem was finally proved in 1980 by Griffiths and Harris (cf. [GH]) using an old idea of Castelnuovo of specializing to a general curve of arithmetic genus g with g nodes.

It is worth to outline the parameter count that prompted Brill and Noether to claim their theorem and which brings into the picture the *Brill-Noether number* $\rho(g, r, d)$ (see also [GriffHa] for this count).

Let C be a general curve of genus $g \geq 3$, in particular, C is non-hyperelliptic. Consider $C \hookrightarrow \mathbb{P}^{g-1}$ canonically embedded and let $D = \sum_{i=1}^d p_i$ be a divisor of degree d on C. Assume D is part of a $\mathfrak{g}^r_d$, i.e. $\dim D \geq r$. Then by Riemann-Roch it follows that D spans a $(d - r - 1)$-plane in $\mathbb{P}^{g-1}$. Since D moves in an r-dimensional family, we get that C has a $\mathfrak{g}^r_d$ if and only if C has an r-dimensional family of d-secant $(d - r - 1)$-planes.

Since the variety of $(d - r - 1)$-planes meeting C at least once has codimension $g - d + r - 1$ in $\mathbb{G}(d - r - 1, g - 1)$, it is natural to expect that the variety of $(d - r - 1)$-planes that are d-secant to C has codimension $d(g - d + r - 1)$. Therefore we expect this variety to be of dimension $\geq r$ if and only if $\dim \mathbb{G}(d - r - 1, g - 1) - d(g - d + r - 1) \geq r$, and this is equivalent with $\rho(g, r, d) \geq 0$.

0.4 Outline of the results

Chapter 1 deals with the geometry of the moduli space of curves of genus 23. It is known that $\mathcal{M}_g$ is of general type for $g \geq 24$ and that for low values of g (conjecturally for all $g \leq 22$), the moduli space $\mathcal{M}_g$ is uniruled. This leaves $\mathcal{M}_{23}$ as an interesting transition case between two extremes: uniruledness and being of general type. The main result is the following:

Theorem *The Kodaira dimension of the moduli space of curves of genus 23 is ≥ 2.*

The proof is based on the study of three explicit divisors on $\mathcal{M}_{23}$ which turn out to be multicanonical. Evidence is presented which suggests that the Kodaira dimension of $\mathcal{M}_{23}$ is actually equal to 2. Degeneration to singular curves and the theory of limit linear series (reviewed in Section 1.3) as well as deformation theory are the tools we use.

Chapter 2 deals with the geography (relative position) of the *Brill-Noether loci*

$$\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g : C \text{ carries a } \mathfrak{g}^r_d\}.$$

We compare different Brill-Noether loci and show that they are in general relative position (transversal) inside $\mathcal{M}_g$, unless there are some obvious containment relations between them. In Section 2.4 we prove under certain numerical conditions the existence of *regular* (generically smooth, of the expected dimension) components of the Hilbert scheme of curves $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$, where $r \geq 3$. The main result of Section 2.5 is the following theorem concerning the gonality of space curves:

Theorem *Let $g \geq 5$ and $d \geq 8$ be integers with g odd and d even, such that $d^2 > 8g$, $4d \leq 3g + 12$, $d^2 - 8g + 8$ is not a square and either $d \leq 18$ or $g < 4d - 31$. Assume that*

$$(d', g') \in \{(d, g), (d+1, g+1), (d+1, g+2), (d+2, g+3)\}.$$

Then there exists a regular component of the scheme $\mathrm{Hilb}_{d',g',3}$, with general point a smooth curve $C' \subseteq \mathbb{P}^3$ of degree d' and genus g' and such that the gonality of C' is $\min(\mathrm{d}' - 4, [(\mathrm{g}' + 3)/2])$.

As a consequence of results from Chapter 2, we find a new proof for our result $\kappa(\mathcal{M}_{23}) \geq 2$.

In Chapter 3 certain aspects of the geometry of the moduli spaces $\mathcal{M}_{g,n}$ of n-pointed curves of genus g are studied. For an integer $g \equiv 1 \bmod 3$ with $g \geq 4$ we set $d := (2g+7)/3$ and we can consider the following divisors:

- On $\overline{\mathcal{M}}_{g,1}$, the closure of the locus HF (resp. CU) consisting of 1-pointed curves $(C, p) \in \mathcal{M}_{g,1}$ such that there exists a $\mathfrak{g}^2_d$ on C with a *hyperflex* (resp. *cusp*) at the point p.
- On $\overline{\mathcal{M}}_{g,2}$, the closure of the locus FL consisting of those $[C, p_1, p_2] \in \mathcal{M}_{g,2}$ such that there exists a $\mathfrak{g}^2_d$ with *flexes* at both p_1 and p_2.

We determine the classes $[\overline{HF}]$, $[\overline{CU}]$ and $[\overline{FL}]$ (in the respective Picard groups).

For an integer $d \geq 3$ we set $g := 2d - 4$. We denote by TR the locus of 1-pointed curves $[C, p] \in \mathcal{M}_{g,1}$ such that there exists a degree d map $f : C \to \mathbb{P}^1$ having triple ramification at p and at some unspecified point $x \in C - \{p\}$. It turns out that $\overline{TR}$ is a divisor on $\overline{\mathcal{M}}_{g,1}$ and in Section 3.6 we compute the class $[\overline{TR}]$ in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,1})$.

We close Chapter 3 by proving in Section 3.7 the following:

Theorem *For $g = 11, 12$ and 15 the universal curve $\mathcal{C}_g$ has Kodaira dimension $-\infty$.*

The first two chapters of this thesis are based on the papers

G. Farkas: *The Geometry of the Moduli Space of Curves of Genus 23*, to appear in *Mathematische Annalen* (also available as math.AG/9907013 preprint).

G. Farkas: *The Geography of Brill-Noether Loci in the Moduli Space of Curves*, preliminary version.

Chapter 1

The geometry of the moduli space of curves of genus 23

1.1 Introduction

The problem of describing the birational geometry of the moduli space $\mathcal{M}_g$ of complex curves of genus g has a long history. Severi already knew in 1915 that $\mathcal{M}_g$ is unirational for $g \leq 10$ (cf. [Sev]; see also [AC1] for a modern proof). In the same paper Severi conjectured that $\mathcal{M}_g$ is unirational for all genera g. Then for a long period this problem seemed intractable (Mumford writes in [Mu], p.51:"Whether more $\mathcal{M}_g$'s, $g \geq 11$, are unirational or not is a very interesting problem, but one which looks very hard too, especially if g is quite large"). The breakthrough came in the eighties when Eisenbud, Harris and Mumford proved that $\mathcal{M}_g$ is of general type as soon as $g \geq 24$ and that the Kodaira dimension of $\mathcal{M}_{23}$ is ≥ 1 (see [HM], [EH3]). We note that $\mathcal{M}_g$ is rational for $g \leq 6$ (see [Dol] for problems concerning the rationality of various moduli spaces).

Severi's proof of the unirationality of $\mathcal{M}_g$ for small g was based on representing a general curve of genus g as a plane curve of degree d with δ nodes; this is possible when $d \geq 2g/3+2$. When the number of nodes is small, i.e. $\delta \leq (d+1)(d+2)/6$, the dominant map from the variety of plane curves of degree d and genus g to $\mathcal{M}_g$ yields a rational parametrization of the moduli space. The two conditions involving d and δ can be satisfied only when $g \leq 10$, so Severi's argument cannot be extended for other genera. However, using much more subtle ideas, Chang, Ran and Sernesi proved the unirationality of $\mathcal{M}_g$ for $g = 11, 12, 13$ (see [CR1], [Se1]), while for $g = 15, 16$ they proved that the Kodaira dimension is $-\infty$ (see [CR2,4]). The remaining cases $g = 14$ and $17 \leq g \leq 23$ are still quite mysterious. Harris and Morrison conjectured in [HMo] that $\mathcal{M}_g$ is uniruled precisely when $g < 23$.

All these facts indicate that $\mathcal{M}_{23}$ is a very interesting transition case. Our main result is the following:

Theorem 1.1 *The Kodaira dimension of the moduli space of curves of genus 23 is ≥ 2.*

We will also present some evidence for the hypothesis that the Kodaira dimension of $\mathcal{M}_{23}$ is actually equal to 2.

1.2 Multicanonical linear systems and the Kodaira dimension of $\mathcal{M}_g$

We study three multicanonical divisors on $\mathcal{M}_{23}$, which are (modulo some boundary components) of Brill-Noether type, and we conclude by looking at their relative position that $\kappa(\mathcal{M}_{23}) \geq 2$.

We review some notations. We shall denote by $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{C}}_g$ the moduli spaces of stable and 1-pointed stable curves of genus g over $\mathbb{C}$. If C is a smooth algebraic curve of genus g, we consider for any r and d, the scheme whose points are the $\mathfrak{g}^r_d$'s on C, that is,

$$G^r_d(C) = \{(\mathcal{L}, V) : \mathcal{L} \in \text{Pic}^d(C), V \subseteq H^0(C, \mathcal{L}), \dim(V) = r+1\},$$

(cf. [ACGH]) and denote the associated Brill-Noether locus in $\mathcal{M}_g$ by

$$\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g : G^r_d(C) \neq \emptyset\},$$

and by $\overline{\mathcal{M}}^r_{g,d}$ its closure in $\overline{\mathcal{M}}_g$.

The distribution of linear series on algebraic curves is governed (to some extent) by the *Brill-Noether number*

$$\rho(g, r, d) := g - (r+1)(g - d + r).$$

The Brill-Noether Theorem asserts that when $\rho(g, r, d) \geq 0$ every curve of genus g possesses a $\mathfrak{g}^r_d$, while when $\rho(g, r, d) < 0$ the general curve of genus g has no $\mathfrak{g}^r_d$'s, hence in this case the Brill-Noether loci are proper subvarieties of $\mathcal{M}_g$. When $\rho(g, r, d) < 0$, the naive expectation that $-\rho(g, r, d)$ is the codimension of $\mathcal{M}^r_{g,d}$ inside $\mathcal{M}_g$, is in general way off the mark, since there are plenty of examples of Brill-Noether loci of unexpected dimension (cf. [EH2]). However, we have Steffen's result in one direction (see [St]):

If $\rho(g, r, d) < 0$ then each component of $\mathcal{M}^r_{g,d}$ has codimension at most $-\rho(g, r, d)$ in $\mathcal{M}_g$.

On the other hand, when the Brill-Noether number is not very negative, the Brill-Noether loci tend to behave nicely. Existence of components of $\mathcal{M}^r_{g,d}$ of the expected dimension has been proved for a rather wide range, namely for those g, r, d such that $\rho(g, r, d) < 0$, and

$$\rho(g, r, d) \geq \begin{cases} -g + r + 3 & \text{if } r \text{ is odd;} \\ -rg/(r+2) + r + 3 & \text{if } r \text{ is even.} \end{cases}$$

We have a complete answer only when $\rho(g, r, d) = -1$. Eisenbud and Harris have proved in [EH2] that in this case $\mathcal{M}^r_{g,d}$ has a unique divisorial component, and using the previously mentioned theorem of Steffen's, we obtain the following result:

If $\rho(g,r,d) = -1$, then $\overline{\mathcal{M}}^r_{g,d}$ is an irreducible divisor of $\overline{\mathcal{M}}_g$.

We will also need Edidin's result (see [Ed2]) which says that for $g \geq 12$ and $\rho(g,r,d) = -2$, all components of $\mathcal{M}^r_{g,d}$ have codimension 2. We can get codimension 1 Brill-Noether conditions only for the genera g for which $g+1$ is composite. In that case we can write

$$g + 1 = (r+1)(s-1), \ s \geq 3$$

and set $d := rs - 1$. Obviously $\rho(g,r,d) = -1$ and $\overline{\mathcal{M}}^r_{g,d}$ is an irreducible divisor. Furthermore, its class has been computed (cf. [EH3]):

$$[\overline{\mathcal{M}}^r_{g,d}] = c_{g,r,d}\left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g-i)\delta_i\right),$$

where $c_{g,r,d}$ is a positive rational number equal to $3\mu/(2g-4)$, with μ being the number of $\mathfrak{g}^r_d$'s on a general pointed curve (C_0, q) of genus $g-2$ with ramification sequence $(0,1,2,\ldots,2)$ at q. For $g = 23$ we have the following possibilities:

$$(r,s,d) = (1,13,12),\ (11,3,32),\ (2,9,17),\ (7,4,24),\ (3,7,20),\ (5,5,24).$$

It is immediate by Serre duality, that cases $(1,13,12)$ and $(11,3,32)$ yield the same divisor on $\mathcal{M}_{23}$, namely the 12-gonal locus $\mathcal{M}^1_{12}$; similarly, cases $(2,9,17)$ and $(7,4,24)$ yield the divisor $\mathcal{M}^2_{17}$ of curves having a $\mathfrak{g}^2_{17}$, while cases $(3,7,20)$ and $(5,5,24)$ give rise to $\mathcal{M}^3_{20}$, the divisor of curves having a $\mathfrak{g}^3_{20}$. Note that when the genus we are referring to is clear from the context, we write $\mathcal{M}^r_d = \mathcal{M}^r_{g,d}$.

By comparing the classes of the Brill-Noether divisors to the class of the canonical divisor $K_{\overline{\mathcal{M}}_{g,reg}} = 13\lambda - 2\delta_0 - 3\delta_1 - \cdots - 2\delta_{[g/2]}$, at least in the case when $g+1$ is composite we can infer that

$$K_{\overline{\mathcal{M}}_{g,reg}} = a[\overline{\mathcal{M}}^r_{g,d}] + b\lambda + (\text{ positive combination of } \delta_0,\ldots,\delta_{[g/2]}),$$

where a is a positive rational number, while $b > 0$ as long as $g \geq 24$ but $b = 0$ for $g = 23$. As it is well-known that λ is big on $\overline{\mathcal{M}}_g$, it follows that $\mathcal{M}_g$ is of general type for $g \geq 24$ and that it has non-negative Kodaira dimension when $g = 23$. Specifically for $g = 23$, we get that there are positive integer constants m, m_1, m_2, m_3 such that:

$$mK = m_1[\overline{\mathcal{M}}^1_{12}] + E,\ mK = m_2[\overline{\mathcal{M}}^2_{17}] + E,\ mK = m_3[\overline{\mathcal{M}}^3_{20}] + E, \tag{1.1}$$

where E is the same positive combination of $\delta_1,\ldots,\delta_{11}$.

Proposition 1.2.1 (Eisenbud-Harris, [EH3]) *There exists a smooth curve of genus 23 that possesses a $\mathfrak{g}^1_{12}$, but no $\mathfrak{g}^2_{17}$. It follows that $\kappa(\mathcal{M}_{23}) \geq 1$.*

Harris and Mumford proved (cf. [HM]) that $\overline{\mathcal{M}}_g$ has only canonical singularities for $g \geq 4$, hence $H^0(\overline{\mathcal{M}}_{g,reg}, nK) = H^0(\widetilde{\mathcal{M}}_g, nK)$ for each $n \geq 0$, with $\widetilde{\mathcal{M}}_g$ a desingularization of $\overline{\mathcal{M}}_g$.

We already know that $\dim(\mathrm{Im}\phi_{mK}) \geq 1$, where $\phi_{mK} : \overline{\mathcal{M}}_{23} -- \rightarrow \mathbb{P}^{\nu}$ is the multicanonical map, m being as in (1.1). We will prove that $\kappa(\overline{\mathcal{M}}_{23}) \geq 2$. Indeed, let us assume that $\dim(\mathrm{Im}\phi_{mK}) = 1$. Denote by $C := \overline{\mathrm{Im}\phi_{mK}}$ the Kodaira image of $\overline{\mathcal{M}}_{23}$. We reach a contradiction by proving two things:

- α) The Brill-Noether divisors $\mathcal{M}^1_{12}, \mathcal{M}^2_{17}$ and $\mathcal{M}^3_{20}$ are mutually distinct.
- β) There exist smooth curves of genus 23 which belong to exactly two of the Brill-Noether divisors from above.

This suffices in order to prove Theorem 1.1: since $\overline{\mathcal{M}}^1_{12}, \overline{\mathcal{M}}^2_{17}$ and $\overline{\mathcal{M}}^3_{20}$ are part of different multicanonical divisors, they must be contained in different fibres of the multicanonical map ϕ_{mK}. Hence there exists different points $x, y, z \in C$ such that

$$\mathcal{M}^1_{12} = \overline{\phi^{-1}(x)} \cap \mathcal{M}_{23}, \ \mathcal{M}^2_{17} = \overline{\phi^{-1}(y)} \cap \mathcal{M}_{23}, \ \mathcal{M}^3_{20} = \overline{\phi^{-1}(z)} \cap \mathcal{M}_{23}.$$

It follows that the set-theoretic intersection of any two of them will be contained in the base locus of $mK_{\overline{\mathcal{M}}_{23}}$. In particular:

$$\mathrm{supp}(\mathcal{M}^1_{12}) \cap \mathrm{supp}(\mathcal{M}^2_{17}) = \mathrm{supp}(\mathcal{M}^2_{17}) \cap \mathrm{supp}(\mathcal{M}^3_{20}) = \mathrm{supp}(\mathcal{M}^3_{20}) \cap \mathrm{supp}(\mathcal{M}^1_{12}). \quad (1.2)$$

and this contradicts β). We complete the proof of α) and β) is Section 1.5.

1.3 Deformation theory for $\mathfrak{g}^r_d$'s and limit linear series

We recall a few things about the variety parametrising $\mathfrak{g}^r_d$'s on the fibres of the universal curve (cf. [AC2]), and then we recap on the theory of limit linear series (cf. [EH1], [Mod]), which is our main technique for the study of $\mathcal{M}_{23}$.

Given g, r, d and a point $[C] \in \mathcal{M}_g$, there is a connected neighbourhood U of $[C]$, a finite ramified covering $h : \mathcal{M} \rightarrow U$, such that $\mathcal{M}$ is a fine moduli space of curves (i.e. there exists $\xi : \mathcal{C} \rightarrow \mathcal{M}$ a universal curve), and a proper variety over $\mathcal{M}$,

$$\pi : \mathcal{G}^r_d \rightarrow \mathcal{M}$$

which parametrizes classes of couples (C, l), with $[C] \in \mathcal{M}$ and $l \in G^r_d(C)$, where we have made the identification $C = \xi^{-1}([C])$.
Let (C, l) be a point of $\mathcal{G}^r_d$ corresponding to a curve C and a linear series $l = (\mathcal{L}, V)$, where $\mathcal{L} \in \mathrm{Pic}^d(C), V \subseteq H^0(C, \mathcal{L})$, and $\dim(V) = r+1$. By choosing a basis in V, one has a morphism $f : C \rightarrow \mathbb{P}^r$. The normal sheaf of f is defined through the exact sequence

$$0 \longrightarrow T_C \longrightarrow f^*(T_{\mathbb{P}^r}) \longrightarrow N_f \longrightarrow 0. \quad (1.3)$$

By dividing out the torsion of N_f one gets to the exact sequence

$$0 \longrightarrow \mathcal{K}_f \longrightarrow N_f \longrightarrow N'_f \longrightarrow 0, \quad (1.4)$$

where the torsion sheaf $\mathcal{K}_f$ (the cuspidal sheaf) is based at those points $x \in C$ where $df(x) = 0$, and N'_f is locally free of rank $r-1$. The tangent space $T_{(C,l)}(\mathcal{G}^r_d)$ fits into an exact sequence (cf. [AC2]):

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{Hom}(V, V) \longrightarrow H^0(C, N_f) \longrightarrow T_{(C,l)}(\mathcal{G}^r_d) \longrightarrow 0, \quad (1.5)$$

from which we have that $\dim T_{(C,l)}(\mathcal{G}^r_d) = 3g - 3 + \rho(g, r, d) + h^1(C, N_f)$.

Proposition 1.3.1 *Let C be a curve and $l \in G^r_d(C)$ a base point free linear series. Then the variety $\mathcal{G}^r_d$ is smooth and of dimension $3g-3+\rho(g,r,d)$ at the point (C,l) if and only if $H^1(C,N_f)=0$.*

Remark: The condition $H^1(C,N_f)=0$ is automatically satisfied for $r=1$ as N_f is a sheaf with finite support. Thus $\mathcal{G}^1_d$ is smooth of dimension $2g+2d-5$. It follows that $\mathcal{G}^1_d$ is birationally equivalent to the d-gonal locus $\mathcal{M}^1_d$ when $d<(g+2)/2$.

In Chapter 2 we will be interseted in the differential $(d\pi)_{(C,l)} : T_{(C,l)}(\mathcal{G}^r_d) \to T_{[C]}(\mathcal{M}_g)$. Let $(C,l) \in \mathcal{G}^r_d$ be a point such that $H^1(C,N_f)=0$ and assume for simplicity that $l=(\mathcal{L},V)$ is a complete, base point free linear series, that is, $V=H^0(C,\mathcal{L})$. By standard Kodaira-Spencer theory (see [AC2] or [Mod]) one has that

$$\mathrm{Im}(d\pi)_{(C,l)} = \mathrm{Im}\{\delta : H^0(C,N_f) \to H^1(C,T_C)\},$$

where δ is the coboundary of the cohomology sequence associated to (1.3). We thus get that $\mathrm{rk}(d\pi)_{(C,l)} = 3g-3-h^1(C,f^*(T_{\mathbb{P}^r}))$. By pulling back to C the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{O}_{\mathbb{P}^r}(1)^{r+1} \longrightarrow T_{\mathbb{P}^r} \longrightarrow 0,$$

we obtain that $H^1(C,f^*T_{\mathbb{P}^r}) \simeq \left(\mathrm{Ker}(\mu_0(C,\mathcal{L}))\right)^\vee$, where

$$\mu_0(C,\mathcal{L}) : H^0(C,\mathcal{L}) \otimes H^0(C,K_C \otimes \mathcal{L}^\vee) \to H^0(C,K_C)$$

is the *Petri map*. We obtain thus that the differential $(d\pi)_{(C,l)}$ has the expected rank $\min(3g-3, 3g-3+\rho(g,r,d))$, if and only if the Petri map is of maximal rank (which means surjective when $\rho(g,r,d)<0$).

It is convenient to have a description of the annihilator $(\mathrm{Im}(d\pi)_{(C,l)})^\perp \subseteq H^0(C,2K_C)$, where we have made the identification $T_{[C]}(\mathcal{M}_g)^\vee = H^0(C,2K_C)$ via Serre duality. We introduce the Gaussian map (cf. [CGGH])

$$\mu_1(C,\mathcal{L}) : \mathrm{Ker}\mu_0(C,\mathcal{L}) \to H^0(C,2K_C),$$

as follows: let us consider the evaluation sequence corresponding to $(C,\mathcal{L})$

$$0 \longrightarrow M_{\mathcal{L}} \longrightarrow H^0(C,\mathcal{L}) \otimes \mathcal{O}_C \longrightarrow \mathcal{L} \longrightarrow 0.$$

We restrict the $\mathbb{C}$-linear map $H^0(C,\mathcal{L}) \otimes \mathcal{O}_C \to \Omega_C \otimes \mathcal{L}$, $s \otimes f \mapsto df \otimes s$, to the kernel $M_{\mathcal{L}}$ and get an $\mathcal{O}_C$-linear map $M_{\mathcal{L}} \to \Omega_C \otimes \mathcal{L}$. If we tensor this map with $\Omega_C \otimes \mathcal{L}^\vee$, then take global sections and finally use that $H^0(C,M_{\mathcal{L}} \otimes \Omega_C \otimes \mathcal{L}^\vee) \simeq \mathrm{Ker}\mu_0(C,\mathcal{L})$, we get the map $\mu_1(C,\mathcal{L}) : \mathrm{Ker}\ \mu_0(C,\mathcal{L}) \to H^0(C,2K_C)$, which we can loosely refer to as the 'derivative' of the Petri map.

The map $\mu_1(C,\mathcal{L})$ can be explicitly described: for an element $s_0 \otimes \eta_0 + \cdots + s_r \otimes \eta_r \in \mathrm{Ker}\mu_0(C,\mathcal{L})$, with $s_i \in H^0(C,\mathcal{L})$ and $\eta_i \in H^0(C,\Omega_C \otimes \mathcal{L}^\vee)$, if we consider the meromorphic functions $f_i = s_i/s_0$ on C, we have that

$$\mu_1(C,\mathcal{L})(s_0 \otimes \eta_0 + \cdots + s_r \otimes \eta_r) = s_0(\eta_1 df_1 + \cdots + \eta_r df_r).$$

An easy calculation shows that

$$(\mathrm{Im}(d\pi)_{(C,l)})^{-} = \mathrm{Im}\mu_1(C,\mathcal{L}) \subseteq H^0(C, 2K_C).$$

Limit linear series try to answer questions of the following kind: what happens to a family of $\mathfrak{g}^r_d$'s when a smooth curve specializes to a reducible curve? Limit linear series solve such problems for a class of reducible curves, those of compact type. A curve C is of *compact type* if its dual graph is a tree. A curve C is *tree-like* if, after deleting edges leading from a node to itself, the dual graph becomes a tree.

Let C be a smooth curve of genus g and $l = (\mathcal{L}, V) \in G^r_d(C)$, $\mathcal{L} \in \mathrm{Pic}^d(C)$, $V \subseteq H^0(C, \mathcal{L})$, and $\dim(V) = r+1$. Fix $p \in C$ a point. By ordering the finite set $\{\mathrm{ord}_p(\sigma)\}_{\sigma\in V}$ one gets the *vanishing sequence* of l at p:

$$a^l(p) : 0 \leq a^l_0(p) < \ldots < a^l_r(p) \leq d.$$

The *ramification sequence* of l at p

$$\alpha^l(p) : 0 \leq \alpha^l_0(p) \leq \ldots \leq \alpha^l_r(p) \leq d - r$$

is defined as $\alpha^l_i(p) = a^l_i(p) - i$ and the *weight* of l at p is

$$w^l(p) = \sum_{i=0}^{r} \alpha^l_i(p).$$

A *Schubert index of type* (r, d) is a sequence of integers $\beta : 0 \leq \beta_0 \leq \ldots \beta_r \leq d - r$. If α and β are Schubert indices of type (r, d) we write $\alpha \leq \beta \iff \alpha_i \leq \beta_i, i = 0, \ldots, r$. The point p is said to be a *ramification point* of l if $w^l(p) > 0$. The linear series l is said to have a *cusp* at p if $\alpha^l(p) \geq (0, 1, \ldots, 1)$. For C a tree-like curve, $p_1, \ldots, p_n \in C$ smooth points and $\alpha^1, \ldots, \alpha^n$ Schubert indices of type (r, d), we define

$$G^r_d(C, (p_1, \alpha^1), \ldots (p_n, \alpha^n)) := \{l \in G^r_d(C) : \alpha^l(p_1) \geq \alpha^1, \ldots, \alpha^l(p_n) \geq \alpha^n\}.$$

This scheme can be realized naturally as a determinantal variety and its expected dimension is

$$\rho(g, r, d, \alpha^1, \ldots, \alpha^n) := \rho(g, r, d) - \sum_{i=1}^{n}\sum_{j=0}^{r} \alpha^i_j.$$

If C is a curve of compact type, a *crude limit* $\mathfrak{g}^r_d$ on C is a collection of ordinary linear series $l = \{l_Y \in G^r_d(Y) : Y \subseteq C \text{ is a component}\}$, satisfying the following compatibility condition: if Y and Z are components of C with $\{p\} = Y \cap Z$, then

$$a^{l_Y}_i(p) + a^{l_Z}_{r-i}(p) \geq d, \text{ for } i = 0, \ldots r.$$

If equality holds everywhere, we say that l is a *refined limit* $\mathfrak{g}^r_d$. The 'honest' linear series $l_Y \in G^r_d(Y)$ is called the Y-aspect of the limit linear series l.

We will often use the additivity of the Brill-Noether number: if C is a curve of compact type, for each component $Y \subseteq C$, let $q_1, \ldots, q_s$ be the points where Y meets the other components of C. Then for any limit $\mathfrak{g}^r_d$ on C we have the following inequality:

$$\rho(g, r, d) \geq \sum_{Y \subseteq C} \rho(l_Y, \alpha^{l_Y}(q_1), \ldots, \alpha^{l_Y}(q_s)), \tag{1.6}$$

with equality if and only if l is a refined limit linear series.

It has been proved in [EH1] that limit linear series arise indeed as limits of ordinary linear series on smooth curves. Suppose we are given a family $\pi : \mathcal{C} \to B$ of genus g curves, where $B = \mathrm{Spec}(R)$ with R a complete discrete valuation ring. Assume furthermore that $\mathcal{C}$ is a smooth surface and that if $0, \eta$ denote the special and generic point of B respectively, the central fibre C_0 is reduced and of compact type, while the generic geometric fibre C_η is smooth and irreducible. If $l_\eta = (\mathcal{L}_\eta, V_\eta)$ is a $\mathfrak{g}^r_d$ on C_η, there is a canonical way to associate a crude limit series l_0 on C_0 which is the limit of l_η in a natural way: for each component Y of C_0, there exists a unique line bundle $\mathcal{L}^Y$ on $\mathcal{C}$ such that

$$\mathcal{L}^Y_{\mathcal{C}_\eta} = \mathcal{L}_\eta \text{ and } \deg_Z(\mathcal{L}^Y_{\ Z}) = 0,$$

for any component Z of C_0 with $Z \neq Y$. (This implies of course that $\deg_Y(\mathcal{L}^Y_{\ Y}) = d$). Define $V^Y = V_\eta \cap H^0(\mathcal{C}, \mathcal{L}^Y) \subseteq H^0(C_\eta, \mathcal{L}_\eta)$. Clearly, V^Y is a free R-module of rank $r+1$. Moreover, the composite homomorphism

$$V^Y(0) \to (\pi_* \mathcal{L}^Y)(0) \to H^0(C_0, \mathcal{L}^Y_{|C_0}) \to H^0(Y, \mathcal{L}^Y_{\ Y})$$

is injective, hence $l_Y = (\mathcal{L}^Y_{|Y}, V^Y(0))$ is an ordinary $\mathfrak{g}^r_d$ on Y. One proves that $l = \{l_Y :$ Y component of $C_0\}$ is a limit linear series.

If C is a reducible curve of compact type, l a limit $\mathfrak{g}^r_d$ on C, we say that l is *smoothable* if there exists $\pi : \mathcal{C} \to B$ a family of curves with central fibre $C = C_0$ as above, and $(\mathcal{L}_\eta, V_\eta)$ a $\mathfrak{g}^r_d$ on the generic fibre C_η whose limit on C (in the sense previously described) is l.

Remark: If a stable curve of compact type C, has no limit $\mathfrak{g}^r_d$'s, then $[C] \notin \overline{\mathcal{M}}^r_{g,d}$. If there exists a smoothable limit $\mathfrak{g}^r_d$ on C, then $[C] \in \overline{\mathcal{M}}^r_{g,d}$.

Now we explain a criterion due to Eisenbud and Harris (cf. [EH1]), which gives a sufficient condition for a limit $\mathfrak{g}^r_d$ to be smoothable. Let l be a limit $\mathfrak{g}^r_d$ on a curve C of compact type. Fix $Y \subseteq C$ a component, and $\{q_1, \ldots, q_s\} = Y \cap \overline{(C - Y)}$. Let

$$\pi : \mathcal{Y} \to B, \ \tilde{q}_i : B \to \mathcal{Y}$$

be the versal deformation space of $(Y, q_1, \ldots q_s)$. The base B can be viewed as a small $(3g(Y)-3+s)$-dimensional polydisk. Using general theory one constructs a proper scheme over B,

$$\sigma : \mathcal{G}^r_d(\mathcal{Y}/B; (\tilde{q}_i, \alpha^{l_Y}(q_i))^s_{i=1}) \to B$$

whose fibre over each $b \in B$ is $\sigma^{-1}(b) = G^r_d(Y_b, (\tilde{q}_i(b), \alpha^{l_Y}(q_i))^s_{i=1})$. One says that l is *dimensionally proper with respect to* Y, if the Y-aspect l_Y is contained in some component $\mathcal{G}$ of $\mathcal{G}^r_d(\mathcal{Y}/B; (\tilde{q}_i, \alpha^{l_Y}(q_i))^s_{i=1})$ of the expected dimension, i.e.

$$\dim \mathcal{G} = \dim B + \rho(l_Y, \alpha^{l_Y}(q_1), \ldots \alpha^{l_Y}(q_s)).$$

One says that l is *dimensionally proper*, if it is dimensionally proper with respect to any component $Y \subseteq C$. The 'Regeneration Theorem' (cf. [EH1]) states that every dimensionally proper limit linear series is smoothable.

The next result is a 'strong Brill-Noether Theorem', i.e. it not only asserts a Brill-Noether type statement, but also singles out the locus where the statement fails.

Proposition 1.3.2 (Eisenbud-Harris) *Let C be a tree-like curve and for any component $Y \subseteq C$, denote by $q_1, \ldots, q_s \in Y$ the points where Y meets the other components of C. Assume that for each Y the following conditions are satisfied:*

a. If $g(Y) = 1$ then $s = 1$.

b. If $g(Y) = 2$ then $s = 1$ and q is not a Weierstrass point.

c. If $g(Y) \geq 3$ then $(Y, q_1, \ldots, q_s)$ is a general s-pointed curve.

Then for $p_1, \ldots p_t \in C$ general points, $\rho(l, \alpha^l(p_1), \ldots, \alpha^l(p_t)) \geq 0$ for any limit linear series on C.

Simple examples involving pointed elliptic curves show that the condition $\rho(g, r, d) \geq \sum_{i=1}^{t} w^l(p_i)$ does not guarantee the existence of a linear series $l \in G^r_d(C)$ with prescribed ramification at general points $p_1, p_2, \ldots, p_t \in C$. The appropriate condition in the pointed case can be given in terms of Schubert cycles. Let $\alpha = (\alpha_0, \ldots, \alpha_r)$ be a Schubert index of type (r, d) and

$$\mathbb{C}^{d+1} = W_0 \supset W_1 \supset \ldots \supset W_{d+1} = 0$$

a decreasing flag of linear subspaces. We consider the Schubert cycle in the Grassmanian,

$$\sigma_\alpha = \{\Lambda \in G(r+1, d+1) : \dim(\Lambda \cap W_{\alpha_i + i}) \geq r + 1 - i, \ i = 0, \ldots, r\}.$$

For a general t-pointed curve $(C, p_1, \ldots, p_t)$ of genus g, and $\alpha^1, \ldots, \alpha^t$ Schubert indices of type (r, d), the necessary and sufficient condition that C has a $\mathfrak{g}^r_d$ with ramification α^i at p_i is that

$$\sigma_{\alpha^1} \cdot \ldots \cdot \sigma_{\alpha^t} \cdot \sigma^g_{(0,1,\ldots,1)} \neq 0 \text{ in } H^*(G(r+1, d+1), \mathbb{Z}). \tag{1.7}$$

In the case $t = 1$ this condition can be made more explicit (cf. [EH3]): a general pointed curve (C, p) of genus g carries a $\mathfrak{g}^r_d$ with ramification sequence $(\alpha_0, \ldots, \alpha_r)$ at p, if and only if

$$\sum_{i=0}^{r} (\alpha_i + g - d + r)_+ \leq g. \tag{1.8}$$

where $x_+ = \max\{x, 0\}$. One can make the following simple but useful observation:

Proposition 1.3.3 *Let (C, p, q) be a general 2-pointed curve of genus g and $(\alpha_0, \dots, \alpha_r)$ a Schubert index of type (r, d). Then C has a $\mathfrak{g}^r_d$ having ramification sequence $(\alpha_0, \dots, \alpha_r)$ at p and a cusp at q if and only if*

$$\sum_{i=0}^{r}(\alpha_i + g + 1 - d + r)_+ \leq g + 1.$$

Proof: The condition for the existence of the $\mathfrak{g}^r_d$ with ramification α at p and a cusp at q is that $\sigma_\alpha \cdot \sigma^{g+1}_{(0,1,\dots,1)} \neq 0$ (cf. (1.7)). According to the Littlewood-Richardson rule (see [F]), this is equivalent with $\sum_{i=0}^{r}(\alpha_i + g + 1 - d + r)_+ \leq g + 1$. □

1.4 A few consequences of limit linear series

We investigate the Brill-Noether theory of a 2-pointed elliptic curve (see also [EH4]), and we prove that $\overline{\mathcal{M}}^r_{g,d} \cap \Delta_1$ is irreducible for $\rho(g, r, d) = -1$.

Proposition 1.4.1 *Let (E, p, q) be a two-pointed elliptic curve. Consider the sequences*

$$a: \; a_0 < a_1 < \dots a_r \leq d, \quad b: \; b_0 < b_1 < \dots b_r \leq d.$$

1. For any linear series $l = (\mathcal{L}, V) \in G^r_d(E)$ one has that $\rho(l, \alpha^l(p), \alpha^l(q)) \geq -r$. Furthermore, if $\rho(l, \alpha^l(p), \alpha^l(q)) \leq -1$, then $p - q \in \mathrm{Pic}^0(\mathrm{E})$ is a torsion class.
2. Assume that the sequences a and b satisfy the inequalities: $d - 1 \leq a_i + b_{r-i} \leq d$, $i = 0, \dots, r$. Then there exists at most one linear series $l \in G^r_d(E)$ such that $a^l(p) = a$ and $a^l(q) = b$. Moreover, there exists exactly one such linear series $l = (\mathcal{O}_E(D), V)$ with $D \in E^{(d)}$, if and only if for each $i = 0, \dots, r$ the following is satisfied: if $a_i + b_{r-i} = d$, then $D \sim a_i\, p + b_{r-i}\, q$, and if $(a_i + 1)\, p + b_{r-i}\, q \sim D$, then $a_{i+1} = a_i + 1$.

Proof: In order to prove 1. it is enough to notice that for dimensional reasons there must be sections $\sigma_i \in V$ such that $\mathrm{div}(\sigma_i) \geq a^l_i(p)\, p + a^l_{r-i}(q)\, q$, therefore, $a^l_i(p) + a^l_{r-i}(q) \leq d$. By adding up all these inequalities, we get that $\rho(l, \alpha^l(p), \alpha^l(q)) \geq -r$. Furthermore, $\rho(l, \alpha^l(p), \alpha^l(q)) \leq -1$ precisely when for at least two values $i < j$ we have equalities $a_i + b_{r-i} = d$, $a_j + b_{r-j} = d$, which means that there are sections $\sigma_i, \sigma_j \in V$ such that $\mathrm{div}(\sigma_i) = a_i\, p + b_{r-i}\, q$, $\mathrm{div}(\sigma_j) = a_j\, p + b_{r-j}\, q$. By subtracting, we see that $p - q \in \mathrm{Pic}^0(E)$ is torsion. The second part of the Proposition is in fact Prop.5.2 from [EH4]. □

Proposition 1.4.2 *Let g, r, d be such that $\rho(g, r, d) = -1$. Then the intersection $\overline{\mathcal{M}}^r_{g,d} \cap \Delta_1$ is irreducible.*

Proof: Let Y be an irreducible component of $\overline{\mathcal{M}}^r_{g,d} \cap \Delta_1$. Either $Y \cap \mathrm{Int}\Delta_1 \neq \emptyset$, hence $Y = \overline{Y \cap \mathrm{Int}\Delta_1}$, or $Y \subseteq \Delta_1 - \mathrm{Int}\Delta_1$. The second alternative never occurs. Indeed, if $Y \subseteq \Delta_1 - \mathrm{Int}\Delta_1$, then since codim $(Y, \overline{\mathcal{M}_g}) = 2$, Y must be one of the irreducible components of $\Delta_1 - \mathrm{Int}\Delta_1$. The components of $\Delta_1 - \mathrm{Int}\Delta_1$ correspond to curves with two nodes. We list these components (see [Ed1]):

- For $1 \leq j \leq g-2$, Δ_{1j} is the closure of the locus in $\overline{\mathcal{M}}_g$ whose general point corresponds to a chain composed of an elliptic curve, a curve of genus $g-j-1$, and a curve of genus j.

- The component Δ_{01}, whose general point corresponds to the union of a smooth elliptic curve and an irreducible nodal curve of genus $g-2$.

- The component $\Delta_{0,g-1}$ whose general point corresponds to the union of a smooth curve of genus $g-1$ and an irreducible rational curve.

As the general point of $\Delta_{1,j}$, $\Delta_{0,1}$ or $\Delta_{0,g-1}$ is a tree-like curve which satisfies the conditions of Prop.1.3.2 it follows that such a curve satisfies the 'strong' Brill-Noether Theorem, hence $\Delta_{1,j} \nsubseteq \overline{\mathcal{M}}^r_{g,d}$, $\Delta_{0,1} \nsubseteq \overline{\mathcal{M}}^r_{g,d}$ and $\Delta_{0,g-1} \nsubseteq \overline{\mathcal{M}}^r_{g,d}$, a contradiction. So, we are left with the first possibility: $Y = \overline{Y \cap \mathrm{Int}\Delta_1}$. We are going to determine the general point $[C] \in Y \cap \mathrm{Int}\Delta_1$. Let $X = C \cup E$, $g(C) = g-1$, E elliptic, $E \cap C = \{p\}$ such that X carries a limit $\mathfrak{g}^r_d$, say l. By the additivity of the Brill-Noether number, we have:

$$-1 = \rho(g,r,d) \geq \rho(l,C,p) + \rho(l,E,p).$$

Since $\rho(l,E,p) \geq 0$, it follows that $\rho(l,C,p) \leq -1$, so $w^{l_C}(p) \geq r$. Let us denote by

$$\beta : \mathcal{C}_{g-1} \times \mathcal{C}_1 \to \mathrm{Int}\Delta_1$$

the natural map given by $\beta([C,p],[E,q]) = [X := C \cup E/p \sim q]$. We claim that if we choose X generically, then $\alpha^{l_C}_0(p) = 0$. If not, p is a base point of l_C and after removing the base point we get that $[C] \in \mathcal{M}^r_{g-1,d-1}$. Note that $\rho(g-1,r,d-1) = -2$, so $\dim \mathcal{M}^r_{g-1,d-1} = 3g-8$ (cf. [Ed2]). If we denote by $\pi : \mathcal{C}_{g-1} \to \mathcal{M}_{g-1}$ the morphism which 'forgets the point', we get that

$$\dim \beta(\pi^{-1}(\mathcal{M}^r_{g-1,d-1}) \times \mathcal{C}_1) = 3g-6 < \dim Y,$$

a contradiction. Hence, for the generic $[X] \in Y$ we must have $\alpha^{l_C}_0(p) = 0$, so $a^{l_E}_r(p) = d$. Since an elliptic curve cannot have a meromorphic function with a single pole, it follows that $a^{l_E}_{r-1}(p) \leq d-2$ and this implies $\alpha^{l_C}(p) \geq (0,1,\ldots,1)$, i.e. l_C has a cusp at p. Thus, if we introduce the notation

$$\mathcal{C}^r_{g-1,d}(0,1,\ldots,1) = \{[C,p] \in \mathcal{C}_{g-1} : G^r_d(C,(p,(0,1,\ldots,1))) \neq \emptyset\},$$

then $Y \subseteq \overline{\beta(\mathcal{C}^r_{g-1,d}(0,1,\ldots,1) \times \mathcal{C}_1)}$. On the other hand, it is known (cf. [EH2]) that $\mathcal{C}^r_{g-1,d}(0,1,\ldots,1)$ is irreducible of dimension $3g-6$ (that is, codimension 1 in $\mathcal{C}_{g-1}$), so we must have $Y = \overline{\beta(\mathcal{C}^r_{g-1,d}(0,1,\ldots,1) \times \mathcal{C}_1)}$, which not only proves that $\overline{\mathcal{M}}^r_{g,d} \cap \Delta_1$ is irreducible, but also determines the intersection. □

1.5 The Kodaira dimension of $\mathcal{M}_{23}$

In this section we prove that $\kappa(\mathcal{M}_{23}) \geq 2$ and we investigate closely the multicanonical linear systems on $\overline{\mathcal{M}}_{23}$. We now describe the three multicanonical Brill-Noether divisors from Section 2.

1.5.1 The divisor $\overline{\mathcal{M}}^1_{12}$

There is a stratification of $\mathcal{M}_{23}$ given by gonality:

$$\mathcal{M}^1_2 \subseteq \mathcal{M}^1_3 \subseteq \ldots \subseteq \mathcal{M}^1_{12} \subseteq \mathcal{M}_{23}.$$

For $2 \leq d \leq g/2+1$ one knows that $\mathcal{M}^1_k = \mathcal{M}^1_{g,k}$ is an irreducible variety of dimension $2g+2d-5$. The general point of $\mathcal{M}^1_{g,d}$ corresponds to a curve having a unique $\mathfrak{g}^1_d$.

1.5.2 The divisor $\overline{\mathcal{M}}^2_{17}$

The *Severi variety* $V_{d,g}$ of irreducible plane curves of degree d and geometric genus g, where $0 \leq g \leq \binom{d-1}{2}$, is an irreducible subscheme of $\mathbb{P}^{d(d+3)/2}$ of dimension $3d+g-1$ (cf. [H], [Mod]). Inside $V_{d,g}$ we consider the open dense subset $U_{d,g}$ of irreducible plane curves of degree d having exactly $\delta = \binom{d-1}{2} - g$ nodes and no other singularities. There is a global normalization map

$$m : U_{d,g} \to \mathcal{M}_g,\ m([Y]) := [\tilde{Y}],\ \tilde{Y} \text{ is the normalization of } Y.$$

When $d-2 \leq g \leq \binom{d-1}{2}, d \geq 5$, $U_{d,g}$ has the expected number of moduli, i.e.

$$\dim m(U_{d,g}) = \min(3g-3, 3g-3+\rho(g,2,d)).$$

In our case we can summarize this as follows:

Proposition 1.5.1 *There is exactly one component of $\mathcal{G}^2_{17}$ mapping dominantly to $\mathcal{M}^2_{17}$. The general element $(C,l) \in \mathcal{G}^2_{17}$ corresponds to a curve C of genus 23, together with a $\mathfrak{g}^2_{17}$ which provides a plane model for C of degree 17 with 97 nodes.*

1.5.3 The divisor $\overline{\mathcal{M}}^3_{20}$

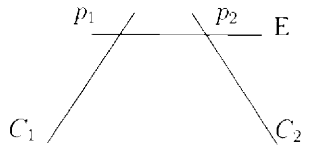
Here we combine the result of Eisenbud and Harris (see [EH2]) about the uniqueness of divisorial components of $\mathcal{G}^r_d$ when $\rho(g,r,d) = -1$, with Sernesi's (see [Se2]) which asserts the existence of components of the Hilbert scheme $H_{d,g}$ parametrizing curves in $\mathbb{P}^3$ of degree d and genus g with the expected number of moduli, for $d-3 \leq g \leq 3d-18, d \geq 9$.

Proposition 1.5.2 *There is exactly one component of $\mathcal{G}^3_{20}$ mapping dominantly to $\mathcal{M}^3_{20}$. The general point of this component corresponds to a pair (C,l) where C is a curve of genus 23 and l is a very ample $\mathfrak{g}^3_{20}$.*

We are going to prove that the Brill-Noether divisors $\overline{\mathcal{M}}^1_{12}, \overline{\mathcal{M}}^2_{17}$ and $\overline{\mathcal{M}}^3_{20}$ are mutually distinct.

Theorem 1.2 *There exists a smooth curve of genus 23 having a $\mathfrak{g}^2_{17}$, but no $\mathfrak{g}^3_{20}$'s. Equivalently, one has $\mathrm{supp}(\mathcal{M}^2_{17}) \not\subseteq \mathrm{supp}(\mathcal{M}^3_{20})$.*

Proof: It suffices to construct a reducible curve X of compact type of genus 23, which has a smoothable limit $\mathfrak{g}^2_{17}$, but no limit $\mathfrak{g}^3_{20}$. If $[C] \in \mathcal{M}_{23}$ is a nearby smoothing of X which preserves the $\mathfrak{g}^2_{17}$, then $[C] \in \mathcal{M}^2_{17} - \mathcal{M}^3_{20}$. Let us consider the following curve:



$$X := C_1 \cup C_2 \cup E.$$

where (C_1, p_1) and (C_2, p_2) are general pointed curves of genus 11, E is an elliptic curve, and $p_1 - p_2$ is a primitive 9-torsion point in $\mathrm{Pic}^0(E)$

Step 1) *There is no limit* $\mathfrak{g}^3_{20}$ *on* X. Assume that l is a limit $\mathfrak{g}^3_{20}$ on X. By the additivity of the Brill-Noether number,

$$-1 \geq \rho(l_{C_1}, p_1) + \rho(l_{C_2}, p_2) + \rho(l_E, p_1, p_2).$$

Since (C_i, p_i) are general points in $\mathcal{C}_{11}$, it follows from Prop.1.3.2 that $\rho(l_{C_i}, p_i) \geq 0$, hence $\rho(l_E, p_1, p_2) \leq -1$. On the other hand $\rho(l_E, p_1, p_2) \geq -3$ from Prop.1.4.1.

Denote by (a_0, a_1, a_2, a_3) the vanishing sequence of l_E at p_1, and by (b_0, b_1, b_2, b_3) that of l_E at p_2. The condition (1.8) for a general pointed curve $[(C_i, p_i)] \in \mathcal{C}_{11}$ to possess a $\mathfrak{g}^3_{20}$ with prescribed ramification at the point p_i and the compatibility conditions between l_{C_i} and l_E at p_i give that:

$$(14 - a_3)_+ + (13 - a_2)_+ + (12 - a_1)_+ + (11 - a_0)_+ \leq 11. \tag{1.9}$$

and

$$(14 - b_3)_+ + (13 - b_2)_+ + (12 - b_1)_+ + (11 - b_0)_+ \leq 11. \tag{1.10}$$

1st case: $\rho(l_E, p_1, p_2) = -3$. Then $a_i + b_{3-i} = 20$, for $i = 0, \ldots, 3$ and it immediately follows that $20(p_1 - p_2) \sim 0$ in $\mathrm{Pic}^0(E)$, a contradiction.
2nd case: $\rho(l_E, p_1, p_2) = -2$. We have two distinct possibilities here: i) $a_0 + b_3 = 20, a_1 + b_2 = 20, a_2 + b_1 = 20, a_3 + b_0 = 19$. Then it follows that $a^{l_E}(p_1) = (0, 9, 18, 19)$ and $a^{l_E}(p_2) = (0, 2, 11, 20)$, while according to (1.9), $a_3 \leq 15$, (because $\rho(l_{C_1}, p_1) \leq 1$), a contradiction. ii) $a_0 + b_3 = 20, a_1 + b_2 = 20, a_2 + b_1 = 19, a_3 + b_0 = 20$. Again, it follows that $a_3 = a_0 + 18 \geq 15$, a contradiction.
3rd case: $\rho(l_E, p_1, p_2) = -1$. Then $\rho(l_{C_i}, p_i) = 0$ and l is a refined limit $\mathfrak{g}^3_{20}$. From (1.9) and (1.10) we must have: $a^{l_E}(p_i) \leq (11, 12, 13, 14)$, $i = 1, 2$. There are four possibilities: i) $a_0 + b_3 = a_1 + b_2 = 20, a_2 + b_1 = a_3 + b_0 = 19$. Then $a_1 = a_0 + 9 \leq 12$, so $b_3 = 20 - a_0 \geq 17$, a contradiction. ii) $a_0 + b_3 = a_2 + b_1 = 20, a_2 + b_1 = a_3 + b_0 = 19$. Then $b_3 = 20 - a_0 \leq 14$, so $a_2 = a_0 + 9 \geq 15$, a contradiction. iii) $a_0 + b_3 = a_3 + b_0 = 20, a_1 + b_2 = a_2 + b_1 = 19$. Then $b_3 = 19 - a_0 \leq 14$, so $a_3 \geq a_0 + 9 \geq 15$, a contradiction. iv) $a_0 + b_3 = a_3 + b_0 =$

$19, a_1 + b_2 = a_2 + b_1 = 20$. Then $b_3 = 19 - a_0 \leq 14$, so $a_2 \geq a_1 + 9 \geq 15$, a contradiction again. We conclude that X has no limit $\mathfrak{g}^3_{20}$.

Step 2) *There exists a smoothable limit* $\mathfrak{g}^2_{17}$ *on* X, *hence* $[X] \in \overline{\mathcal{M}}^2_{17}$. We construct a limit linear series l of type $\mathfrak{g}^2_{17}$ on X, aspect by aspect: on C_i take $l_{C_i} \in G^2_{17}(C_i)$ such that $a^{l_{C_i}}(p_i) = (4, 9, 13)$. Note that in this case $\sum_{j=0}^{r}(\alpha_j + g - d + r)_+ = g$, so (1.8) ensures the existence of such a $\mathfrak{g}^2_{17}$. On E we take $l_E = |V_E|$, where $|V_E| \subseteq |4p_1 + 13p_2| = |4p_2 + 13p_1|$ is a $\mathfrak{g}^2_{17}$ with vanishing sequence $(4, 8, 13)$ at p_i. Prop.1.4.1 ensures the existence of such a linear series. In this way l is a refined limit $\mathfrak{g}^2_{17}$ on X with $\rho(l_{C_i}, p_i) = 0, \rho(l_E, p_1, p_2) = -1$. We prove that l is dimensionally proper. Let $\pi_i : \mathcal{C}_i \to \Delta_i$, $\tilde{p}_i : \Delta_i \to \mathcal{C}_i$, be the versal deformation of $[(C_i, p_i)] \in \mathcal{C}_{11}$, and $\sigma_i : \mathcal{G}^2_{17}(\mathcal{C}_i/\Delta_i, (\tilde{p}_i, (4, 8, 11))) \to \Delta_i$ the projection.

Since being general is an open condition, we have that σ_i is surjective and dim $\sigma_i^{-1}(t) = \rho(l_{C_i}, p_i) = 0$, for each $t \in \Delta_i$, therefore

$$\dim\, \mathcal{G}^2_{17}(\mathcal{C}_i/\Delta_i, (\tilde{p}_i, (4, 8, 11))) = \dim\, \Delta_i + \rho(l_{C_i}, p_i) = 31.$$

Next, let $\pi : \mathcal{C} \to \Delta$, $\tilde{p}_1, \tilde{p}_2 : \Delta \to \mathcal{C}$ be the versal deformation of (E, p_1, p_2). We prove that

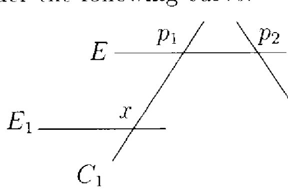
$$\dim\, \mathcal{G}^2_{17}(\mathcal{C}/\Delta, (\tilde{p}_i, (4, 7, 11))) = \dim\, \Delta + \rho(l_E, p_1, p_2) = 1.$$

This follows from Prop.1.4.1, since a 2-pointed elliptic curve $(E_t, \tilde{p}_1(t), \tilde{p}_2(t))$ has at most one $\mathfrak{g}^2_{17}$ with ramification $(4, 7, 11)$ at both $\tilde{p}_1(t)$ and $\tilde{p}_2(t)$, and exactly one when $9(\tilde{p}_1(t) - \tilde{p}_2(t)) \sim 0$. Hence $\mathrm{Im}\mathcal{G}^2_{17}(\mathcal{C}/\Delta, (\tilde{p}_i, (4, 7, 11))) = \{t \in \Delta : 9(\tilde{p}_1(t) - \tilde{p}_2(t)) \sim 0 \text{ in } \mathrm{Pic}^0(E_t)\}$, which is a divisor on Δ, so the claim follows and l is a dimensionally proper $\mathfrak{g}^2_{17}$. □

A slight variation of the previous argument gives us:

Proposition 1.5.3 *We have* $\mathrm{supp}(\overline{\mathcal{M}}^2_{17} \cap \Delta_1) \neq \mathrm{supp}(\overline{\mathcal{M}}^3_{20} \cap \Delta_1)$.

Proof: We construct a curve $[Y] \in \Delta_1 \subseteq \overline{\mathcal{M}}_{23}$ which has a smoothable limit $\mathfrak{g}^2_{17}$ but no limit $\mathfrak{g}^3_{20}$. Let us consider the following curve:



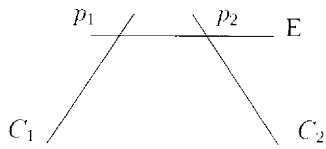
$$Y := C_1 \cup C_2 \cup E_1 \cup E,$$

where (C_2, p_2) is a general point of $\mathcal{C}_{11}$, (C_1, p_1, x) is a general 2-pointed curve of genus 10, (E_1, x) is general in $\mathcal{C}_1$, E is an elliptic curve, and $p_1 - p_2 \in \mathrm{Pic}^0(E)$ is a primitive 9-torsion. In order to prove that Y has no limit $\mathfrak{g}^3_{20}$, one just has to take into account that according to Prop.1.3.3, the condition for a general 1-pointed curve (C, z) of genus g, to have a $\mathfrak{g}^r_d$ with ramification α at z is the same with the condition for a general 2-pointed curve (D, x, y) of genus $g-1$ to have a $\mathfrak{g}^r_d$ with ramification α at x and a cusp at y. Therefore we can repeat what we did in the proof of Theorem 1.2. Next, we construct l, a smoothable limit $\mathfrak{g}^2_{17}$ on Y: take $l_{C_2} \in G^2_{17}(C_2, (p_2, (4, 8, 11)))$, $l_E = |V_E \subseteq |4p_1 + 13p_2|$, with $\alpha^{l_E}(p_i) = (4, 7, 11)$, on E_1 take $l_{E_1} = |4x + 13x|$, and finally on C_1 take l_{C_1} such that $\alpha^{l_{C_1}}(p_1) = (4, 8, 11)$, $\alpha^{l_{C_1}}(x) = (0, 0, 1)$. Prop.1.3.3 ensures the existence of l_{C_1}. Clearly, l is a refined limit $\mathfrak{g}^2_{17}$ and the proof that it is smoothable is all but identical to the one in the last part of Theorem 1.2. □

The other cases are settled by the following:

Theorem 1.3 *There exists a smooth curve of genus 23 having a $\mathfrak{g}^1_{12}$ but having no $\mathfrak{g}^2_{17}$ nor $\mathfrak{g}^3_{20}$. Equivalently,* $\mathrm{supp}(\mathcal{M}^1_{12}) \not\subseteq \mathrm{supp}(\mathcal{M}^2_{17})$ *and* $\mathrm{supp}(\mathcal{M}^1_{12}) \not\subseteq \mathrm{supp}(\mathcal{M}^3_{20})$.

Proof: We take the curve considered in [EH3]:



$$Y := C_1 \cup C_2 \cup E,$$

where (C_i, p_i) are general points of $\mathcal{C}_{11}$, E is elliptic and $p_1 - p_2 \in \mathrm{Pic}^0(E)$ is a primitive 12-torsion. Clearly Y has a (smoothable) limit $\mathfrak{g}^1_{12}$: on C_i take the pencil $|12p_i|$, while on E take the pencil spanned by $12p_1$ and $12p_2$. It is proved in [EH3] that Y has no limit $\mathfrak{g}^2_{17}$'s and similarly one can prove that Y has no limit $\mathfrak{g}^3_{20}$'s either. We omit the details.□

Now we are going to prove that equation (1.2)

$$\mathrm{supp}(\mathcal{M}^1_{12}) \cap \mathrm{supp}(\mathcal{M}^2_{17}) = \mathrm{supp}(\mathcal{M}^2_{17}) \cap \mathrm{supp}(\mathcal{M}^3_{20}) = \mathrm{supp}(\mathcal{M}^3_{20}) \cap \mathrm{supp}(\mathcal{M}^1_{12})$$

is impossible, and as explained before, this will imply that $\kappa(\mathcal{M}_{23}) \geq 2$. The main step in this direction is the following:

Proposition 1.5.4 *There exists a stable curve of compact type of genus 23 which has a smoothable limit $\mathfrak{g}^3_{20}$, a smoothable limit $\mathfrak{g}^2_{15}$ (therefore also a $\mathfrak{g}^2_{17}$), but has generic gonality, that is, it does not have any limit $\mathfrak{g}^1_{12}$.*

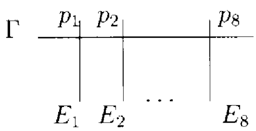
Intermezzo: Before proceeding with the proof let us discuss a possible way to construct curves of genus 23 with such special Brill-Noether properties. Since we are looking for curves C of genus 23 with a $\mathfrak{g}^2_{15}$, a possibility is to start with a (smooth) plane curve $\Gamma \subseteq \mathbb{P}^2$ of degree $d < 15$ and obtain C from Γ by several geometrical operations. We take $\Gamma \subseteq \mathbb{P}^2$ smooth of degree d and pick general points $p_i, q_i \in \Gamma$, for $1 \leq i \leq \delta$.

Let us denote by $\overline{C}$ the curve obtained from Γ by identifying p_i and q_i and by $\nu : \Gamma \to \overline{C}$ the normalization map, hence $\nu(p_i) = \nu(q_i) = s_i$ for $1 \leq i \leq \delta$. There exists a *generalized* $\mathfrak{g}^2_{d+\delta}$ on the integral curve $\overline{C}$ which corresponds to a torsion-free rank one sheaf on $\overline{C}$.

Note that since $\overline{C}$ is irreducible the variety of torsion-free rank one sheaves on $\overline{C}$ is a compactification of $\mathrm{Pic}(\overline{C})$. The generalized $\mathfrak{g}^2_{d-\delta}$ is obtained from the (unique) $\mathfrak{g}^2_d$ on Γ by adding the nodes s_i as base points, hence we have that $\nu^*(\mathfrak{g}^2_{d+\delta}) = \mathfrak{g}^2_d(\sum_{i=1}^{\delta}(p_i + q_i))$.

Using results from [Ta], it is not difficult to show that the $\mathfrak{g}^2_{d+\delta}$ on $\overline{C}$ is smoothable, that is, $[\overline{C}] \in \overline{\mathcal{M}}^2_{p_a(\overline{C}),d+\delta}$. By solving the equations $d + \delta = 15$ and $\binom{d-1}{2} + \delta = 23$ we get $d = 7$ and $\delta = 8$, so we could start with a smooth plane septic $\Gamma \subseteq \mathbb{P}^2$ and identify 8 pairs of general points $p_i, q_i \in \Gamma$, $i = 1, \ldots, 8$. The resulting curve $\overline{C}$, of genus 23, will have a smoothable $\mathfrak{g}^2_{15}$. Letting the points p_i, q_i come together in pairs, we obtain a curve of genus 23 with 8 cusps. From the point of view of the Stable Reduction Theorem (see [Mod]) this is the same as attaching 8 elliptic tails to Γ at the cusps.

Proof We shall consider the following stable curve X of genus 23:



$$X := \Gamma \cup E_1 \cup \ldots \cup E_8,$$

where the E_i's are elliptic curves, $\Gamma \subseteq \mathbb{P}^2$ is a general smooth plane septic and the points of attachment $\{p_i\} = \Gamma \cup E_i$ are general points of Γ.

Step 1) *There is no limit* $\mathfrak{g}^1_{12}$ *on* X. Assume that l is a limit $\mathfrak{g}^1_{12}$ on X. Since the elliptic curves E_i cannot have meromorphic functions with a single pole, we have that $a^{l_{E_i}}(p_i) \leq (10, 12)$, hence $\alpha^{l_\Gamma}(p_i) \geq (0, 1)$, that is, l_Γ has a cusp at p_i for $i = 1, \ldots, 8$. We now prove that Γ has no $\mathfrak{g}^1_{12}$'s with cusps at the points p_i.

First, we notice that $\dim G^1_{12}(\Gamma) = \rho(15, 1, 12) = 7$. Indeed, if we assume that $\dim G^1_{12}(\Gamma) \geq 8$, by applying Keem's Theorem (cf. [ACGH], p.200) we would get that Γ possesses a $\mathfrak{g}^1_4$, which is impossible since $\mathrm{gon}(\Gamma) = 6$. (In general, if $Y \subseteq \mathbb{P}^2$ is a smooth plane curve, $\deg(Y) = d$, then gon $(Y) = d - 1$, and the $\mathfrak{g}^1_{d-1}$ computing the gonality is cut out by the lines passing through a point $p \in Y$, see [ACGH].) Next, we define the variety

$$\Sigma = \{(l, q_1, \ldots, q_8) \in G^1_{12}(\Gamma) \times \Gamma^8 : \alpha^l(q_i) \geq (0, 1), i = 1, \ldots, 8\}$$

and denote by $\pi_1 : \Sigma \to G^1_{12}(\Gamma)$ and $\pi_2 : \Sigma \to \Gamma^8$ the two projections. For any $l \in G^1_{12}(\Gamma)$, the fibre $\pi_1^{-1}(l)$ is finite, hence $\dim \Sigma = \dim G^1_{12}(\Gamma) = 7$, which shows that π_2 cannot be surjective and this proves our claim.

Step 2) *There exists a smoothable limit* $\mathfrak{g}^2_{15}$ *on* X, *hence* $[X] \in \overline{\mathcal{M}}^2_{15}$. We construct l, a limit $\mathfrak{g}^2_{15}$ on X as follows: on Γ there is a (unique) $\mathfrak{g}^2_7$, and we consider $l_\Gamma = \mathfrak{g}^2_7(p_1 + \cdots + p_8)$, i.e. the $\Gamma-$ aspect l_Γ is obtained from the $\mathfrak{g}^2_7$ by adding the base points $p_1, \ldots, p_8$. Clearly $a^{l_\Gamma}(p_i) = (1, 2, 3)$ for each i. On E_i we take $l_{E_i} = \mathfrak{g}^2_3(12p_i)$ for $i = 1, \ldots, 8$, where $\mathfrak{g}^2_3$ is a complete linear series of the form $|2p_i + x_i|$, with $x_i \in E_i - \{p_i\}$. Furthermore, $a^{l_{E_i}}(p_i) = (12, 13, 14)$, so $l = \{l_\Gamma, l_{E_i}\}$ is a refined limit $\mathfrak{g}^2_{15}$ on X. One sees that $\rho(l_{E_i}, \alpha^{l_{E_i}}(p_i)) = 1$ for all i, $\rho(l_\Gamma, \alpha^{l_\Gamma}(p_1), \ldots, \alpha^{l_\Gamma}(p_8)) = -15$, and $\rho(l) = -7$. We now prove that l is dimensionally proper.

Let $\pi_i : \mathcal{C}_i \to \Delta_i$, $\tilde{p}_i : \Delta_i \to \mathcal{C}_i$ be the versal deformation space of (E_i, p_i), for $i = 1, \ldots, 8$. There is an obvious isomorphism over Δ_i

$$\mathcal{G}^2_{15}(\mathcal{C}_i/\Delta_i, (\tilde{p}_i, (12, 12, 12))) \simeq \mathcal{G}^2_3(\mathcal{C}_i/\Delta_i, (\tilde{p}_i, 0)).$$

If $\sigma_i : \mathcal{G}^2_3(\mathcal{C}_i/\Delta_i, (\tilde{p}_i, 0)) \to \Delta_i$ is the natural projection, then for each $t \in \Delta_i$, the fibre $\sigma_i^{-1}(t)$ is isomorphic to $\pi_i^{-1}(t)$, the isomorphism being given by

$$\pi_i^{-1}(t) \ni q \mapsto |2\tilde{p}_i(t) + q| \in G^2_3(\pi_i^{-1}(t)).$$

Thus, $\mathcal{G}^2_3(\mathcal{C}_i/\Delta_i, (\tilde{p}_i, 0))$ is a smooth irreducible surface, which shows that l is dimensionally proper w.r.t. E_i. Next, let us consider $\pi : \mathcal{X} \to \Delta$, $\tilde{p}_1, \ldots, \tilde{p}_8 : \Delta \to \mathcal{X}$, the versal deformation of $(\Gamma, p_1, \ldots, p_8)$. We have to prove that

$$\dim \mathcal{G}^2_{15}(\mathcal{X}/\Delta, (\tilde{p}_i, (1, 1, 1))) = \dim \Delta + \rho(l_\Gamma, \alpha^{l_\Gamma}(p_i)) = 35.$$

There is an isomorphism over Δ,

$$\mathcal{G}^2_{15}(\mathcal{X}/\Delta, (\tilde{p}_i, (1, 1, 1))) \simeq \mathcal{G}^2_7(\mathcal{X}/\Delta, (\tilde{p}_i, 0)).$$

If $\pi_0 : \mathcal{C} \to \mathcal{M}$ is the versal deformation space of Γ, then we denote by $\mathcal{G}^2_7 \to \mathcal{M}$ the scheme parametrizing $\mathfrak{g}^2_7$'s on curves of genus 15 'nearby' Γ (See Section 1.3 for this notation). Clearly $\mathcal{G}^2_7(\mathcal{X}/\Delta, (\tilde{p}_i, 0)) \simeq \mathcal{G}^2_7 \times_{\mathcal{M}} \Delta$, so it suffices to prove that $\mathcal{G}^2_7$ has the expected dimension at the point $(\Gamma, \mathfrak{g}^2_7)$. For this we use Prop.1.3.1. We have that $N_{\Gamma/\mathbb{P}^2} = \mathcal{O}_\Gamma(7)$, $K_\Gamma = \mathcal{O}_\Gamma(4)$, hence

$$H^1(\Gamma, N_{\Gamma/\mathbb{P}^2}) \simeq H^0(\Gamma, \mathcal{O}_\Gamma(-3))^\vee = 0,$$

so l is dimensionally proper w.r.t. Γ as well. We conclude that l is smoothable.

Step 3) *There exists a smoothable limit* $\mathfrak{g}^3_{20}$ *on* X, *that is* $[X] \in \overline{\mathcal{M}}^3_{20}$. First we notice that there is an isomorphism $\Gamma \xrightarrow{\sim} G^1_6(\Gamma)$, given by

$$\Gamma \ni p \mapsto |\mathfrak{g}^2_7 - p| \in G^1_6(\Gamma).$$

Consequently, there is a 2-dimensional family of $\mathfrak{g}^3_{12}$'s on Γ, of the form $\mathfrak{g}^3_{12} = \mathfrak{g}^1_6 + \mathfrak{h}^1_6 = |2\mathfrak{g}^2_7 - p - q|$, where $p, q \in \Gamma$. Pick $l_0 = l_0' + l_0''$, with $l_0', l_0'' \in G^1_6(\Gamma)$, a general $\mathfrak{g}^3_{12}$ of this type. We construct l, a limit $\mathfrak{g}^3_{20}$ on X, as follows: the Γ-aspect is given by $l_\Gamma = l_0(p_1 + \cdots p_8)$, and because of the generality of the chosen l_0 we have that $\rho(l_\Gamma, \alpha^{l_\Gamma}(p_1), \ldots, \alpha^{l_\Gamma}(p_8)) = -9$. The E_i-aspect is given by $l_{E_i} = \mathfrak{g}^3_4(16p_i)$, where $\mathfrak{g}^3_4 = |3p_i + x_i|$, with $x_i \in E_i - \{p_i\}$, for $i = 1, \ldots, 8$. It is clear that $\rho(l_{E_i}, \alpha^{l_{E_i}}(p_i)) = 1$ and that $l' = \{l_\Gamma, l_{E_i}\}$ is a refined limit $\mathfrak{g}^3_{20}$ on X.

In order to prove that l' is dimensionally proper, we first notice that l' is dimensionally proper w.r.t. the elliptic tails E_i. We now prove that l' is dimensionally proper w.r.t. Γ. As in the previous step, we consider $\pi : \mathcal{X} \to \Delta$, $\tilde{p}_1, \ldots, \tilde{p}_8 : \Delta \to \mathcal{X}$, the versal deformation of $(\Gamma, p_1, \dots, p_8)$ and $\pi_0 : \mathcal{C} \to \mathcal{M}$, the versal deformation space of Γ. There is an isomorphism over Δ

$$\mathcal{G}^3_{20}(\mathcal{X}/\Delta, (\tilde{p}_1, \alpha^{l_\Gamma}(p_1)), \dots, (\tilde{p}_8, \alpha^{l_\Gamma}(p_8))) \simeq \mathcal{G}^3_{12}(\mathcal{C}/\mathcal{M}) \times_{\mathcal{M}} \Delta.$$

It suffices to prove that $\mathcal{G}^3_{12} = \mathcal{G}^3_{12}(\mathcal{C}/\mathcal{M})$ has a component of the expected dimension passing through (Γ, l_0). In this way, the genus 23 problem is turned into a deformation theoretic problem in genus 15. Denote as usual by $\sigma : \mathcal{G}^3_{12} \to \mathcal{M}$ the natural projection. According to Prop.1.3.1 it will be enough to exhibit an element $(C, l) \in \mathcal{G}^3_{20}$, sitting in the same component as (Γ, l_0), such that the linear system l is base point free and simple, and if $\phi_1 : C \to \mathbb{P}^3$ is the map induced by l, then $H^1(C, N_{\phi_1}) = 0$. Certainly we cannot take C to be a smooth plane septic because in this case $H^1(C, N_{\phi_1}) \neq 0$, as one can easily see. Instead, we consider the 6-gonal locus in a neighbourhood of the point $[\Gamma] \in \mathcal{M}_{15}$, or equivalently, the 6-gonal locus in $\mathcal{M}$, the versal deformation space of Γ. One has the projection $\mathcal{G}^1_6 \to \mathcal{M}$ and the scheme $\mathcal{G}^1_6$ is smooth (and irreducible) of dimension $37 (= 2g + 2d - 5; g = 15, d = 6)$. We denote by

$$\mu : \mathcal{G}^1_6 \times_{\mathcal{M}} \mathcal{G}^1_6 \to \mathcal{M}, \quad \mu([C, l, l']) = [C].$$

There is a stratification of $\mathcal{M}$ given by the number of pencils: for $i \geq 0$ we define,

$$\mathcal{M}(i)^0 := \{[C] \in \mathcal{M} : C \text{ possesses } i \text{ mutually independent, base-point-free } \mathfrak{g}^1_6\text{'s }\},$$

and $\mathcal{M}(i) := \overline{\mathcal{M}(i)^0}$. The strata $\mathcal{M}(i)^0$ are constructible subsets of $\mathcal{M}$, the first stratum $\mathcal{M}(1) = \text{Im }(\mathcal{G}^1_6)$ is just the 6-gonal locus; the stratum $\mathcal{M}(2)$ is irreducible and dim $\mathcal{M}(2) = g + 4d - 7 = 32$ (cf. [AC1]). We denote by $\mathcal{M}_{sept} := \overline{m(U_{7,15}) \cap \mathcal{M}}$ the closure of the locus of smooth plane septics in $\mathcal{M}$, and by $\mathcal{M}_{oct} := \overline{m(U_{8,15}) \cap \mathcal{M}}$ the closure of the locus of curves which are normalizations of plane octics with 6 nodes. Since the Severi varieties $U_{7,15}$ and $U_{8,15}$ are irreducible, so are the loci $\mathcal{M}_{sept}$ and $\mathcal{M}_{oct}$. Furthermore dim $\mathcal{M}_{sept} = 27$ and dim $\mathcal{M}_{oct} = 30$. We prove that $\mathcal{M}_{sept} \subseteq \mathcal{M}_{oct}$. Indeed, let us pick $Y \subseteq \mathbb{P}^2$ a smooth plane septic, and $L \subseteq \mathbb{P}^2$ a general line, $L \cdot Y = p_1 + \cdots + p_7$. Denote $Z := C \cup L$, deg $(Z) = 8, p_a(Z) = 21$. We consider the node p_7 unassigned, while $p_1, \dots p_6$ are assigned. By using [Ta] Theorem 2.13, there exists a flat family of plane curves $\pi : \mathcal{Z} \to B$ and a point $0 \in B$, such that $Z_0 = \pi^{-1}(0) = Z$, while for $0 \neq b \in B$, the fibre Z_b is an irreducible octic with nodes $p_1(b), \dots p_6(b)$, and such that $p_i(b) \to p_i$, when $b \to 0$, for $i = 1, \dots, 6$. If $\mathcal{Z}' \to B$ is the family resulting by normalizing the surface $\mathcal{Z}$, and $\eta : \mathcal{Z}'' \to B$ is the stable family associated to the semistable family $\mathcal{Z}' \to B$, then we get that $\eta^{-1}(0) = Y$, while $\eta^{-1}(b)$ is the normalization of Z_b for $b \neq 0$. This proves our contention.

Since $\mathcal{M}_{oct}$ is irreducible there is a component $\mathcal{A}$ of $\mathcal{G}^1_6 \times_{\mathcal{M}} \mathcal{G}^1_6$, such that $\mu(\mathcal{A}) \supseteq \mathcal{M}_{oct}$. The general point of $\mathcal{A}$ corresponds to a curve C and two base-point-free pencils $l', l'' \in G^1_6(C)$ such that if $f' : C \to \mathbb{P}^1$ and $f'' : C \to \mathbb{P}^1$ are the corresponding morphisms, then

$$\phi = (f', f'') : C \to \mathbb{P}^1 \times \mathbb{P}^1$$

is birational. Since $[\Gamma] \in \mu(\mathcal{A})$ we can assume that $[\Gamma, l_0', l_0''] \in \mathcal{A}$. As a matter of fact, we can start the construction of a limit $\mathfrak{g}^3_{20}$ on the genus 23 curve $X = \Gamma \cup E_1 \cup \ldots \cup E_8$, by taking any pair $(l_0', l_0'') \in G^1_6(\Gamma) \times G^1_6(\Gamma)$, such that $\dim |l_0' + l_0''| = 3$, the argument does not change.

We denote by $\eta : \mathcal{A} \to \mathcal{G}^3_{12}$ the map given by $\eta(C, l', l'') := (C, l' + l'')$. The fact that η maps to $\mathcal{G}^3_{12}$ follows from the base-point-free-pencil-trick.

We are going to show that given a general point $[C] \in \mathcal{M}_{oct}$, and $(C, l, l') \in \mu^{-1}([C])$, the condition $H^1(C, N_{\phi_1}) = 0$ is satisfied, hence $\mathcal{G}^3_{12}$ is smooth of the expected dimension at the point $(C, l+l')$. This will prove the existence of a component of $\mathcal{G}^3_{12}$ passing through (Γ, l_0) and having the expected dimension. We take $\overline{C} \subseteq \mathbb{P}^2$, a general point of $U_{8,15}$, with nodes $p_1, \ldots, p_6 \in \mathbb{P}^2$ in general position. Theorem 3.2 from [AC1] ensures that there exists a plane octic having 6 prescribed nodes in general position. Let $\nu : C \to \overline{C}$ be the normalization map, $\nu^{-1}(p_i) = q_i' + q_i''$ for $i = 1, \ldots, 6$. Choose two nodes, say p_1 and p_2, and denote by $\mathfrak{g}^1_6 = |H - q_1' - q_1''|$ and $\mathfrak{h}^1_6 = |H - q_2' - q_2''|$, the linear series obtained by projecting $\overline{C}$ from p_1 and p_2 respectively. Here $H \in |\nu^*\mathcal{O}_{\mathbb{P}^2}(1)|$ is an arbitrary line section of C. The morphism induced by $(\mathfrak{g}^1_6, \mathfrak{h}^1_6)$ is denoted by $\phi : C \to \mathbb{P}^1 \times \mathbb{P}^1$ and $\phi_1 = s \circ \phi : C \to \mathbb{P}^3$, with $s : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ the Segre embedding. There is an exact sequence over C

$$0 \longrightarrow N_\phi \longrightarrow N_{\phi_1} \longrightarrow \phi^* N_{\mathbb{P}^1 \times \mathbb{P}^1 / \mathbb{P}^3} \longrightarrow 0. \tag{1.11}$$

We can argue as in [AC2] p.473, that for a general $(C, \mathfrak{g}^1_6, \mathfrak{h}^1_6)$ with $[C] \in \mathcal{M}_{oct}$, we have $h^1(C, N_\phi) = 0$. Indeed, let us denote by $\mathcal{A}_0$ the open set of $\mathcal{A}$ corresponding to points (X, l, l') such that $\chi : X \to \mathbb{P}^1 \times \mathbb{P}^1$, the morphism associated to the pair of pencils (l, l') is birational, and by $\mathcal{U} \subseteq \mathcal{A}_0$ the variety of those points $(X, l, l') \in \mathcal{A}_0$ such that $H^1(X, N_\chi) \neq 0$. Define

$$\mathcal{V} := \{x = (X, l, l', \mathcal{F}, \mathcal{F}') : (X, l, l') \in \mathcal{U},\ \mathcal{F} \text{ is a frame for } l,\ \mathcal{F}' \text{ is a frame for } l'\}.$$

We may assume that for a generic $x \in \mathcal{U}$, the corresponding pencils l and l' are base-point-free. Suppose that $\mathcal{U}$ has a component of dimension α. For any $x \in \mathcal{V}$,

$$T_x(\mathcal{V}) \subseteq H^0(X, N_\chi), \text{ and } \dim T_x(\mathcal{V}) \geq \alpha + 2 \dim PGL(2) = \alpha + 6.$$

If K_χ is the cuspidal sheaf of χ and $N_\chi' = N_\chi / K_\chi$, then according to [AC1] Lemma 1.4, for a general point $x \in \mathcal{V}$ one has that,

$$T_x(\mathcal{V}) \cap H^0(X, K_\chi) = 0,$$

from which it follows that $\alpha \leq g - 6$. If not, one would have that $h^0(X, N_\chi') \geq g + 1$, and therefore by Clifford's Theorem, $h^1(X, N_\chi) = h^1(X, N_\chi') = 0$, which contradicts the definition of $\mathcal{U}$. Since clearly $\dim \mathcal{M}_{oct} > g - 6$, we can assume that $h^1(C, N_\phi) = 0$, for the general $[C] \in \mathcal{M}_{oct}$. Therefore, by taking cohomology in (1.11), we get that

$$H^1(C, N_{\phi_1}) = H^1(C, \mathcal{O}_C(2)).$$

where $\mathcal{O}_C(1) = \phi_1^*\mathcal{O}_{\mathbb{P}^3}(1)$. By Serre duality,

$$H^1(C, \mathcal{O}_C(2)) = 0 \iff K_C - 2\mathfrak{g}_6^1 - 2\mathfrak{h}_6^1 = \emptyset. \qquad (1.12)$$

Since $K_C = 5H - \sum_{i=1}^6 (q_i' + q_i'')$, equation (1.12) becomes

$$H - q_1' - q_1'' + q_2' + q_2'' - \sum_{i=3}^{6} (q_i' + q_i'') = \emptyset. \qquad (1.13)$$

If $L = \overline{p_1p_2} \subseteq \mathbb{P}^2$, we can write $\nu^*(L) = q_1' + q_1'' + q_2' + q_2'' + x + y + z + t$, and (1.13) is rewritten as

$$2H - x - y - z - t - \sum_{i=3}^{6} (q_i' + q_i'') = \emptyset.$$

So, one has to show that there are no conics passing through the nodes p_3, p_4, p_5 and p_6 and also through the points in $L \cdot \overline{C} - 2p_1 - 2p_2$. Because $[\overline{C}] \in U_{8,15}$ is general we may assume that x, y, z and t are distinct, smooth points of $\overline{C}$. Indeed, if the divisor $x + y + z + t$ on $\overline{C}$ does not consist of distinct points, or one of its points is a node, we obtain that $\overline{C}$ has intersection number 8 with the line L at 5 points or less. But according to [DH], the locus in the Severi variety

$$\{[X] \in U_{d,g} : X \text{ has total intersection number } m+3 \text{ with a line at } m \text{ points }\}$$

is a divisor on $U_{d,g}$, so we may assume that $[\overline{C}]$ lies outside this divisor. Now, if x, y, z and t are distinct and smooth points of $\overline{C}$, a conic satisfying (1.13) would necessarily be a degenerate one, and one gets a contradiction with the assumption that the nodes $p_1, \ldots, p_6$ of $\overline{C}$ are in general position. ☐

Remark: We have a nice geometric characterization of some of the strata $\mathcal{M}_i$. First, by using Zariski's Main Theorem for the birational projection $\mathcal{G}_6^1 \to \mathcal{M}(1)$, one sees that $[C] \in \mathcal{M}(1)_{sing}$ if and only if either $[C] \in \mathcal{M}(2)^0$, or C possesses a $\mathfrak{g}_6^1$ such that $\dim |2\mathfrak{g}_6^1| \geq 3$. In the latter case, the $\mathfrak{g}_6^1$ is a specialization of 2 different $\mathfrak{g}_6^1$'s in some family of curves, hence $\mathcal{M}(2) = \mathcal{M}(1)_{sing}$ (cf [Co2]). As a matter of fact, Coppens has proved that for $4 \leq k \leq [(g+1)/2]$ and $8 \leq g \leq (k-1)^2$, there exists a k-gonal curve of genus g carrying exactly 2 linear series $\mathfrak{g}_k^1$, so the general point of $\mathcal{M}(2)$ corresponds to a curve C of genus 15, having exactly 2 base-point-free $\mathfrak{g}_6^1$'s. Furthermore, using Coppens' classification of curves having many pencils computing the gonality (see [Co1]), we have that $\mathcal{M}(6) = \mathcal{M}_{oct}$, and $\mathcal{M}(i) = \mathcal{M}_{sept}$, for each $i \geq 7$.

Now we are in a position to complete the proof of Theorem 1.1:

Proof of Theorem 1.1 According to (1.2), it will suffice to prove that there exists a smooth curve $[Y] \in \mathcal{M}_{23}$ which carries a $\mathfrak{g}_{20}^3$, a $\mathfrak{g}_{17}^2$ but has no $\mathfrak{g}_{12}^1$'s. In the proof of Prop.1.5.4 we constructed a stable curve of compact type $[X] \in \overline{\mathcal{M}}_{23}$ such that $[X] \in \overline{\mathcal{M}}_{17}^2 \cap \overline{\mathcal{M}}_{20}^3$, but $[X] \notin \overline{\mathcal{M}}_{12}^1$. If we prove that $[X] \in \overline{\mathcal{M}_{17}^2 \cap \mathcal{M}_{20}^3}$, that is, there are smoothings of X which preserve both the $\mathfrak{g}_{17}^2$ and the $\mathfrak{g}_{20}^3$, we are done. One can write $\overline{\mathcal{M}}_{17}^2 \cap \overline{\mathcal{M}}_{20}^3 = Y_1 \cup \ldots \cup Y_s$, where Y_i are irreducible codimension 2 subvarieties of $\overline{\mathcal{M}}_{23}$. Assume that $[X] \in Y_1$. If $Y_1 \cap \mathcal{M}_{23} \neq \emptyset$, then $[X] \in Y_1 = \overline{Y_1 \cap \mathcal{M}_{23}} \subseteq \overline{\mathcal{M}^2_{17} \cap \mathcal{M}^3_{20}}$, and the conclusion follows. So we may assume that $Y_1 \subseteq \overline{\mathcal{M}}_{23} - \mathcal{M}_{23}$. Because $[X] \in \Delta_1 - \bigcup_{j\neq 1} \Delta_j$, we must have $Y \subseteq \Delta_1$. It follows that $\overline{\mathcal{M}}^2_{17} \cap \Delta_1$ and $\overline{\mathcal{M}}^3_{20} \cap \Delta_1$ have Y_1 as a common component. According to Prop. 1.4.2, both intersections $\overline{\mathcal{M}}^2_{17} \cap \Delta_1$ and $\overline{\mathcal{M}}^3_{20} \cap \Delta_1$ are irreducible, hence $\overline{\mathcal{M}}^2_{17} \cap \Delta_1 = \overline{\mathcal{M}}^3_{20} \cap \Delta_1 = Y_1$, which contradicts Prop. 1.5.3. Theorem 1.1 now follows. □

1.6 The slope conjecture and $\mathcal{M}_{23}$

In this final section we explain how the slope conjecture in the context of $\mathcal{M}_{23}$ implies that $\kappa(\mathcal{M}_{23}) = 2$, and then we present evidence for this.

The slope of $\overline{\mathcal{M}}_g$ is defined as $s_g := \inf \{a \in \mathbb{R}_{>0} : |a\lambda - \delta| \neq \emptyset\}$, where $\delta = \delta_0 + \delta_1 + \cdots + \delta_{[g/2]}$, $\lambda \in \mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{R}$. Since λ is big, it follows that $s_g < \infty$. If $\mathbb{E}$ is the cone of effective divisors in $\mathrm{Div}(\overline{\mathcal{M}}_g) \otimes \mathbb{R}$, we define the slope function $s : \mathbb{E} \to \mathbb{R}$ by the formula

$$s_D := \inf \{a/b : a, b > 0 \text{ such that } \exists c_i \geq 0 \text{ with } [D] = a\lambda - b\delta - \sum_{i=0}^{[g/2]} c_i\delta_i\}.$$

for an effective divisor D on $\overline{\mathcal{M}}_g$. Clearly $s_g \leq s_D$ for any $D \in \mathbb{E}$. When $g+1$ is composite, we obtain the estimate $s_g \leq 6 + 12/(g+1)$ by using the Brill-Noether divisors $\overline{\mathcal{M}}^r_{g,d}$, with $\rho(g, r, d) = -1$.

Conjecture 1 ([HMo]) *We have that $s_g \geq 6 + 12/(g+1)$ for each $g \geq 3$, with equality when $g+1$ is composite.*

Harris and Morrison also stated (in a somewhat vague form) that for composite $g+1$, the Brill-Noether divisors not only minimize the slope among all effective divisors, but they also single out those irreducible $D \in \mathbb{E}$ with $s_D = s_g$.

The slope conjecture has been proved for $3 \leq g \leq 11$, $g \neq 10$ (cf. [HMo], [CR3,4], [Tan]). A strong form of the conjecture holds for $g = 3$ and $g = 5$: on $\overline{\mathcal{M}}_3$ the only irreducible divisor of slope $s_3 = 9$ is the hyperelliptic divisor, while on $\overline{\mathcal{M}}_5$ the only irreducible divisor of slope $s_5 = 8$ is the trigonal divisor (cf. [HMo]). Conjecture 1 would imply that $\kappa(\mathcal{M}_g) = -\infty$ for all $g \leq 22$. For $g = 23$, we rewrite (1.1) as

$$nK_{\overline{\mathcal{M}}_{23}} = \frac{n}{c_{23,r,d}}[\overline{\mathcal{M}}^r_{g,d}] + 8n\,\delta_1 + \sum_{i=2}^{11} \frac{(i(23-i)-4)}{2} n\,\delta_i \quad (n \geq 1). \tag{1.14}$$

(see Section 1.2 for the coefficients $c_{g,r,d}$). As Harris and Morrison suggest, we can ask the question whether the class of any $D \in \mathbb{E}$ with $s_D = s_g$ is (modulo a sum of boundary components Δ_i) proportional to $[\overline{\mathcal{M}}^r_{23,d}]$, and whether the sections defining (multiples of) $\overline{\mathcal{M}}^r_{23,d}$ form a maximal algebraically independent subset of the canonical ring $R(\overline{\mathcal{M}}_{23})$.

If so, it would mean that the boundary divisor $8n\delta_1 + (1/2)\sum_{i=2}^{11} n(i(23-i)-4)\delta_i$ is a fixed part of $nK_{\overline{\mathcal{M}}_{23}}$. Moreover, using our independence result for the three Brill-Noether divisors, it would follow that $h^0(\overline{\mathcal{M}}_{23}, nK_{\overline{\mathcal{M}}_{23}})$ grows quadratically in n, for n sufficiently high and sufficiently divisible, hence $\kappa(\overline{\mathcal{M}}_{23}) = 2$. We would also have that $\Sigma \cap \mathcal{M}_{23} = \mathcal{M}_{12}^1 \cap \mathcal{M}_{17}^2 \cap \mathcal{M}_{20}^3$, with Σ the common base locus of all the linear systems $nK_{\overline{\mathcal{M}}_{23}}$.

Evidence for these facts is of various sorts: first, one knows (cf. [Tan], [CR3]) that $|nK_{\overline{\mathcal{M}}_{23}}|$ has a large fixed part in the boundary: for each $n \geq 1$, every divisor in $nK_{\overline{\mathcal{M}}_{23}}$ must contain Δ_i with multiplicity $16n$ when $i = 1$, $19n$ when $i = 2$, and $(21-i)n$ for $i = 3, \ldots, 9$ or 11. The results for Δ_1 and Δ_2 are optimal since these multiplicities coincide with those in (1.14). Note that $[\Delta_1] = 2\delta_1$.

Next, one can show that certain geometric loci in $\mathcal{M}_{23}$ which are contained in all three Brill-Noether divisors, are contained in Σ as well. The method is based on the trivial observation that for a family $f : X \to B$ of stable curves of genus 23 with smooth general member, if $B.K_{\overline{\mathcal{M}}_{23}} < 0$ (or equivalently, slope$(X/B) = \delta_B/\lambda_B > 13/2$), then $\phi(B) \subseteq \Sigma$, where $\phi : B \to \overline{\mathcal{M}}_{23}$, $\phi(b) = [X_b]$, is the associated moduli map. We have that:

• One can fill up the d-gonal locus $\overline{\mathcal{M}}_d^1$ with families $f : X \to B$ of stable curves of genus g such that slope(X/B) is $8 + 4/g$ in the hyperelliptic case, $> 7 + 6/g$ in the trigonal and $> 6 + 12/(g+1)$ in the tetragonal case (cf. [Sta]). For $g = 23$ it follows that $\mathcal{M}_4^1 \subseteq \Sigma$. Note that this result is not optimal if we believe the slope conjecture since we know that $\mathcal{M}_8^1 \subseteq \mathcal{M}_{12}^1 \cap \mathcal{M}_{17}^2 \cap \mathcal{M}_{20}^3$. (The inclusion $\mathcal{M}_8^1 \subseteq \mathcal{M}_{20}^3$ is a particular case of a result from [CM].)

• We take a pencil of nodal plane curves of degree d with f assigned nodes in general position such that $\binom{d-1}{2} - f = 23$, and with b base points, where $4f + b = d^2$. After blowing-up the base points, we have a pencil $Y \to \mathbb{P}^1$ with fibre $[Y_t] \in \overline{\mathcal{M}}_d^2$. For this pencil $\lambda_{\mathbb{P}^1} = \chi(\mathcal{O}_Y) + 23 - 1 = 23$ and $\delta_{\mathbb{P}^1} = c_2(Y) + 88 = 91 + b + f$. The condition $\delta_{\mathbb{P}^1}/\lambda_{\mathbb{P}^1} > 13/2$ is satisfied precisely when $d \leq 10$, hence taking into account that such pencils fill up $\mathcal{M}_d^2$, we obtain that $\mathcal{M}_{10}^2 \subseteq \Sigma$. Note that $\mathcal{M}_{10}^2 \subseteq \mathcal{M}_8^1$, and as mentioned above, the 8-gonal locus is contained in the intersection of the Brill-Noether divisors.

• In a similar fashion we can prove that $\mathcal{M}_{23,\gamma}(2)$, the locus of curves of genus 23 which are double coverings of curves of genus γ is contained in Σ for $\gamma \leq 5$.

The fact that the slopes of other divisors on $\overline{\mathcal{M}}_{23}$ (or on $\overline{\mathcal{M}}_g$ for arbitrary g) consisting of curves with special geometric characterization, are larger than $6 + 12/(g+1)$, lends further support to the slope hypothesis. In another paper we will compute the class of other divisors on $\overline{\mathcal{M}}_{23}$: the closure in $\overline{\mathcal{M}}_{23}$ of the locus

$$\{[C] \in \mathcal{M}_{23} : C \text{ possesses a } \mathfrak{g}_{13}^1 \text{ with two different triple points}\},$$

and the closure of the locus

$$\{[C] \in \mathcal{M}_{23} : C \text{ has a } \mathfrak{g}_{18}^2 \text{ with a 5-fold point, i.e. } \exists D \in C^{(5)} \text{ such that } \mathfrak{g}_{18}^2(-D) = \mathfrak{g}_{13}^1\}.$$

In each case we will show that the slope estimate holds.

Chapter 2

The geography of Brill-Noether loci in the moduli space of curves

2.1 Introduction

We start by explaining the meaning of the word 'geography' from the title of this chapter. Many papers have been published where people studied the geography of certain mathematical objects (e.g. surfaces), meaning that they looked at them from the point of view of a naturalist. Our understanding of the term 'geography' is rather different: we study the position of the Brill-Noether loci $\mathcal{M}^r_{g,d} = \{[C] \in \mathcal{M}_g : C \text{ carries a } \mathfrak{g}^r_d\}$ on the 'map' of the moduli space of curves. We compare different Brill-Noether loci, look whether they meet transversally (or are in general relative position) inside $\mathcal{M}_g$, or describe their position with respect to other distinguished loci in $\mathcal{M}_g$ (e.g. loci of curves sitting on certain surfaces). In most cases we prove that two such loci in $\mathcal{M}_g$ are as transversal (or intersect as properly) as possible, unless there are some obvious containment relations. The general philosophy is that there are no ways (except the obvious ones) to construct linear series on curves with specific properties.

This chapter consists of relatively independent sections. After Section 2.1 in which we set up the necessary techniques, we ask in Section 2.3 whether the only constraints on the possible $\mathfrak{g}^r_d$'s on a general k-gonal curve C of genus g are related to the $\mathfrak{g}^1_k$ on C (as it is the case for hyperelliptic and trigonal curves). We prove that a general k-gonal curves C of genus g, where k is rather high with respect to g, has no other linear series with negative Brill-Noether number except $\mathfrak{g}^1_k$ and $|K_C - \mathfrak{g}^1_k|$. In Section 2.4 we show that by imposing two distinct conditions on a curve C of genus g (the existence of a pencil $\mathfrak{g}^1_k$ and of an embedding $C \subseteq \mathbb{P}^r$ of degree d, where $r \geq 3$ and $\rho(g, r, d) = -1$), we bring down accordingly the number of moduli such curves depend on. Section 2.5 deals with the problem of computing the gonality of space curves: we show that for a wide range of d and g such that $\rho(g, 3, d) < 0$ one can find smooth curves $C \subseteq \mathbb{P}^3$ of degree d and genus g which fill up a component of the Hilbert scheme $\mathrm{Hilb}_{d,g,3}$ and for which $\mathrm{gon}(C) = \min([(g+3)/2], d-4)$: if $d-4 < [(g+3)/2]$, every pencil computing the gonality is given by the planes through a 4-secant line to C. Finally, in Section 2.6 we ask what kind of surfaces can contain a Brill-Noether general curve.

2.2 Deformations of maps and smoothing of algebraic space curves

We review some facts about deformations of maps and smoothing of reducible, nodal curves in $\mathbb{P}^r$. These techniques together with the theory of limit linear series already discussed in the previous chapter will be our main tools throughout this chapter. We start by describing the deformation theory of maps between (possibly singular) complex algebraic varieties. Our main reference is [Ran] (certain aspects of the theory are well treated in [Mod] as well).

Let $f : X \to Y$ be a morphism between complex projective varieties. We denote by $\mathrm{Def}(X, f, Y)$ the space of first-order deformations of the map f, while the space of first-order deformations of X (resp. Y) is denoted by $\mathrm{Def}(X)$ (resp. $\mathrm{Def}(Y)$). The standard identification $\mathrm{Def}(X) = \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$ is obtained by associating to any first-order deformation $\tilde{X}$ of X the class of the extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega_{\tilde{X}} \otimes \mathcal{O}_X \longrightarrow \Omega_X \longrightarrow 0.$$

The deformation space $\mathrm{Def}(X, f, Y)$ fits in the following exact sequence:

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X) \longrightarrow \mathrm{Def}(X, f, Y) \longrightarrow \mathrm{Def}(X) \oplus \mathrm{Def}(Y) \longrightarrow \mathrm{Ext}^1_f(\Omega_Y, \mathcal{O}_X). \quad (2.1)$$

The second arrow is given by the natural forgetful maps, the space $\mathrm{Hom}_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X) = H^0(X, f^*T_Y)$ parametrizes first-order deformations of $f : X \to Y$ when both X and Y are fixed, while for A, B, respectively $\mathcal{O}_X$ and $\mathcal{O}_Y$-modules, $\mathrm{Ext}^i_f(B, A)$ denotes the derived functor of $\mathrm{Hom}_f(B, A) = \mathrm{Hom}_{\mathcal{O}_X}(f^*B, A) = \mathrm{Hom}_{\mathcal{O}_Y}(B, f_*A)$. Under reasonable assumptions (trivially satisfied when f is a finite map between nodal curves) one has that $\mathrm{Ext}^1_f(\Omega_Y, \mathcal{O}_X) = \mathrm{Ext}^1(f^*\Omega_Y, \mathcal{O}_X)$. Using (2.1) it follows that when X is smooth and irreducible and Y is rigid (e.g. a product of projective spaces) $\mathrm{Def}(X, f, Y) = H^0(X, N_f)$, with N_f the normal sheaf of the map f (see Chapter 1 for the definition).

Next, we recount some basic facts about moduli spaces of maps from curves to projective varieties. For Y a smooth, projective variety and $\beta \in H_2(Y, \mathbb{Z})$, one can consider the Kontsevich moduli space $\overline{\mathcal{M}}_g(Y, \beta)$ of stable maps $f : C \to Y$ from reduced, connected, nodal curves of genus g to Y, such that $f_*([C]) = \beta$ (see [FP] for the construction of these moduli spaces). If $f : C \to Y$ is a point of $\overline{\mathcal{M}}_g(Y, \beta)$ with C smooth, $\deg(f) = 1$ and f has no cusps (i.e. it is an immersion), then by Riemann-Roch $\chi(C, N_f) = \dim(Y)\,(1-g) + 3g - 3 - \beta \cdot K_Y$. Because $T_{[f]}(\overline{\mathcal{M}}_g(Y, \beta)) = H^0(C, N_f)$, the number

$$\dim(Y)\,(1-g) + 3g - 3 - \beta \cdot K_Y$$

is called the *expected dimension* of the Kontsevich moduli space. If there exists a point $[f] \in \overline{\mathcal{M}}_g(Y, \beta)$, with C smooth, $\deg(f) = 1$ and $H^1(C, N_f) = 0$, then every class in $H^0(C, N_f)$ is unobstructed, f is an immersion (cf. [AC1] Lemma 1.4) and $\overline{\mathcal{M}}_g(Y, \beta)$ is smooth of the expected dimension at the point $[f]$. An irreducible component of $\overline{\mathcal{M}}_g(Y,3)$ which has the expected dimension and is generically smooth, is said to be *regular*.

We now describe a few smoothing techniques of algebraic curves in $\mathbb{P}^r, r \geq 3$ (cf. [HH], [Se2]). Let X be a nodal curve in $\mathbb{P}^r$, with $p_a(X) = g, \deg(X) = d$. We say that X is *smoothable in* $\mathbb{P}^r$ if there exists a flat family of curves $\{X_t\}$ in $\mathbb{P}^r$ over a smooth and irreducible base, with the general fibre X_t smooth while the special fibre X_0 is X. In other words, if $\mathrm{Hilb}_{d,g,r}$ denotes the Hilbert scheme of curves in $\mathbb{P}^r$ of degree d and (arithmetic) genus g, then X is smoothable in $\mathbb{P}^r$ if and only if the point $[X]$ belongs to a component of $\mathrm{Hilb}_{d,g,r}$ whose general member corresponds to a smooth curve.
For $X \subseteq \mathbb{P}^r$ a nodal curve with normal sheaf $N_X = N_{X/\mathbb{P}^r}$, one has the exact sequence

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^r} \otimes \mathcal{O}_X \longrightarrow N_X \longrightarrow T^1_X \longrightarrow 0,$$

where T^1_X is the Lichtenbaum-Schlessinger cotangent sheaf based on $\mathrm{Sing}(X)$ and which describes deformations of the nodes of X. The basic smoothing criterion is the following result of Hartshorne and Hirschowitz:

Proposition 2.2.1 *Let $X \subseteq \mathbb{P}^r$ be a nodal curve. Assume $H^1(X, N_X) = 0$ and that for each $p \in$ Sing(X), the map $H^0(X, N_X) \to H^0(T^1_{X,p})$ is surjective (that is, non-zero). Then X is smoothable in $\mathbb{P}^r$ and the Hilbert scheme is smooth of the expected dimension $\chi(X, N_X) = (r+1)d - (r-3)(g-1)$ at the point $[X]$.*

We will be interested in smoothing curves $X \subseteq \mathbb{P}^r$ which are unions of two curves C and E meeting quasi-transversally at a finite set Δ. For such a curve one has the Mayer-Vietoris sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \oplus \mathcal{O}_E \longrightarrow \mathcal{O}_\Delta \longrightarrow 0, \tag{2.2}$$

as well as the exact sequences

$$0 \longrightarrow \mathcal{O}_E(-\Delta) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0, \tag{2.3}$$

and

$$0 \longrightarrow \Omega_E \longrightarrow \omega_X \longrightarrow \Omega_C(\Delta) \longrightarrow 0, \tag{2.4}$$

where ω_X is the dualizing sheaf of X. We will also need the following results:

Proposition 2.2.2 *Let $C \subseteq \mathbb{P}^r$ be a smooth curve with $H^1(C, N_C) = 0$.*

1. *(Sernesi) Let $H \subseteq \mathbb{P}^r$ be a hyperplane transversal to C and $Q \subseteq H$ a smooth, irreducible, rational curve of degree $r-1$ meeting C quasi-transversally in $\leq r+2$ points. Then $X = C \cup Q$ is smoothable and $H^1(X, N_X) = 0$.*

2. *(Ballico-Ellia) Let $p_1, \dots, p_{r+2} \in C$ be $r+2$ points in general linear position and $E \subseteq \mathbb{P}^r$ a smooth rational curve of degree r which meets C quasi-transversally at $p_1, \dots, p_{r+2}$. Then $X = C \cup E$ is smoothable and $H^1(X, N_X) = 0$.*

3. *(Ballico-Ellia) Assume $r = 3$ and let $L \subseteq \mathbb{P}^3$ be a line meeting C quasi-transversally at $k \leq 3$ points. If $k = 3$ assume furthermore that not all tangent lines to C at the points in $L \cap C$ lie in the same plane. Then $X = C \cup L$ is smoothable and $H^1(X, N_X) = 0$.*

2.3 Linear series on k-gonal curves

For a smooth curve C of genus g one defines the *gonality sequence* $(d_1, d_2, \dots, d_r, \dots)$ by $d_r := \min\{d \in \mathbb{Z}_{\geq 1} : d \leq g-1, \exists \text{ a } \mathfrak{g}^r_d \text{ on } C\}$. This sequence is strictly increasing and clearly $d_r \leq r d_1$. The first term d_1 is just the *gonality* of C, while obviously $d_r = g + r$ for $r \geq g$, so we will restrict ourselves to the first $g-1$ terms of the sequence. The Brill-Noether Theorem tells us that $d_r \leq \lceil r(g-r+2)/(r+1) \rceil$ and we have equality when C is a general curve of genus g. The terms of the gonality sequence can be easily computed for various classes of curves (hyperelliptic, trigonal, smooth plane curves). In order to find the d_r's for a curve C, it suffices to look only at the set of linear series

$$S(C) = \{|D| : D \in \text{Div}(C), \deg(D) \leq g-1, h^0(D) \geq 2, h^1(D) \geq 2\}.$$

This is because any $\mathfrak{g}^r_d$ with $d \geq 2g-1$ is non-special, hence $r = d - g$, while for $d \leq 2g-2$ by interchanging if necessary $\mathfrak{g}^r_d$ by $|K_C - \mathfrak{g}^r_d|$, we land eventually in the range $d \leq g-1$.

We would like to determine the sequence $(d_1, d_2, \dots)$ for a general k-gonal curve of genus g when $k < (g+2)/2$ (i.e. $\rho(g, 1, k) < 0$). Coppens and Martens (cf. [CM]) have investigated how the existence of a $\mathfrak{g}^1_k$ on a curve C can be used to produce special linear series on C with negative Brill-Noether number (i.e. the ones you cannot expect to find on a general curve of genus g). Under certain numerical constraints, a general k-gonal curve C of genus g carries linear series $\mathfrak{g}^r_d = |(r-f)\mathfrak{g}^1_k + E|$ (which we shall call *Segre linear series*, see the motivation below), where $0 < f \leq k-2$ and $E \in \text{Div}(C), E \geq 0$. For $r = 2$ one recovers a famous result of Beniamino Segre (see [AC1]): A general nonhyperelliptic k-gonal curve C of genus g has a linear series $\mathfrak{g}^2_d = |\mathfrak{g}^1_k + E|$, with $E \geq 0$, when $d \geq (g+k+2)/2$, and which provides a plane model Γ of C with an ordinary $(d-k)$-fold singularity p and nodes as other singularities. The original $\mathfrak{g}^1_k$ can be retrieved by projecting Γ from p. When $r = 3$ and $k \geq 4$, a general k-gonal curve of genus g has a linear series $\mathfrak{g}^3_d = |\mathfrak{g}^1_k + E|$ when $d \geq (2g+k+6)/3$. Thus for a general $[C] \in \mathcal{M}^1_{g,k}$ one has that $d_2(C) \leq \lceil (g+k+3)/2 \rceil$ and $d_3(C) \leq \lceil (2g+k+8)/3 \rceil$ and we expect to have equality in the case when the Segre linear series have negative Brill-Noether number. This would certainly be the case if the following two expectations were true:

- α) For a general $[C] \in \mathcal{M}^1_{g,k}$, the Segre linear series are of minimal degree among those $\mathfrak{g}^r_d = |D| \in S(C)$ for which $|D - \mathfrak{g}^1_k| \neq 0$. (This is known to be true at least when $r = 2$ and $2k > \lceil (g+k+3)/2 \rceil$, see [CKM] Proposition 1.1).

- *3)* If $\mathfrak{g}^r_d \in S(C)$ and $|\mathfrak{g}^r_d - \mathfrak{g}^1_k| = \emptyset$, then $\rho(g, r, d) \geq 0$. (This holds when $r = 1$ (cf. [AC1]) and for $k \leq 4$ (cf. [CM])).

We are going to prove that these expectations hold in the case when the curve C is of relatively high gonality (but still non-generic):

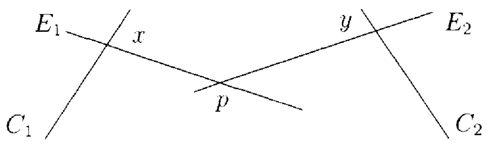
Theorem 2.1 *Let g and k be positive integers such that $-3 \leq \rho(g, 1, k) < 0$. Assume furhermore that $k \geq 6$ when $\rho(g, 1, k) = -3$. Then a general k-gonal curve of genus g has no $\mathfrak{g}^r_d$'s with negative Brill-Noether number except $\mathfrak{g}^1_k$ and $|K_C - \mathfrak{g}^1_k|$. In other words, the k-gonal locus $\mathcal{M}^1_{g,k}$ is not contained in any other proper Brill-Noether locus $\mathcal{M}^r_{g,d}$.*

Remark: A general k-gonal curve of genus g with $\rho(g, 1, k) < 0$ has a unique pencil $\mathfrak{g}^1_k$ (cf. [AC1]) so there is no ambiguity when we speak of "the $\mathfrak{g}^1_k$ of a general k-gonal curve".

Proof: We will make use of the theory of limit linear series. In each case we construct k-gonal curves of compact type that do not possess any limit $\mathfrak{g}^r_d$ with $r \geq 2, d \leq g-1$ and $\rho(g, r, d) < 0$. Using the fact that the k-gonal locus $\mathcal{M}^1_{g,k}$ is irreducible we obtain the conclusion for a general $[C] \in \mathcal{M}^1_{g,k}$.

The case $\rho(g, 1, k) = -1$ (when $\mathcal{M}^1_{g,k}$ is an irreducible divisor in $\mathcal{M}_g$) is settled using the curves constructed in the proof of Theorem 1.2. Since the proof goes along the same lines we skip the details.

Assume now that $\rho(g, 1, k) = -2$. Because any component of $\mathcal{M}^r_{g,d}$ has codimension ≥ 3 when $\rho(g, r, d) \leq -3$ (cf. [Ed2]), it suffices to construct a k-gonal curve of genus g having no $\mathfrak{g}^r_d$'s when $\rho(g, r, d) \in \{-1, -2\}$.
Let us consider the following curve of genus $2k$,



$$X := C_1 \cup C_2 \cup E_1 \cup E_2,$$

where (C_1, x) and (C_2, y) are general pointed curves of genus $k-1$, E_i are elliptic curves and $x - p \in \mathrm{Pic}^0(E_1)$ is a primitive k-torsion as it is $p - y \in \mathrm{Pic}^0(E_2)$. It is straightforward to construct a limit $\mathfrak{g}^1_k$ on X: on C_1 take the pencil $|kx|$, on C_2 take the pencil $|ky|$, on E_1 the pencil $\langle kx, kp\rangle$, spanned by kx and kp, while on E_2 the pencil $\langle kp, ky\rangle$.
Assume now that there is a limit $\mathfrak{g}^r_d$ on X, say l, with $r \geq 2, d \leq 2k-1$ and $\rho(g, r, d) < 0$. From (1.6), we have that

$$-1 \geq \rho(l_X) \geq \rho(l_{C_1}, x) + \rho(l_{C_2}, y) + \rho(l_{E_1}, x, p) + \rho(l_{E_2}, p, y).$$

Because of Prop. 1.3.2 one has $\rho(l_{C_1}, x) \geq 0$ and $\rho(l_{C_2}, y) \geq 0$. Moreover, we have that $\rho(E_1, x, p) \geq -1$ and $\rho(E_2, p, y) \geq -1$. Indeed, if say $\rho(E_1, x, p) \leq -2$, then by denoting

by $(a_0, \dots, a_r)$ the vanishing sequence of l_{E_1} at x and by $(b_0, \dots, b_r)$ that of l_{E_1} at p, it would follow that for at least 3 indices $i < j < k$ there are equalities $a_i + b_{r-i} = a_j + b_{r-j} = a_k + b_{r-k} = d$, from which $k | (a_j - a_i)$ and $k | (a_k - a_i)$, hence $d \geq a_k \geq a_i + 2k \geq 2k$, which is a contradiction since we assumed $d \leq 2k - 1$.
This implies that there are essentially two cases to consider:

1. $\rho(l_{C_1}, x) = \rho(l_{C_2}, y) = 0, \rho(l_{E_1}, x, p) = -1, \rho(l_{E_2}, p, y) = 0.$

2. $\rho(l_{C_1}, x) = 0, \rho(l_{C_2}, y) = 1, \rho(l_{E_1}, x, p) = \rho(l_{E_2}, p, y) = -1.$

In both cases l is a refined limit $\mathfrak{g}^r_d$. The other possibilities can either be dismissed right away (when one of the adjusted Brill-Noether numbers is ≥ 2), or are equivalent to the cases just mentioned.

Let us first settle case 1. By using (1.8),

$$\sum_{i=0}^{r}(k - 1 - a_i^{l_{E_1}}(x) + i)_+ = \sum_{i=0}^{r}(k - 1 - a_i^{l_{E_1}}(x) + i) = k - 1, \text{ hence}$$

$$a_i^{l_{E_1}}(x) \leq k - 1 + i, \quad \text{and similarly} \quad a_i^{l_{E_2}}(y) \leq k - 1 + i, \text{ for all } \ i = 0, \dots, r. \qquad (2.5)$$

Since on E_2 we have inequalities $a_i^{l_{E_2}}(p) + a_{r-i}^{l_{E_2}}(y) \geq d - 2$, for all i (otherwise once again we would clash with the assumption $d \leq 2k - 1$), we eventually obtain that

$$a_i^{l_{E_1}}(p) \leq k + i + 1, \quad \text{for all } \ i = 0, \dots, r. \qquad (2.6)$$

Since $\rho(l_{E_1}, x, p) = -1$, there must be indices $i < j$ with $a_i^{l_{E_1}}(x) + a_{r-i}^{l_{E_1}}(p) = a_j^{l_{E_1}}(x) + a_{r-j}^{l_{E_1}}(p) = d$, from where we get that $a_j^{l_{E_1}}(x) - a_i^{l_{E_1}}(x) = k$. Then, because of (2.5) and (2.6) we can write

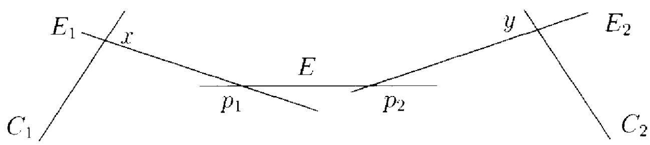
$$d - k - r + i - 1 \leq d - a_{r-i}^{l_{E_1}}(p) = a_i^{l_{E_1}}(x) \leq j - 1,$$

hence $2r + k \geq d$. Combine this with $\rho(2k, r, d) \geq -2$ to get that $k \leq (r^2 + r + 2)/(r - 1)$. But we also have that $2k \geq r^2 + r + 1$ (because $\rho(2k, r, d) \geq -r$ and $d \leq 2k - 1$), so all in all, we end up with $r^3 - 2r^2 - 2r - 5 \leq 0$, which can be possible only for $r \leq 3$. When $r \in \{2, 3\}$, by plugging in one of the previous inequalities we have that $8 \leq g = 2k \leq 16$. But these cases can be disposed of easily. First, notice that when $g \in \{10, 12, 16\}$, since $g + 1$ is prime, we have no codimension one Brill-Noether condition on $\mathcal{M}_g$. To treat one of the remaining cases when we do have a codimension one Brill-Noether locus, take for example $g = 14$ and $r = 2$, hence $d = 11$. In this case the inequalities (2.6) can be improved and this leads to a contradiction: since on E_1 we have that $\rho(l_{E_1}, x, p) = -1$, exactly two of the numbers $a_i^{l_{E_1}}(x) + a_{2-i}^{l_{E_1}}(p)$ are equal to 11 while the remaining one is equal to 10. There are three cases and each can be dismissed swiftly.

We turn to case 2. Use again (1.8) to obtain that $a_r^{l_{E_2}}(y) \leq k + r$. Moreover

$a_r^{l_{E_2}}(y) + a_0^{l_{E_2}}(p) \geq d-1$, hence $a_i^{l_{E_1}}(p) \leq k+i+1$, for all $i = 0,\dots,r$, hence we have obtained again (2.6), while the inequalities $a_i^{l_{E_1}}(x) \leq k-i+1$ for $i=0,\dots,r$ still hold, so the previous argument can be repeated here as well.

We treat now the case $\rho(g,1,k) = -3$, that is $g = 2k+1$, with $k \geq 6$. Note that when $k=5$, Segre's Theorem gives a $\mathfrak{g}^2_9 = |\mathfrak{g}^1_5 + E|$, with $E \geq 0$, on a general 5-gonal curve of genus 11, i.e. the 5-gonal locus $\mathcal{M}^1_{11,5}$ is contained in the Brill-Noether divisor $\mathcal{M}^2_{11,9}$. The idea is the same, but the computations are a bit more cumbersome. We use the following curve:



$$X := C_1 \cup C_2 \cup E_1 \cup E_2 \cup E,$$

where (C_1,x) and (C_2,y) are general pointed curves of genus $k-1$, the curves E, E_1, E_2 are all elliptic, and the differences $x - p_1 \in \mathrm{Pic}^0(E_1), p_1 - p_2 \in \mathrm{Pic}^0(E)$, and $p_2 - y \in \mathrm{Pic}^0(E_2)$ are all primitive k-torsions. Just as in the previous case, it is clear that X possesses a limit $\mathfrak{g}^1_k$. Assume now by contradiction that there exists l a limit $\mathfrak{g}^r_d$ on X, with $r \geq 2, d \leq g-1$ and $\rho(g,r,d) < 0$. There are many cases to consider, but it is clear that in order to maximize the chances for such a limit $\mathfrak{g}^r_d$ to exist, the adjusted Brill-Noether numbers must be as evenly distributed and as close to 0 as possible: a very positive Brill-Noether number on one component, implies by (1.6) very negative Brill-Noether numbers on other components (2-pointed elliptic curves) and this immediately yields a contradiction. We will only treat one case the other being similar. Assume $\rho(l_{C_1},x) = \rho(l_{C_2},y) = \rho(l_{E_1},x,p_1) = \rho(l_{E_2},p_2,y) = 0$, and $\rho(l_E,p_1,p_2) = -1$. Then by (1.8) we have that $a_r^{l_{E_1}}(x) \leq k+r-1$ and $a_r^{l_{E_2}}(y) \leq k+r-1$. Since $a_r^{l_{E_1}}(x) + a_0^{l_{E_1}}(p_1) \geq d-2$ and $a_r^{l_{E_2}}(y) + a_0^{l_{E_2}}(p_2) \geq d-2$, we get that

$$a_i^{l_E}(p_1) \leq i+k+1 \quad \text{and} \quad a_i^{l_E}(p_2) \leq i+k+1, \quad \text{for } i = 0,\dots,r. \tag{2.7}$$

As in the case $\rho = -2$, we can conlude from (2.7) that $2r+k+2 \geq d$. This we combine with $\rho(g,r,d) \geq -r$ to obtain that $k \leq (r^2+3r+2)/(r-1)$. Also $2k \geq r^2+r$ (just put together $d \leq 2k$ and $\rho(g,r,d) \geq -r$), and in the end we get that $r^3 - 2r^2 - 7r - 4 \leq 0 \Leftrightarrow r \leq 4$. The case $r=4$ can be dismissed though right away, because then all inequalities we have writen down become equalities, hence $g = 21, d = 20$, and $\rho(g,r,d) = -4$, contradiction since we assumed $\rho(g,r,d) = -1$. When $r \leq 3$ we have that $k \leq 10$. In these particular cases however, we can improve the inequalities (2.7) (which we have watered down to obtain an argument working for general r), and we easily reach a contradiction. □

Remark: One can try to extend these results for more negative values of $\rho(g,1,k)$. The cases $\rho = -4$ (resp. $\rho = -5$) could be handled by slightly modifying the curves used for treating the cases $\rho = -2$ (resp. $\rho = -3$): require that the points $x \in C_1$ and $y \in C_2$ are ordinary Weierstrass points instead of general points. We have checked that for $g \leq 23$ Theorem 2.1 still holds when $\rho \in \{-4, -5\}$. For instance we get that the general 10-gonal curve of genus 23 does not possess any $\mathfrak{g}^r_d$'s with $r \geq 2$, $d \leq 22$ and negative Brill-Noether number. In these cases however, computations become horrendous therefore we think that limit linear series cannot provide the full answer to problem *3*).

2.4 Existence of regular components of moduli spaces of maps to $\mathbb{P}^1 \times \mathbb{P}^r$

In this section we construct regular components of the moduli space $\overline{\mathcal{M}}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$ of stable maps $f: C \to \mathbb{P}^1 \times \mathbb{P}^r$ of bidegree (k, d), in the case $k \geq r + 2, d \geq r \geq 3$, and $\rho(g, r, d) < 0$.

The spaces $\mathcal{M}_g(\mathbb{P}^r, d)$ (or the Hilbert schemes $\mathrm{Hilb}_{d,g,r}$ of curves $C \subseteq \mathbb{P}^r, \deg(C) = d, p_a(C) = g$) have been the subject of much study in the past 20 years. For instance, in the case of curves in $\mathbb{P}^3$ one knows that for each g there is $D(g) \in \mathbb{Z}$ such that for any $d \geq D(g)$, there exists a curve $C \subseteq \mathbb{P}^3$ of genus g and degree d, with $H^1(C, N_{C/\mathbb{P}^3}(-2)) = 0$ (so also $H^1(C, N_{C/\mathbb{P}^3}) = 0$). The numbers $D(g)$ satisfy the estimate limsup $D(g)g^{-2/3} \leq (9/8)^{1/3}$ (cf. [EllH]). Therefore, when (asymptotically) $d \geq g^{2/3}(9/8)^{1/3}$, there are regular components of $\mathcal{M}_g(\mathbb{P}^3, d)$ whose general points correspond to embeddings $C \hookrightarrow \mathbb{P}^3$. In the case $\rho(g, r, d) \geq 0$ there is a unique (regular) component of $\mathcal{M}_g(\mathbb{P}^r, d)$ which dominates $\mathcal{M}_g$ and whose general point corresponds to a non-degenerate map to $\mathbb{P}^r$ (i.e. the image is not contained in a hyperplane). This follows from the fact that $G^r_d(C)$ is irreducible for general C when $\rho(g, r, d) \geq 1$ (see [ACGH]); when $\rho(g, r, d) = 0$ an extra monodromy argument is needed.

When the target space is $\mathbb{P}^1 \times \mathbb{P}^1$, Arbarello and Cornalba proved that any component of $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^1, (d, h))$, when $2 \leq g, d, h$, having general points corresponding to birational maps $C \to \mathbb{P}^1 \times \mathbb{P}^1$, is regular: as a matter of fact, it is not hard to see that there is exactly one such component. More generally, the methods from [AC1] can be used succesfully in order to compute dim $\mathcal{M}_g(Y, \beta)$ when Y is a smooth surface: if $M \subseteq \mathcal{M}_g(Y, \beta)$ is a component of dimension $\geq g + 1$ and containing a point $[f : C \to Y]$ with $\deg(f) = 1$, then M is regular. One uses here in an essential way the fact that the normal sheaf N_f is of rank 1, hence the Clifford Theorem gives a straightforward condition for the vanishing of $H^1(C, N_f)$, so this techniques cannot be applied for handling moduli spaces of maps to higher dimensional target spaces Y.

Although we only treat the case of curves mapping into $\mathbb{P}^1 \times \mathbb{P}^r$ when $r \geq 3$, it will be clear that our methods can be also applied to study regular components of the moduli space of curves sitting on the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2$.

We start our study of moduli spaces of maps into $\mathbb{P}^1 \times \mathbb{P}^r$. Fix integers $g \geq 0, d \geq r \geq 3$ and $k \geq 2$, as well as C a smooth curve of genus g with maps $f_1 : C \to \mathbb{P}^1, f_2 : C \to \mathbb{P}^r$, such that $\deg(f_1) = k$, $\deg(f_2(C)) = d$ and f_2 is generically injective. Let us denote by $f : C \to \mathbb{P}^1 \times \mathbb{P}^r$ the product map.

There is a commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & T_C & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & T_C & \longrightarrow & f^*(T_{\mathbb{P}^1\times\mathbb{P}^r}) & \longrightarrow & N_f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow = & & \downarrow & & \\
0 & \longrightarrow & T_C \oplus T_C & \longrightarrow & f_1^*(T_{\mathbb{P}^1}) \oplus f_2^*(T_{\mathbb{P}^r}) & \longrightarrow & N_{f_1} \oplus N_{f_2} & \rightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & & .
\end{array}
$$

By taking cohomology in the last column, we see that the condition $H^1(C, N_f) = 0$ is equivalent with $H^1(C, N_{f_1}) = 0$ (trivial), $H^1(C, N_{f_2}) = 0$, and

$$\mathrm{Im}\{\delta_1 : H^0(C, N_{f_1}) \to H^1(C, T_C)\} + \mathrm{Im}\{\delta_2 : H^0(C, N_{f_2}) \to H^1(C, T_C)\} = H^1(C, T_C), \tag{2.8}$$

where the condition (2.8) is equivalent (cf. Chapter 1) with

$$(d\pi_1)_{[f_1]}\ \big(T_{[f_1]}(\mathcal{M}_g(\mathbb{P}^1, k))\big) + (d\pi_2)_{[f_2]}\ \big(T_{[f_2]}(\mathcal{M}_g(\mathbb{P}^r, d))\big) = T_{[C]}(\mathcal{M}_g), \tag{2.9}$$

(we assume that the curve C has no automorphisms, otherwise we work in the versal deformation space of C, it makes no difference). The projections $\pi_1 : \mathcal{M}_g(\mathbb{P}^1, k) \to \mathcal{M}_g$ and $\pi_2 : \mathcal{M}_g(\mathbb{P}^r, d) \to \mathcal{M}_g$ are the natural forgetful maps. Slightly abusing the terminology, if C is a smooth curve and $(l_1, l_2) \in G^1_k(C) \times G^r_d(C)$ is a pair of base point free linear series on C, we say that (C, l_1, l_2) satisfies (2.9) if (C, f_1, f_2) satifies (2.9), where f_1 and f_2 are maps associated to l_1 and l_2.

We prove the existence of regular components of $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$ inductively, using the following:

Proposition 2.4.1 *Fix positive integers g, r, d and k with $d \geq r \geq 3, k \geq r + 2$ and $\rho(g, r, d) < 0$. Let us assume that $C \subseteq \mathbb{P}^r$ is a smooth nondegenerate curve of degree d and genus g, such that $h^1(C, N_C) = 0, h^0(C, \mathcal{O}_C(1)) = r + 1$ and the Petri map*

$$\mu_0(C) = \mu_0(C, \mathcal{O}_C(1)) : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1)) \to H^0(C, K_C)$$

is surjective. Assume furthermore that C possesses a simple base point free pencil $\mathfrak{g}^1_k$ say l, such that $|\mathcal{O}_C(1)|(-l) = \emptyset$ and $(C, l, |\mathcal{O}_C(1)|)$ satisfies (2.9).

Then there exists a smooth nondegenerate curve $Y \subseteq \mathbb{P}^r$ with $g(Y) = g + r + 1$, $\deg(Y) = d + r$ and a simple base point free pencil $l' \in G^1_k(Y)$, so that Y enjoys exactly the same properties: $h^1(Y, N_Y) = 0$, $h^0(Y, \mathcal{O}_Y(1)) = r + 1$, the Petri map $\mu_0(Y)$ is surjective, $|\mathcal{O}_Y(1)|(-l') = \emptyset$ and $(Y, l', |\mathcal{O}_Y(1)|)$ satifies (2.9).

Proof: We first construct a reducible k-gonal nodal curve $X \subseteq \mathbb{P}^r$, with $p_a(X) = g+r+1$, $\deg(X) = d+r$, having all the required properties, then we prove that X can be smoothed in $\mathbb{P}^r$ preserving all properties we want.

Let $f_1 : C \to \mathbb{P}^1$ be the degree k map corresponding to the pencil l. The covering f_1 is simple (i.e. over each branch point $\lambda \in \mathbb{P}^1$ there is only one ramification point $x \in f_1^{-1}(\lambda)$ and $e_x(f_1) = 2$), hence the monodromy of f_1 is the full symmetric group. Then since $|\mathcal{O}_C(1)|(-l) = \emptyset$, we have that for a general $\lambda \in \mathbb{P}^1$ the fibre $f_1^{-1}(\lambda) = p_1 + \cdots + p_k$ consists of k distinct points in general linear position. Let $\Delta = \{p_1, \ldots, p_{r+2}\}$ be a subset of $f_1^{-1}(\lambda)$ and let $E \subseteq \mathbb{P}^r$ be a rational normal curve ($\deg(E) = r$) passing through $p_1, \ldots, p_{r+2}$. (Through any $r+3$ points in general linear position in $\mathbb{P}^r$, there passes a unique rational normal curve, so we have picked E out of a 1-dimensional family of curves through the chosen points $p_1, \ldots, p_{r+2}$). Let $X := C \cup E$, with C and E meeting quasi-transversally at Δ. Of course $p_a(X) = g+r+1$ and $\deg(X) = d+r$. Note that $\rho(g, r, d) = \rho(g+r+1, r, d+r)$.

We first prove that $[X] \in \overline{\mathcal{M}}^1_{g+r+1,k}$ (that is, X is k-gonal), by constructing an admissible covering of degree k having as domain a curve X', stably equivalent to X. Let $X' := X \cup D_{r+3} \cup \ldots \cup D_k$, where $D_i \simeq \mathbb{P}^1$ and $D_i \cap X = \{p_i\}$, for $i = r+3, \ldots, k$. Take $Y := (\mathbb{P}^1)_1 \cup_\lambda (\mathbb{P}^1)_2$ a union of two lines identified at λ. We construct a degree k admissible covering $f' : X' \to Y$ as follows: take $f'_{_C} = f_1 : C \to (\mathbb{P}^1)_1$, $f'_{_E} = f_2 : E \to (\mathbb{P}^1)_2$ a map of degree $r+2$ sending the points $p_1, \ldots, p_{r+2}$ to λ, and finally $f'_{_{D_i}} : D_i \simeq (\mathbb{P}^1)_2$ isomorphisms sending p_i to λ. Clearly f' is an admissible covering, so X which is stably equivalent to X' is a k-gonal curve.

Let us consider now the space $\overline{\mathcal{H}}_{g+r+1,k}$ of Harris-Mumford admissible coverings of degree k (cf. [HM]) and denote by $\pi_1 : \overline{\mathcal{H}}_{g+r+1,k} \to \overline{\mathcal{M}}_{g+r+1}$ the natural projection which sends a covering to the stable model of its domain. If we assume that $\mathrm{Aut}(C) = \{Id_C\}$ (which we can safely do), then also $\mathrm{Aut}(f') = \{Id_{X'}\}$, so $[f']$ is a smooth point of $\overline{\mathcal{H}}_{g+r+1,k}$. We compute the differential of the map π_1 at $[f']$. We notice that $T_{[f']}(\overline{\mathcal{H}}_{g+r+1,k}) = \mathrm{Def}(X', f', Y) = \mathrm{Def}(X, f, Y)$, where $f = f'_{_X} : X \to Y$. The differential $(d\pi_1)_{[f']}$ is just the forgetful map $\mathrm{Def}(X, f, Y) \to \mathrm{Def}(X)$ and from the sequence (2.1) we get that $\mathrm{Im}(d\pi_1)_{[f']} = u_1^{-1}(\mathrm{Im}\, u_2)$, where $u_1 : \mathrm{Def}(X) \to \mathrm{Ext}^1(f^*\Omega_Y, \mathcal{O}_X)$ and $u_2 : \mathrm{Def}(Y) \to \mathrm{Ext}^1(f^*\Omega_Y, \mathcal{O}_X)$ are the dual maps of $u_1^\vee : H^0(X, \omega_X \otimes f^*\Omega_Y) \to H^0(X, \omega_X \otimes \Omega_X)$ and $u_2^\vee : H^0(X, \omega_X \otimes f^*\Omega_Y) \to H^0(Y, \omega_Y \otimes \Omega_Y)$ (the last one induced by the trace map $\mathrm{tr} : f_*\omega_X \to \omega_Y$). Starting with the exact sequence on X,

$$0 \longrightarrow \mathrm{Tors}(\omega_X \otimes \Omega_X) \longrightarrow \omega_X \otimes \Omega_X \longrightarrow \Omega_C^{\otimes 2}(\Delta) \oplus \Omega_E^{\otimes 2}(\Delta) \longrightarrow 0,$$

we can write the following commutative diagram of sequences

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H^0(\mathrm{Tors}(\omega_X \otimes f^*\Omega_Y)) & \hookrightarrow & H^0(\omega_X \otimes f^*\Omega_Y) & \twoheadrightarrow & H^0(2K_C - R_1 + \Delta) \oplus H^0(2K_E - R_2 + \Delta) \\
\downarrow {\scriptstyle (u_1^\vee)_{\mathrm{tors}}} & & \downarrow {\scriptstyle u_1^\vee} & & \downarrow \\
H^0(\mathrm{Tors}(\omega_X \otimes \Omega_X)) & \hookrightarrow & H^0(\omega_X \otimes \Omega_X) & \twoheadrightarrow & H^0(2K_C + \Delta) \oplus H^0(2K_E + \Delta)
\end{array}$$

where R_1 (resp. R_2) is the ramification divisor of the map f_1 (resp. f_2). Taking into account that $H^0(E, 2K_E - R_2 + \Delta) = 0$ and that $H^0(Y, \omega_Y \otimes \Omega_Y) = H^0(\mathrm{Tors}(\omega_Y \otimes \Omega_Y))$, we obtain that

$$\mathrm{Im}(d\pi_1)_{[f']} = \big(H^0(C, 2K_C - R_1 + \Delta) \oplus \mathrm{Ker}(u_2^\vee)_{\mathrm{tors}}\big)^\perp, \tag{2.10}$$

where $(u_2^\vee)_{\mathrm{tors}} : H^0(\mathrm{Tors}(\omega_X \otimes f^*\Omega_Y)) \to H^0(\mathrm{Tors}(\omega_Y \otimes \Omega_Y))$ is the restriction of $u_2^\vee$. The space $\mathrm{Ker}(u_2^\vee)_{\mathrm{tors}}$ is just a hyperplane in $H^0(\mathrm{Tors}(\omega_X \otimes f^*\Omega_Y)) \simeq \mathbb{C}^{r+2}$.

Remark: Since $\Delta \in C_{r+2}$ was chosen generically in a fibre of the $\mathfrak{g}_k^1$ on C, it follows from Riemann-Roch that $h^0(C, 2K_C - R_1 + \Delta) = g - 2k + 3 + r = \mathrm{codim}(\overline{\mathcal{M}}^1_{g+r+1,k}, \overline{\mathcal{M}}_{g+r+1})$. If C has only finitely many $\mathfrak{g}_k^1$'s the fibre of the map $\pi_1 : \overline{\mathcal{H}}_{g+r+1,k} \to \overline{\mathcal{M}}_{g+r+1}$ over the point $[X]$ is $(r+1)$-dimensional: the fibre is basically the space of degree $r+1$ maps $f_2 : E \to \mathbb{P}^1$ such that $f_2(p_1) = \ldots = f_2(p_{r+2}) = \lambda$. Moreover, if we assume (in the case $g > 2k-2$) that $[C]$ is a smooth point of the locus $\mathcal{M}^1_{g,k}$ (which happens precisely when C has only one $\mathfrak{g}_k^1$ and $\dim|2\mathfrak{g}_k^1| = 2$), then we have for the tangent cone

$$TC_{[X]}(\overline{\mathcal{M}}^1_{g+r+1,k}) = \bigcup\{\mathrm{Im}(d\pi_1)_z : z \in \pi_1^{-1}([X])\} = H^0(C, 2K_C - R_1 + \Delta)^\perp,$$

which shows that $[X]$ is a smooth point of the locus $\overline{\mathcal{M}}^1_{g+r+1,k}$.

We compute now the differential

$$(d\pi_2)_{[X]} : T_{[X]}(\mathrm{Hilb}_{d+r,g+r+1,r}) \to T_{[X]}(\overline{\mathcal{M}}_{g+r+1}),$$

which is the same thing as the differential at the point $[X \hookrightarrow \mathbb{P}^r]$ of the projection $\pi_2 : \overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r, d+r) \to \overline{\mathcal{M}}_{g+r+1}$. We start by noticing that X is smoothable in $\mathbb{P}^r$ and that $H^1(X, N_X) = 0$ (apply Prop.2.2.2). We also have that X is embedded in $\mathbb{P}^r$ by a complete linear system, that is $h^0(X, \mathcal{O}_X(1)) = r+1$. Indeed, on one hand, since X is nondegenerate, $h^0(X, \mathcal{O}_X(1)) \geq h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r(1)}) = r+1$; on the other hand from the sequence (2.3) we have that $h^0(X, \mathcal{O}_X(1)) \leq h^0(C, \mathcal{O}_C(1)) = r+1$.

If X is embedded in $\mathbb{P}^r$ by a complete linear system, we know (cf. Section 1.3) that

$$\mathrm{Im}(d\pi_2)_{[X]} = (\mathrm{Im}\mu_1(X))^\perp,$$

where $\mu_1(X) : \mathrm{Ker}\mu_0(X) \to H^0(X, \omega_X \otimes \Omega_X)$ is the 'derivative' of the Petri map $\mu_0(X) : H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \omega_X(-1)) \to H^0(X, \omega_X)$. We compute the kernel of $\mu_0(X)$ and show that $\mu_0(X)$ is surjective in a way that resembles the proof of Prop.2.3 in [Se2].

From the sequence (2.4) we obtain $H^0(X, \omega_X) = H^0(C, K_C + \Delta)$, while from (2.3) we have that $H^0(X, \mathcal{O}_X(1)) = H^0(E, \mathcal{O}_E(1))$ (keeping in mind that $H^0(C, \mathcal{O}_C(1)(-\Delta)) = 0$, as $p_1, \ldots, p_{r+2}$ are in general linear position). Finally, using (2.4) again, we have that $H^0(X, \omega_X(-1)) = H^0(C, K_C(-1)+\Delta)$. Therefore we can write the following commutative diagram:

$$\begin{array}{ccc}
H^0(C,\mathcal{O}_C(1))\otimes H^0(C,K_C(-1)) & \stackrel{\mu_0(C)}{\longrightarrow} & H^0(C,K_C)\\
\downarrow & & \downarrow\\
H^0(C,\mathcal{O}_C(1))\otimes H^0(C,K_C(-1)+\Delta) & \longrightarrow & H^0(C,K_C+\Delta)\\
\downarrow = & & \downarrow =\\
H^0(X,\mathcal{O}_X(1))\otimes H^0(X,\omega_X(-1)) & \stackrel{\mu_0(X)}{\longrightarrow} & H^0(X,\omega_X)
\end{array}$$

.

It follows that $\mathrm{Ker}\mu_0(C)\subseteq \mathrm{Ker}\mu_0(X)$. By using Corollary 1.6 from [CR], our assumptions ($\mu_0(C)$ surjective and $\mathrm{card}(\Delta)\geq 4$) enable us to conclude that $\mu_0(X)$ is surjective too. Then $\mathrm{Ker}\mu_0(C)=\mathrm{Ker}\mu_0(X)$ for dimension reasons, hence also $\mathrm{Im}\mu_1(X)=\mathrm{Im}\mu_1(C)\subseteq H^0(C,2K_C)\subseteq H^0(X,\omega_X\otimes\Omega_X)$. We thus get that $\mathrm{Im}(d\pi_2)_{[X]}=(\mathrm{Im}\mu_1(X))^{\perp}=(\mathrm{Im}\mu_1(C))^{\perp}$.

The assumption that (C,f_1,f_2) satisfies (2.9) can be rewritten by passing to duals as

$$H^0(C,2K_C-R_1)^{\perp}+(\mathrm{Im}\mu_1(C))^{\perp}=H^1(C,T_C)\Longleftrightarrow H^0(C,2K_C-R_1)\cap\mathrm{Im}\mu_1(C)=0.$$

Then it follows that $\mathrm{Im}\mu_1(X)\cap\big(H^0(C,2K_C-R_1+\Delta)\otimes\mathrm{Ker}((u_2^{\vee})_{\mathrm{tors}})\big)=0$, which is the same thing as

$$(d\pi_1)_{[f']}\ (T_{[f']}(\overline{\mathcal{H}}_{g+r+1,k}))+(d\pi_2)_{[X\hookrightarrow\mathbb{P}^r]}\ (T_{[X\hookrightarrow\mathbb{P}^r]}(\overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r,d+r)))=\mathrm{Ext}^1(\Omega_X,\mathcal{O}_X). \tag{2.11}$$

This means that the images of $\overline{\mathcal{H}}_{g+r+1,k}$ under the map π_1 and of $\overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r,d+r)$ under the map π_2, meet transversally at the point $[X]\in\overline{\mathcal{M}}_{g+r+1}$.

Claim: The curve X can be smoothed in such a way that the $\mathfrak{g}^1_k$ and the very ample $\mathfrak{g}^r_{d+r}$ are preserved (while (2.11) is an open condition on $\overline{\mathcal{H}}_{g+r+1,k}\times\overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r,d+r)$).

Indeed, the tangent directions that fail to smooth at least one node of X are those in $\bigcup_{i=1}^{r+2}H^0(\mathrm{Tors}_{p_i}(\omega_X\otimes\Omega_X))^{\perp}$, whereas the tangent directions that preserve both the $\mathfrak{g}^1_k$ and the $\mathfrak{g}^r_{d+r}$ are those in

$$\big((\mathrm{Im}\mu_1(C)+H^0(C,2K_C-R_1+\Delta))\otimes\mathrm{Ker}(u_2^{\vee})_{\mathrm{tors}}\big)^{\perp}.$$

Since obviously $H^0(\mathrm{Tors}_{p_i}(\omega_X\otimes\Omega_X))\not\subseteq\mathrm{Ker}(u_2^{\vee})_{\mathrm{tors}}$ for $i=1,\ldots,r+2$, by moving in a suitable direction in the tangent space at $[f']$ of $\pi_1^{-1}\pi_2(\overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r,d+r))$, we finally obtain a curve $Y\subseteq\mathbb{P}^r$ with $g(Y)=g+r+1,\deg(Y)=d+r$ and satisfying all the required properties. □

In order to use Prop.2.4.1 as the inductive step $(g,d)\mapsto(g+r+1,d+r)$ in the construction of regular components of $\mathcal{M}_g(\mathbb{P}^1\times\mathbb{P}^r,(k,d))$, we need curves $C\subseteq\mathbb{P}^r$ with all the properties listed in the statement of the Proposition (so that we can start the induction). We are able to construct such curves when $\rho(g,r,d)=-1$, i.e. when $\mathcal{M}^r_{g,d}$ is a divisor in $\mathcal{M}_g$.

Theorem 2.2 *Let $r \geq 3, s \geq (2r+1)/(r-1)$ and $k \geq 3$ be integers such that*

$$(rs+s-1)/2 \leq k \leq rs-r-1.$$

Then for any integers d, g such that $\rho(g,r,d) = -1$ and $g \geq (r+1)(s-1)-1$ there exists a regular component of the moduli space of maps $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k,d))$.

Remarks: 1. Theorem 2.2 actually provides regular components of the Hilbert scheme of curves of bidegree (k,d) in $\mathbb{P}^1 \times \mathbb{P}^r$, where k and d are as above.
2. In the case $g = 23$ (extensively treated in Chapter 1), the theorem provides regular components of $\mathcal{M}_{23}(\mathbb{P}^1 \times \mathbb{P}^3, (k,20))$ when $k \geq 8$.

Proof: We set $g_0 = (r+1)(s-1)-1$ and $d_0 = rs-1$. One checks that $\rho(g_0, r, d_0) = -1$ and any solution (g,d) of the equation $\rho(g,r,d) = -1$ with $g \geq g_0$, can be obtained from (g_0, d_0) by applying several times the transformation $(g,d) \mapsto (g+r+1, d+r)$. According to Prop.2.4.1 it suffices to construct a smooth curve $C \subseteq \mathbb{P}^r$ of genus g_0 and degree d_0, with $h^1(C, N_C) = 0, h^0(C, \mathcal{O}_C(1)) = r+1$, having the Petri map $\mu_0(C)$ surjective, and also carrying a simple base point free pencil $\mathfrak{g}^1_k = |Z|$ such that $2\mathfrak{g}^1_k$ is non-special and $|\mathcal{O}_C(1)|(-\mathfrak{g}^1_k) = \emptyset$.

For such a triple $(C, \mathfrak{g}^1_k, \mathcal{O}_C(1))$, condition (2.9) also required in Prop.2.4.1 is immediately satisfied: if $f_1 : C \to \mathbb{P}^1$ is the map corresponding to $\mathfrak{g}^1_k$, we know (cf. Section 1.3) that $(d\pi_1)_{[f_1]}\big(T_{[f_1]}(\mathcal{M}_{g_0}(\mathbb{P}^1, k))\big) = H^0(C, K_C - 2Z)^{\perp} = H^1(C, T_C)$, because $|2Z|$ is non-special, so (2.9) follows at once.

It is more convenient to replace the projection $\mathcal{M}_{g_0}(\mathbb{P}^1, k) \to \mathcal{M}_{g_0}$ by the surjective proper map $\pi : \mathcal{G}^1_k \to \mathcal{M}_{g_0}$, given by $\pi(C, l) = [C]$, where $l \in G^1_k(C)$. Of course π does not exist quite as it stands, instead one should replace $\mathcal{M}_{g_0}$ by a finite cover over which the universal curve has a section, but we can safely ignore this minor nuisance. The map π is surjective (and with connected fibres) because $\rho(g_0, 1, k) \geq r-1$. Theorem 6.1 from [Se2] ensures the existence of an irreducible, smooth, open subset U of $\mathcal{M}_{g_0}(\mathbb{P}^r, d_0)$, of the expected dimension, such that all points of U correspond to embeddings of smooth curves $C \hookrightarrow \mathbb{P}^r$, with $h^1(C, N_C) = 0$, $h^0(C, \mathcal{O}_C(1)) = r+1$ and $\mu_0(C)$ surjective. Since $\mathcal{M}^r_{g_0, d_0}$ is irreducible (cf. Chapter 1), it follows that the natural projection $\pi_2 : U \to \mathcal{M}^r_{g_0, d_0}$ is dominant.

We now find a curve C having the properties listed above. For a start, we notice that it is enough to find one curve $[C_0] \in \mathcal{M}^{r_0}_{g_0, d_0}$ possessing a simple, complete, base point free $\mathfrak{g}^1_k$ such that $2\mathfrak{g}^1_k$ is non-special, because then, by semicontinuity we get the same properties for a general point of U. To find one particular such curve we proceed as follows: take C_0 a general $(r+1)$-gonal curve of genus g_0. These curves will have rather few moduli ($r+1 < [(g+3)/2]$) but we still have that $[C_0] \in \mathcal{M}^r_{g_0, d_0}$. Indeed, according to [CM] p. 348, we can construct a complete, birationally very ample $\mathfrak{g}^r_{d_0} = \mathfrak{g}^1_{r+1} + F$ on C_0, where F is an effective divisor on C_0 with $h^0(C_0, F) = 1$. Using Corollary 2.2.3 from [CKM] we find that C_0 also possesses a complete, simple, base-point-free $\mathfrak{g}^1_k$ which is not composed with the $\mathfrak{g}^1_{r+1}$ computing $\mathrm{gon}(C_0)$, and such that $2\mathfrak{g}^1_k$ is non-special. Since these are open conditions they will hold generically along a component of $G^1_k(C_0)$. Applying semicontinuity, for a general element $[C] \in \mathcal{M}^r_{g_0, d_0}$ (hence also for a general element $[C] \in U$) the variety $G^1_k(C)$ will contain a component A with general point $l \in A$ being simple, base point free and with $2l$ non-special.

We claim that there exists a pencil $l \in A$ having the properties listed above and moreover $|\mathcal{O}_C(1)(-l)| = \emptyset$. Suppose not. Then if we denote by $V^{r-1}_{d_0-k}(|\mathcal{O}_C(1)|)$ the variety of effective divisors of degree $d_0 - k$ on C imposing $\leq r-1$ conditions on $|\mathcal{O}_C(1)|$, we have that

$$\dim V^{r-1}_{d_0-k}(|\mathcal{O}_C(1)|) \geq \dim A \geq \rho(g_0, 1, k) \geq r-1,$$

the last inequality being the only point where we need the assumption $k \geq (rs+s-1)/2$. Therefore $C \subseteq \mathbb{P}^r$ has at least ∞^{r-1} (d_0-k)-secant $(r-2)$-planes, hence also at least ∞^{r-1} r-secant $(r-2)$-planes (because $d_0 - k \geq r$). This last statement clearly contradicts the Uniform Position Theorem (see [ACGH], p. 112). All in all, the general point $[C] \in U$ enjoys all properties required to make Prop.2.4.1 work. □

Remarks: 1. We could apply Prop.2.4.1 and get regular components of the moduli space $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$ for lower values of $\rho(g, r, d)$ (and not only when $\rho(g, r, d) = -1$), if we knew that the $(r+1)$-gonal locus $\mathcal{M}^1_{g,r+1}$ is contained in *every* component of $\mathcal{M}^r_{g,d}$ (or at least in a component of $\mathcal{M}^r_{g,d}$ with the expected number of moduli). No such result appears to be known at the moment (except in the case $\rho(g, r, d) = -1$).

2. Let us fix g, k such that $\rho(g, 1, k) \geq 0$. One knows (cf. [ACGH]) that if $l \in G^1_k(C)$ is a complete, base point free pencil, then $\dim T_l(G^1_k(C)) = \rho(g, 1, k) + h^1(C, 2l)$. Therefore if A is a component of $G^1_k(C)$ such that $\dim A = \rho(g, 1, k)$ and the general $l \in A$ is base point free such that $2l$ is special, then A is nonreduced. We ask the following question: what is the dimension of the locus

$$M := \{[C] \in \mathcal{M}_g : \text{ every component of } G^1_k(C) \text{ is nonreduced } \}?$$

A result of Coppens (cf. [Co4]) says that for a curve C, if the scheme $W^1_k(C)$ is reduced and of dimension $\rho(g, 1, k)$, then the scheme $W^1_{k+1}(C)$ is reduced too and of dimension $\rho(g, 1, k+1)$. Therefore it would make sense to determine $\dim(M)$ when $\rho(g, 1, k) \in \{0, 1\}$ (depending on the parity of g). We suspect that M depends on very few moduli. A suitable upper bound for $\dim(M)$ would rule out the possibility of a component of $\mathcal{M}^r_{g,d}$ being contained in M (we have the lower bound $3g-3+\rho(g, r, d)$ for all components of $\mathcal{M}^r_{g,d}$) and we could apply Prop.2.4.1 without having to resort to Corollary 2.2.3 from [CKM].

2.5 The gonality of space curves

2.5.1 Preliminaries

The gonality of a curve is perhaps the second most natural invariant of a curve: it gives an indication of how far from being rational a curve is, in a way different from what the genus does. For $g \geq 3$ we consider the stratification of $\mathcal{M}_g$ given by gonality:

$$\mathcal{M}^1_{g,2} \subseteq \mathcal{M}^1_{g,3} \subseteq \ldots \subseteq \mathcal{M}^1_{g,k} \subseteq \ldots \subseteq \mathcal{M}_g.$$

where $\mathcal{M}^1_{g,k} = \mathcal{M}_g$ for $k \geq [(g+3)/2]$. The number $[(g+3)/2]$ is thus the generic gonality for curves of genus g. We want to study the relative position of the Brill-Noether loci $\mathcal{M}^r_{g,d}$ (with $r \geq 3, \rho(g,r,d) < 0$) and the k-gonal loci $\mathcal{M}^1_{g,k}$ (where $k < (g+2)/2$). More precisely, we would like to know the gonality of a general point of $\mathcal{M}^r_{g,d}$. Since the geometry of the loci $\mathcal{M}^r_{g,d}$ is, as we already pointed out in Section 1.2, very messy (existence of many components, some unreduced and/or of unexpected dimension), we will content ourselves with computing gon(C) when $[C]$ is a general point of a 'genuine' component of $\mathcal{M}^r_{g,d}$ (i.e. a component which is generically smooth, with general point corresponding to a curve with a very ample $\mathfrak{g}^r_d$).

The same problem for $r = 2$ has already been solved by M. Coppens (cf. [Co3]):

Proposition 2.5.1 *Let $\nu : C \to \Gamma$ be the normalization of a general, irreducible plane curve of degree d with $\delta = g - \binom{d-1}{2}$ nodes. Assume that $0 < \delta < (d^2 - 7d + 18)/2$. Then* gon$(C) = d - 2$.

Remarks: 1. The result says that there are no $\mathfrak{g}^1_{d-3}$'s on C. On the other hand a $\mathfrak{g}^1_{d-2}$ is given by the lines through a node of Γ.
2. The condition $\delta < (d^2 - 7d + 18)/2$ is equivalent with $\rho(g,1,d-3) < 0$. This is the range in which the problem is non-trivial: if $\rho(g,1,d-3) \geq 0$, the Brill-Noether Theorem provides $\mathfrak{g}^1_{d-3}$'s on C.

For $r \geq 3$ we could expect a similar result. Let $C \subseteq \mathbb{P}^r$ be a suitably general smooth curve of genus g and degree d, with $\rho(g,r,d) < 0$. We can always assume that $d \leq g - 1$ (by duality $\mathfrak{g}^r_d \mapsto K_C - \mathfrak{g}^r_d$ we can always land in this range). One can expect that a $\mathfrak{g}^1_k$ computing gon(C) is of the form $\mathfrak{g}^r_d(-D) = \{E - D : E \in \mathfrak{g}^r_d, E \geq D\}$ for some effective divisor D on C. Since the expected dimension of the variety of e-secant $(r-2)$-plane divisors

$$V^{r-1}_e(\mathfrak{g}^r_d) := \{D \in C_e : \dim \mathfrak{g}^r_d(-D) \geq 1\}$$

is $2r - 2 - e$ (cf. [ACGH]), we may ask whether C has finitely many $(2r-2)$-secant $(r-2)$-planes (and no $(2r-1)$-secant $(r-2)$-planes at all). This is known to be true for curves with general moduli, that is, when $\rho(g,r,d) \geq 0$ (cf. [Hirsch]): for instance a smooth curve $C \subseteq \mathbb{P}^3$ with general moduli has only finitely many 4-secant lines and no 5-secant lines. However, no such principle appears to be known for curves with special moduli.

Definition: We call the number $\min(d - 2r + 2, [(g+3)/2])$ the *expected gonality* of a smooth nondegenerate curve $C \subseteq \mathbb{P}^r$ of degree d and genus g.

The main result of this section is the following:

Theorem *Let $g \geq 5$ and $d \geq 8$ be integers, g odd, d even, such that $d^2 > 8g$, $4d < 3g+12$, $d^2 - 8g + 8$ is not a square and either $d \leq 18$ or $g < 4d - 31$. If*

$$(d', g') \in \{(d,g), (d+1, g+1), (d+1, g+2), (d+2, g+3)\},$$

then there exists a regular component of Hilb$_{d',g',3}$ *whose general point $[C']$ is a smooth curve such that* gon$(C') = \min(d' - 4, [(g'+3)/2])$.

One can approach this problem from a different angle: find recipes to compute the gonality of various classes of curves $C \subseteq \mathbb{P}^r$. Our knowledge in this respect is very scant: we know how to compute the gonality of extremal curves $C \subseteq \mathbb{P}^r$ (that is, curves attaining the Castelnuovo bound, see [ACGH]) and the gonality of complete intersections in $\mathbb{P}^3$ (cf. [Ba]): If $C \subseteq \mathbb{P}^3$ is a smooth complete intersection of type (a, b) then $\mathrm{gon}(C) = ab - l$, where l is the degree of a maximal linear divisor on C. Hence an effective divisor $D \subseteq C$ computing $\mathrm{gon}(C)$ (that is $\deg(D) = \mathrm{gon}(C)$ and $h^0(C, D) \geq 2$), is residual to a linear divisor of degree l in a plane section of C. Of course, we know $\mathrm{gon}(C)$ in a few other cases: It is a classical result that the gonality of a smooth plane C curve of degree d is $d-1$ and every $\mathfrak{g}^1_{d-1}$ on C is of the form $|\mathcal{O}_C(1)(-p)|$, where $p \in C$. If C is a smooth curve of type (a, b) on a smooth quadric surface in $\mathbb{P}^3$, then $\mathrm{gon}(C) = \min(a, b)$, i.e. the gonality is computed by a ruling. One gets a similar result for a curve sitting on a Hirzebruch surface. Finally, in [Pa] there is a rather surprising lower bound for the gonality of a smooth curve $C \subseteq \mathbb{P}^r$ in terms of the Seshadri constant of C, which is an invariant measuring the positivity of $\mathcal{O}_{\mathbb{P}^r}(1)$ in a neighbourhood of C.

2.5.2 Linear systems on smooth quartic surfaces in $\mathbb{P}^3$

We recall a few basic facts about linear systems on $K3$ surfaces (cf. [SD]). Let S be a smooth $K3$ surface. For an effective divisor $D \subseteq S$, we have $h^1(S, D) = h^0(D, \mathcal{O}_D) - 1$. If $C \subseteq S$ is an irreducible curve then $H^1(S, C) = 0$, and by Riemann-Roch we have that

$$\dim|C| = 1 + \frac{C^2}{2} = p_a(C).$$

In particular $C^2 \geq -2$ for every irreducible curve C.

Proposition 2.5.2 *Let S be a $K3$ surface. We have the following equivalences:*

1. $C^2 = -2 \iff \dim |C| = 0 \iff C$ *is a smooth, rational curve.*
2. $C^2 = 0 \iff \dim |C| = 1 \iff p_a(C) = 1$.

For a $K3$ surface one also has a 'strong Bertini' Theorem:

Proposition 2.5.3 *Let $\mathcal{L}$ be a line bundle on a $K3$ surface S. Then $|\mathcal{L}|$ has no base points outside its fixed components. Moreover, if $\mathrm{bs}\,|\mathcal{L}| = \emptyset$ then either*

- $\mathcal{L}^2 > 0$, $h^1(S, \mathcal{L}) = 0$ *and the general member of $|\mathcal{L}|$ is a smooth, irreducible curve of genus $\mathcal{L}^2/2 + 1$, or*
- $\mathcal{L}^2 = 0$ *and $\mathcal{L} = \mathcal{O}_S(kE)$, where $k \in \mathbb{Z}_{\geq 1}$, $E \subseteq S$ is an irreducible curve with $p_a(E) = 1$. We have that $h^0(S, \mathcal{L}) = k + 1$, $h^1(S, \mathcal{L}) = k - 1$ and all divisors in $|\mathcal{L}|$ are of the form $E_1 + \cdots + E_k$ with $E_i \sim E$.*

We are interested in space curves sitting on $K3$ surfaces and the starting point is Mori's Theorem (cf. [Mo]): if $d > 0$, $g \geq 0$, there is a smooth curve $C \subseteq \mathbb{P}^3$ of degree d and genus g, lying on a smooth quartic surface S, if and only if (1) $g = d^2/8 + 1$, or (2) $g < d^2/8$ and $(d, g) \neq (5, 3)$. Moreover, we can choose S such that $\mathrm{Pic}(S) = \mathbb{Z}H = \mathbb{Z}(4/d)C$ in case (1) and such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, with $H^2 = 4, C^2 = 2g - 2$ and $H \cdot C = d$, in case (2). In each case H denotes a plane section of S. Note that from the Hodge Index Theorem one has the necessary condition

$$(C \cdot H)^2 - H^2C^2 = d^2 - 8(g-1) \geq 0.$$

We will repeatedly use the following observation:

Proposition 2.5.4 *Let $S \subseteq \mathbb{P}^3$ be a smooth quartic surface with a smooth curve $C \subseteq S$ such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$ and assume that S has no (-2) curves. For a divisor $D \subseteq S$ we have that:*

1. *D is effective $\iff D^2 \geq 0$ and $D \cdot H > 2$.*

2. *If $D^2 = 0$ and $D \cdot H > 2$, then $D = kE$, where E is an irreducible curve of genus 1 and $h^0(S, D) = k + 1$, $h^1(S, D) = k - 1$.*

3. *If $D^2 > 0$ and $D \cdot H > 2$, then the general element of $|D|$ is smooth and irreducible.*

Remarks: a) The first part of Proposition 2.5.4 is based on the fact that if $D \subseteq S$ is a curve with $\deg(D) = D \cdot H \leq 2$, then $h^0(S, D) = 1$, i.e. D is isolated. But every isolated curve is a (-2) curve and we have assumed that there are no such curves on S.
b) If $S \subseteq \mathbb{P}^3$ is a smooth quartic surface with Picard number 2 as above, S has no (-2) curves when the equation

$$2m^2 + mnd + (g-1)n^2 = -1, \tag{2.12}$$

has no solutions $m, n \in \mathbb{Z}$. This is the case for instance when d is even and g is odd.

2.5.3 Brill-Noether special linear series on curves on $K3$ surfaces

The study of special linear series on curves lying on $K3$ surfaces began with Lazarsfeld's proof of the Brill-Noether-Petri Theorem (cf. [La]). He noticed that there is no Brill-Noether type obstruction to embed a curve in a $K3$ surface: if $C_0 \subseteq S$ is a smooth curve of genus $g \geq 2$ on a $K3$ surface such that $\mathrm{Pic}(S) = \mathbb{Z}C_0$, then the general curve $C \in |C_0|$ satisfies the Brill-Noether-Petri Theorem, that is, for any line bundle A on C, the Petri map $\mu_0(C, A) : H^0(C, A) \otimes H^0(C, K_C \otimes A^\vee) \to H^0(C, K_C)$ is injective. We mention that Petri's Theorem implies (trivially) the Brill-Noether Theorem.

The general philosophy when studying linear series on a $K3$-section $C \subseteq S$ of genus $g \geq 2$, is that the type of a Brill-Noether special $\mathfrak{g}^r_d$ often does not depend on C but only on its linear equivalence class in S, i.e. a $\mathfrak{g}^r_d$ on C with $\rho(g, r, d) < 0$ is expected to propagate to all smooth curves $C' \in |C|$. This expectation, in such generality, is perhaps a bit too optimistic, but it was proved to be true for the Clifford index of a curve (see [GL]): for $C \subseteq S$ a smooth $K3$-section of genus $g \geq 2$, one has that $\mathrm{Cliff}(C') = \mathrm{Cliff}(C)$ for every smooth curve $C' \in |C|$. Furthermore, if $\mathrm{Cliff}(C) < [(g-1)/2]$ (the generic value of the Clifford index), then there exists a line bundle $\mathcal{L}$ on S such that for all smooth $C' \in |C|$ the restriction $\mathcal{L}_{C'}$ computes $\mathrm{Cliff}(C')$. Recall that the *Clifford index* of a curve C of genus g is defined as

$$\mathrm{Cliff}(C) := \min\{\mathrm{Cliff}(D) : D \in \mathrm{Div}(C), h^0(D) \geq 2, h^1(D) \geq 2\},$$

where for a divisor D on C, we have $\mathrm{Cliff}(D) = \deg(D) - 2(h^0(D) - 1)$. Note that in the definition of $\mathrm{Cliff}(C)$ the condition $h^1(D) \geq 2$ can be replaced with $\deg(D) \leq g-1$. Another invariant of a curve is the *Clifford dimension* of C defined as

$$\text{Cliff-dim}(C) := \min\{r \geq 1 : \exists \mathfrak{g}^r_d \text{ on } C \text{ with } d \leq g-1, \text{ such that } d - 2r = \mathrm{Cliff}(C)\}.$$

Curves with Clifford dimension ≥ 2 are rare: smooth plane curves are precisely the curves of Clifford dimension 2, while curves of Clifford dimension 3 occur only in genus 10 as complete intersections of two cubic surfaces in $\mathbb{P}^3$.

Harris and Mumford during their work in [HM] conjectured that the gonality of a $K3$-section should stay constant in a linear system: if $C \subseteq S$ carries an exceptional $\mathfrak{g}^1_d$ then every smooth $C' \in |C|$ carries an equally exceptional $\mathfrak{g}^1_d$. This conjecture was later disproved by Donagi and Morrison (cf. [DMo]) who showed that the gonality can vary in a linear system: Consider the following situation: let $\pi : S \to \mathbb{P}^2$ be a $K3$ surface, double cover of $\mathbb{P}^2$ branched along a smooth sextic and let $\mathcal{L} = \pi^*\mathcal{O}_{\mathbb{P}^2}(3)$. The genus of a smooth $C \in |\mathcal{L}|$ is 10. The general $C \in |\mathcal{L}|$ carries a very ample $\mathfrak{g}^2_6$, hence $\mathrm{gon}(C) = 5$. On the other hand, any curve in the codimension 1 linear system $|\pi^* H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))|$ is bielliptic, therefore has gonality 4. Under reasonable assumptions this turns out to be the only counterexample to the Harris-Mumford conjecture. Ciliberto and Pareschi proved (in [CilP]) that if $C \subseteq S$ is such that $|C|$ is base-point-free and ample, then either $\mathrm{gon}(C') = \mathrm{gon}(C)$ for all smooth $C' \in |C|$, or (S, C) are as in the previous counterexample. Although $\mathrm{gon}(C)$ can drop as C varies in a linear system, base point free $\mathfrak{g}^1_d$'s on $K3$-sections do propagate:

Proposition 2.5.5 (Donagi-Morrison) *Let S be a $K3$ surface, $C \subseteq S$ a smooth, non-hyperelliptic curve and $|Z|$ a complete, base point free $\mathfrak{g}^1_d$ on C such that $\rho(g, 1, d) < 0$. Then there is an effective divisor $D \subseteq S$ such that:*

- $h^0(S, D) \geq 2$, $h^0(S, C - D) \geq 2$, $\deg_C(D_C) \leq g - 1$.
- $\mathrm{Cliff}(C', D_{C'}) \leq \mathrm{Cliff}(C, Z)$, *for any smooth* $C' \in |C|$.
- *There is* $Z_0 \in |Z|$, *consisting of distinct points such that* $Z_0 \subseteq D \cap C$.

2.5.4 The gonality of curves on quartic surfaces

For a wide range of d and g we construct curves $C \subseteq \mathbb{P}^3$ of degree d and genus g having the expected gonality. We start with the case when g is odd and d is even when we can realize our curves as sections of smooth quartic surfaces.

Theorem 2.3 *Let $g \geq 5, d \geq 8$ be integers, g odd, d even, such that $d^2 > 8g$, $4d < 3g+12$ and $d^2 - 8g + 8$ is not a square. Then there exists a smooth curve $C \subseteq \mathbb{P}^3$ of degree d and genus g such that $\text{gon}(C) = \min(d-4, [(g+3)/2])$. If $\text{gon}(C) = d - 4 < [(g+3)/2]$, every $\mathfrak{g}^1_{d-4}$ computing the gonality is given by the planes through a 4-secant line to C. Moreover, C has only finitely many 4-secant lines, finitely many tangential trisecants and no 5-secant lines.*

Proof: By Mori's Theorem, for such d and g, there exists a smooth quartic surface $S \subseteq \mathbb{P}^3$ and $C \subseteq S$ a smooth curve of degree d and genus g such that $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, where H is a plane section. The conditions d and g are subject to, ensure that S does not contain (-2) curves or genus 1 curves (the existence of a curve with self-intersection 0 would imply that $d^2 - 8g + 8$ is a square).

We prove first that Cliff-dim$(C) = 1$. It suffices to show that $C \subseteq S$ is an ample divisor, because then by using Prop.3.3 from [CilP] we obtain that either Cliff-dim$(C) = 1$ or C is a smooth plane sextic, $g = 10$ and (S, C) are as in Donagi-Morrison's example (then Cliff-dim$(C) = 2$). The latter case obviously does not happen.

We prove that $C \cdot D > 0$ for any effective divisor $D \subseteq S$. Let $D \sim mH + nC$, with $m, n \in \mathbb{Z}$, such a divisor. Then $D^2 = 4m^2 + 2mnd + n^2(2g-2) \geq 0$ and $D \cdot H = 4m + dn > 2$. The case $m \leq 0, n \leq 0$ is impossible, while the case $m \geq 0, n \geq 0$ is trivial. Let us assume $m > 0, n < 0$. Then $D \cdot C = md + n(2g-2) > -2n(d^2/8 - g + 1) + d/2 > 0$, because $d^2/8 > g$. In the remaining case $m < 0, n > 0$ we have that $nD \cdot C \geq -mD \cdot H > 0$, so C is ample by Nakai-Moishezon.

Our assumptions imply that $d \leq g - 1$, so $\mathcal{O}_C(1)$ is among the line bundles from which Cliff(C) is computed. We get thus the following estimate on the gonality of C:

$$\text{gon}(C) = \text{Cliff}(C) + 2 \leq \text{Cliff}(C, H_C) + 2 = d - 4,$$

which yields $\text{gon}(C) \leq \min(d-4, [(g+3)/2])$.

Assume now that $\text{gon}(C) < [(g+3)/2]$. We will then show that $\text{gon}(C) = d - 4$. Let $|Z|$ be a complete, base point free pencil computing gon(C). By applying Prop.2.5.5, there exists an effective divisor $D \subseteq S$ satisfying

$$h^0(S, D) \geq 2, h^0(S, C-D) \geq 2, \deg(D_C) \leq g-1, \ \text{gon}(C) = \text{Cliff}(D_C) + 2 \text{ and } Z \subseteq D \cap C.$$

We consider the exact cohomology sequence:

$$0 \to H^0(S, D - C) \to H^0(S, D) \to H^0(C, D_C) \to H^1(S, D - C).$$

Since $C - D$ is effective and $\nsim 0$, one sees that $D - C$ cannot be effective, so $H^0(S, D - C) = 0$. The surface S does not contain (-2) curves, so $|C - D|$ has no fixed components; the equation $(C-D)^2=0$ has no solutions, therefore $(C-D)^2>0$ and the general element of $C-D$ is smooth and irreducible. Then it follows that $H^1(S, D-C) = H^1(S, C-D)^\vee = 0$. Thus $H^0(S, D) = H^0(C, D_C)$ and

$$\mathrm{gon}(C) = 2 + \mathrm{Cliff}(D_C) = 2 + D\cdot C - 2\dim D \; = D\cdot C - D^2.$$

We consider the following family of effective divisors

$$\mathcal{A} := \{D \in \mathrm{Div}(S) : h^0(S, D) \geq 2, h^0(S, C-D) \geq 2, \; C\cdot D \leq g-1\},$$

and since we already know that $d-4 \geq \mathrm{gon}(C) \geq \alpha$, where $\alpha = \min\{D\cdot C - C^2 : D\in\mathcal{A}\}$, we are done if we prove that $\alpha \geq d-4$. Take $D\in\mathcal{A}$, such that $D\sim mH+nC$, $m,n\in\mathbb{Z}$. The conditions $D^2>0, D\cdot C\leq g-1$ and $2<D\cdot H<d-2$ (use Prop.2.5.4 for the last inequality) can be rewritten as

$$2m^2+mnd+n^2(g-1)>0 \text{ (i)}, \; 2<4m+nd<d-2 \text{ (ii)}, \; md+(2n-1)(g-1)\leq 0 \text{ (iii)}.$$

We have to prove that for any $D\in\mathcal{A}$ the following inequality holds

$$f(m,n) = D\cdot C - D^2 = -4m^2+m(d-2nd)+(n-n^2)(2g-2) \geq f(1,0) = d-4.$$

We solve this standard calculus problem. Denote by $a := (d+\sqrt{d^2-8g+8}\;)/4$ and $b := (d-\sqrt{d^2-8g+8}\;)/4$. We dispose first of the case $n<0$. Assuming $n<0$, from (i) we have that either $m<-bn$ or $m>-an$. If $m<-bn$ from (ii) we obtain that $2<n(d-4b)<0$, because $n<0$ and $d-4b=\sqrt{d^2-8g+8}>0$, so we have reached a contradiction.

We assume now that $n<0$ and $m>-an$. From (iii) we get that $m\leq (g-1)(1-2n)/d$. If $-an>(g-1)(1-2n)/d$ we are done because there is no $m\in\mathbb{Z}$ satisfying (i), (ii) and (iii), while in the other case for any $D\in\mathcal{A}$ with $D\sim mH+nC$, one has the inequalities

$$f(m,n) > f(-an,n) = (2g-2-ad)n = \frac{(d^2-8g+8)+d\sqrt{d^2-8g+8}}{4}(-n) > d-4.$$

unless $n=-1$ and $d^2-8g<8$ (which forces $d^2-8g=4$). In this last case we obtain $m\geq (d+4)/4$ so $f(m,-1)\geq f((d+4)/4,-1)>d-4$.

The case $n>0$ can be treated rather similarly. From (i) we get that either $m<-an$ or $m>-bn$. The first case can be dismissed immediately. When $m>-bn$ we use that for any $D\in\mathcal{A}$ with $D\sim mH+nC$,

$$f(m,n) \geq \min\{f(-(g-1)(2n-1)/d, n), \max\{f(-bn,n), f((2-nd)/4, n)\}\}.$$

Elementary manipulations give that

$$f(-(g-1)(2n-1)/d, n) = (g-1)/2\,[(2n-1)^2(d^2-8g+8)/d^2+1] \geq d-4$$

(use that $d^2>8g$ and $d\leq g-1$). Note that we have equality if and only if $n=1, m=-1$ and $d=g-1$. This possibility is compatible with the other conditions only for $g\in$

$\{11, 13, 15\}$.

Furthermore $f(-bn, n) = n(2g - 2 - bd) \geq 2g - 2 - bd$ and $2g - 2 - bd \geq d - 4 \Leftrightarrow 4 \leq \sqrt{d^2 - 8g + 8} \leq d - 4$. When this does not happen we proceed as follows: if $\sqrt{d^2 - 8g + 8} > d - 4$ then if $n = 1$ we have that $m > -b > -1$, that is $m \geq 0$, but this contradicts (ii). When $n \geq 2$, we have $f((2 - nd)/4, n) = [(d^2 - 8g + 8)(n^2 - n) + (2d - 4)]/4 \geq d - 4$. Finally, the remaining possibility $4 > \sqrt{d^2 - 8g + 8}$ can be disposed of easily by an ad-hoc argument (our assumptions force in this case $d^2 - 8g = 4$).

All this leaves us with the case $n = 0$, when $f(m, 0) = -4m^2 + md$. Clearly $f(m, 0) \geq f(1, 0)$ for all m complying with (i),(ii) and (iii).

Thus we proved that $\text{gon}(C) = d - 4$. We have equality $D \cdot C - D^2 = d - 4$ where $D \in \mathcal{A}$, if and only if $D = H$ or in the case $d = g - 1$, $g \in \{11, 13, 15\}$ also when $D = C - H$. It is easy to show that if $d = g - 1$ then $K_C = 2H_C$, therefore we can always assume that the divisor on S cutting a $\mathfrak{g}^1_{d-4}$ on C is the plane section of S. Since $Z \subseteq H \cap C$, if we denote by Δ the residual divisor of Z in $H \cap C$, we have that $h^0(C, H_C - \Delta) = 2$, so Δ spans a line and $|Z|$ is given by the planes through the 4-secant line $\langle \Delta \rangle$. This shows that every pencil computing $\text{gon}(C)$ is given by the planes through a 4-secant line.

There are a few ways to see that C has only finitely many 4-secant lines. The shortest is to invoke Theorem 3.1 from [CilP]: since $\text{gon}(C') = d - 4$ is constant as C' varies in $|C|$, for the general smooth curve $C' \in |C|$ one has $\dim W^1_{d-4}(C') = 0$. Thus C has only finitely many 4-secant lines and no 5-secant lines. Note that the last part of this assertion can also be seen directly using Bezout's Theorem: if $L \subseteq \mathbb{P}^3$ were a 5-secant line to C, then $L \subseteq S$, but S contains no lines. Finally, C has only finitely many tangential trisecants because C is nondegenerate and we can apply a result from [Kaji]. □

Remarks: 1. One can find quartic surfaces $S \subseteq \mathbb{P}^3$ containing a smooth curve C of degree d and genus g in the case $g = d^2/8 + 1$ (which is outside the range Theorem 2.3 deals with). Then $d = 4m, g = 2m^2 + 1$ with $m \geq 1$ and C is a complete intersection of type $(4, m)$. For such a curve, $\text{gon}(C) = d - l$, where l is the degree of a maximal linear divisor on C (cf. [Ba]). If S is picked sufficiently general so that it contains no lines, by Bezout, C cannot have 5-secant lines so $\text{gon}(C) = d - 4$ in this case too.

2. Mori's Theorem can be extended to curves sitting on $K3$ surfaces which are embedded in higher dimensional projective spaces: for $r \geq 3, d > 0, g \geq 0$ such that $g < d^2/(4r - 4)$ and $(d, g) \neq (2r - 1, r)$, there exists a $K3$ surface $S \subseteq \mathbb{P}^r$ of degree $2r - 2$ containing a smooth curve C of degree d and genus g and such that $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, where H is a hyperplane section of S (see [Kn]). It seems very likely (although I have only checked several particular cases) that under the same conditions (i.e. S contains no genus 0 or genus 1 curves) the analogue of Theorem 2.3 still holds, that is $\text{gon}(C) = \min([(g + 3)/2], d - 2r + 2)$.

We want to find out when the curves constructed in Theorem 2.3 correspond to 'good points' of $\text{Hilb}_{d,g,3}$. We have the following:

Proposition 2.5.6 *Let $C \subseteq S \subseteq \mathbb{P}^3$ be a smooth curve sitting on a quartic surface such that $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$ with H being a plane section and assume furthermore that S contains no (-2) curves. Then $H^1(C, N_{C/\mathbb{P}^3}) = 0$ if and only if $d \leq 18$ or $g < 4d - 31$.*

Proof: We use the exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow N_{S/\mathbb{P}^3} \otimes \mathcal{O}_C \longrightarrow 0. \tag{2.13}$$

where $N_{S/\mathbb{P}^3} \otimes \mathcal{O}_C = \mathcal{O}_C(4)$ and $N_{C/S} = K_C$. We claim that there is an isomorphism $H^1(C, N_{C/\mathbb{P}^3}) = H^1(C, \mathcal{O}_C(4))$. Suppose this is not the case. Then the injective map $H^1(C, K_C) \to H^1(C, N_{C/\mathbb{P}^3})$ provides a splitting of the sequence (2.13) and by using Proposition 3.25 from [Mod] we obtain that C is a complete intersection with S. This is clearly a contradiction.

We have isomorphisms $H^1(C, 4H_C) = H^2(S, 4H - C) = H^0(S, C - 4H)^\vee$. According to Prop.2.5.4 the divisor $C - 4H$ is effective if and only if $(C-4H)^2 \geq 0$ and $(C-4H)\cdot H > 2$, from which the conclusion follows. ❑

We need to determine the gonality of nodal curves not of compact type and which consist of two components (like those appearing in Prop.2.2.2). The following result is intuitively clear if one uses admissible coverings:

Proposition 2.5.7 *Let $C = C_1 \cup_\Delta C_2$ be a quasi-transversal union of two smooth curves C_1 and C_2 meeting at a finite set Δ. Denote by $g_1 = g(C_1), g_2 = g(C_2), \delta = \mathrm{card}(\Delta)$. Let us assume that C_1 has only finitely many pencils $\mathfrak{g}^1_d$, where $\delta \leq d$ and that the points of Δ do not occur in the same fibre of one of these pencils. Then $\mathrm{gon}(C) \geq d+1$. Moreover if $\mathrm{gon}(C) = d+1$ then either (1) C_2 is rational and there is a degree d map $f_1 : C_1 \to \mathbb{P}^1$ and a degree 1 map $f_2 : C_2 \to \mathbb{P}^1$ such that $f_1|_\Delta = f_2|_\Delta$, or (2) there is a $\mathfrak{g}^1_{d+1}$ on C_1 containing Δ in a fibre.*

Proof: For the proof we use Section 2 of [EH1] (the one which works for nodal curves not necessarily of compact type). We briefly reviewed this in Chapter 1 (see also [Est] for a clear account on limit linear series on (general) reducible nodal curves). Let us assume that C is k-gonal, that is, a limit of smooth k-gonal curves. Then there exists a family of curves $\pi : \mathcal{C} \to B$, with $B = \mathrm{Spec}(R)$, R being a discrete valuation ring, such that the central fibre C_0 is C, the generic fibre C_η is smooth ($\eta \in B$ is the generic point) and there is a $\mathfrak{g}^1_k$ on C_η, which as in Chapter 1 we denote by $l_\eta = (\mathcal{L}_\eta, V_\eta)$, where $V_\eta \subseteq \pi_*\mathcal{L}_\eta$ is a vector bundle of rank 2. To the family of pencils l_η we can associate a limit linear series on C as follows (cf. [Est]): there are unique line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathcal{C}$ such that:

1. $\mathcal{L}_1$ and $\mathcal{L}_2$ are extensions of $\mathcal{L}_\eta$: $\mathcal{L}_i|_{C_\eta} = \mathcal{L}_\eta$, for $i = 1, 2$.

2. If $V_{\mathcal{L}_i} := V_\eta \cap \pi_*\mathcal{L}_i \subseteq \pi_*\mathcal{L}_i$, then the map $V_{\mathcal{L}_1}(0) \to H^0(C_1, \mathcal{L}_1(0)|_{C_1})$ is injective and the map $V_{\mathcal{L}_1}(0) \to H^0(C_2, \mathcal{L}_1(0)|_{C_2})$ is $\neq 0$. Similarly, $V_{\mathcal{L}_2}(0) \to H^0(C_2, \mathcal{L}_2(0)|_{C_2})$ is injective and $V_{\mathcal{L}_2}(0) \to H^0(C_1, \mathcal{L}_2(0)|_{C_1})$ is $\neq 0$.

Note that in the case of curves of compact type, it was possible to get for each component of the special fibre one extension of $\mathcal{L}_\eta$ whose restriction had degree k on the chosen component and degree 0 on all the other components of the special fibre; obviously we cannot expect something like this for arbitrary nodal curves. We also point out that even in the case when C is of compact type the extensions $\mathcal{L}_1$ and $\mathcal{L}_2$ may differ from Eisenbud-Harris' extensions: this happens when there is some ramification at the nodes of C.

Let us denote by $l \geq 0$ the unique integer such that $\mathcal{L}_1 = \mathcal{L}_2(lC_2)$ and by $d_1 = \deg_{C_1}(\mathcal{L}_1 \ _{C_1}), d_2 = \deg_{C_2}(\mathcal{L}_2 \ _{C_2})$. Then $d_1 + d_2 = k + l\delta$.

We show first that $k \geq d+1$. Suppose $k = d$. Then $d_1 = d, d_2 = l\delta \geq 1$ (because $h^0(C_2, \mathcal{L}_2(0)_{C_2}) \geq 2$), and then $(\mathcal{L}_1(0)_{C_1}, V_{\mathcal{L}_1}(0))$ is one of the finitely many $\mathfrak{g}^1_d$'s on C_1. From 2. we have that $1 \leq h^0(C_1, \mathcal{L}_2(0)_{C_1}) \leq h^0(C_1, \mathcal{L}_1(0)_{C_1} - \Delta)$, that is, Δ is contained in a fibre of a $\mathfrak{g}^1_d$ on C_1, a contradiction.

Assume now $k = d+1$. There are two cases to consider: (i). $d_1 = d, d_2 = l\delta + 1$ which forces $l = 0$ (if $l \geq 1$ once again Δ would be entirely contained in a fibre of a $\mathfrak{g}^1_d$ on C_1), hence $\mathcal{L}_2 = \mathcal{L}_1$, so we have only one line bundle on C which gives a degree $d+1$ map $C_1 \cup_\Delta C_2 \to \mathbb{P}^1$, which is case (1) of Prop.2.5.7 (ii). $d_1 = d+1, d_2 = l\delta$. Again, the condition $h^0(C_2, \mathcal{L}_2(0)_{C_2}) \geq 2$ gives $l \geq 1$, hence $1 \leq h^0(C_1, \mathcal{L}_2(0)_{C_1}) \leq h^0(C_1, \mathcal{L}_1(0)_{C_1} - \Delta)$, which yields case (2) of Prop.2.5.7. ☐

Theorem 2.3 provides space curves of expected gonality when d is even and g is odd. Naturally, we would like to have such curves when d and g have other parities as well. We will achieve this by attaching to a 'good' curve of expected gonality, either a 2 or 3-secant line or a 4-secant conic.

Theorem 2.4 *Let $g \geq 5, d \geq 8$ be integers with g odd and d even, such that $d^2 > 8g, 4d < 3g+12$, $d^2 - 8g + 8$ is not a square and either $d \leq 18$ or $g < 4d - 31$. If*

$$(d', g') \in \{(d, g), (d+1, g+1), (d+1, g+2), (d+2, g+3)\},$$

then there exists a regular component of $\mathrm{Hilb}_{d', g', 3}$ with general point $[C']$ a smooth curve such that $\mathrm{gon}(C') = \min(\mathrm{d}' - 4, [(\mathrm{g}' + 3)/2])$.

Proof: For d and g as in the statement we know by Theorem 2.3 and Prop.2.5.6 that there exists a smooth, nondegenerate curve $C \subseteq \mathbb{P}^3$ of degree d and genus g, with $\mathrm{gon}(C) = \min(d-4, [(g+3)/2])$ and $H^1(C, N_{C/\mathbb{P}^3}) = 0$. We can also assume that C sits on a smooth quartic surface S and $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$. Moreover, in the case $d - 4 < [(g+3)/2]$ the curve C has only finitely many $\mathfrak{g}^1_{d-4}$'s, all given by planes through a 4-secant line.

i) Let us settle first the case $(d', g') = (d+1, g+1)$. Take $p, q \in C$ general points, $L = \overline{pq} \subseteq \mathbb{P}^3$ and $X := C \cup L$. By Prop.2.2.2 X is smoothable and $H^1(X, N_X) = 0$. If $d - 4 < [(g+3)/2]$, then since C has only finitely many $\mathfrak{g}^1_{d-4}$'s, by applying Prop. 2.5.7 we get that $\mathrm{gon}(X) = d - 3$. In the case $d - 4 \geq [(g+3)/2]$ we just notice that $\mathrm{gon}(X) \geq \mathrm{gon}(C) = [(g'+3)/2]$.

ii) Next, we tackle the case $(d', g') = (d+1, g+2)$. Assume first that $d - 4 < [(g+3)/2] \Leftrightarrow d' - 4 < [(g'+3)/2]$. Let $p \in C$ be a general point. The image of the projection $\pi_p : C \to \mathbb{P}^2$ from p is a plane curve of degree $d-1$ having only nodes as singularities, which means that C has no *stationary trisecants* through p (i.e. trisecants $\overline{pqq'}$ such that $T_q(C)$ and $T_{q'}(C)$ meet), because a stationary trisecant would correspond to a tacnode of $\pi_p(C)$.

Pick L one of the $\binom{d-2}{2}-g$ trisecants through p and consider $X := C \cup L$. The conditions required by Prop.2.2.2 (part 3) being satisfied, X is smoothable and $H^1(X, N_X) = 0$. To conclude that $\text{gon}(X) = d-3$, we have to show that there is no $\mathfrak{g}^1_{d-4}$ on C containing $L \cap C$ in a fibre. A line in $\mathbb{P}^3$ (hence also a 4-secant line to C) can meet only finitely many trisecants. Indeed, assuming that $m \subseteq \mathbb{P}^3$ is a line meeting infinitely many trisecants, by considering the correspondence

$$T = \{(p,t) \in C \times m : \exists l \text{ a trisecant to } C \text{ passing through } p \text{ and } t\},$$

the projection $\pi_2 : T \to m$ yields a $\mathfrak{g}^1_3$ on T, hence C is trigonal as well, a contradiction. Since p and L have been chosen generally we may assume that L does not meet any of the 4-secant lines.

In the remaining case $d-4 \geq [(g+3)/2]$ we apply Theorem 2.3 to obtain a smooth curve $C_1 \subseteq \mathbb{P}^3$ of degree d and genus $g+2$ such that $\text{gon}(C_1) = (g+5)/2$ and $H^1(C_1, N_{C_1}) = 0$. We take $X_1 := C_1 \cup L_1$ with L_1 being a general 1-secant line to C_1. Then X_1 is smoothable and $\text{gon}(X_1) = \text{gon}(C_1) = (g+5)/2$.

iii) Finally, we turn to the case $(d', g') = (d+2, g+3)$. Take $H \subseteq \mathbb{P}^3$ a general plane meeting C in d distinct points in general linear position and pick 4 of them: $p_1, p_2, p_3, p_4 \in C \cap H$. Choose $Q \subseteq H$ a general conic such that $Q \cap C = \{p_1, p_2, p_3, p_4\}$. Prop.2.2.2 ensures that $X := C \cup Q$ is smoothable and $H^1(X, N_X) = 0$.

Assume first that $d'-4 \leq [(g'+3)/2]$. We claim that $\text{gon}(X) \geq \text{gon}(C)+2$. According to Prop. 2.5.7 the opposite could happen only in 2 cases: a) there exists a $\mathfrak{g}^1_{d-3}$ on C, say $|Z|$, such that $|Z|(-p_1-p_2-p_3-p_4) \neq \emptyset$. b) there exists a degree $d-4$ map $f : C \to \mathbb{P}^1$ and a degree 1 map $f' : Q \to \mathbb{P}^1$ such that $f(p_i) = f'(p_i)$, for $i = 1, \ldots, 4$.

Assume that a) does happen. We denote by $U = \{D \in C_4 : |\mathcal{O}_C(1)|(-D) \neq \emptyset\}$ the irreducible 3-fold of divisors of degree 4 spanning a plane and also consider the correspondence

$$\Sigma = \{(l, D) \in G^1_{d-3}(C) \times U : l(-D) \neq \emptyset\},$$

with the projection $\pi_1 : \Sigma \to G^1_{d-3}(C)$. We have that $\dim \Sigma \geq 3$. There are two possibilities: a_1) There is $l \in \pi_1(\Sigma)$ such that $|\mathcal{O}_C(1)|(-l) = \emptyset$. Then $\pi_1^{-1}(l)$ is finite hence $\dim G^1_{d-3}(C) \geq 3$. By using the theory of excess linear series (cf. [ACGH]) we get that $\dim G^1_{d-4}(C) \geq 1$, a contradiction. a_2) For all $l \in \pi_1(\Sigma)$ we have that $|\mathcal{O}_C(1)|(-l) \neq 0$ and then C has ∞^2 trisecants. But a nondegenerate curve in $\mathbb{P}^3$ can have at most ∞^1 trisecants, so by picking the plane section $H \cap C$ generally we can assume that a_2) does not happen either.

We now rule out case b). Suppose that b) does happen and denote by $L \subseteq \mathbb{P}^3$ the 4-secant line corresponding to f. Let $\{p\} = L \cap H$, and pick $l \subseteq H$ a general line. As Q was a general conic through $p_1, \ldots, p_4$ we may assume that $p \notin Q$. The map $f' : Q \to l$ is (up to a projective isomorphism of l) the projection from a point $q \in Q$, while $f(p_i) = \overline{p_i p} \cap l$, for $i = 1, \ldots, 4$. By Steiner's Theorem from classical projective geometry, the condition $(f(p_1)f(p_2)f(p_3)f(p_4)) = (f'(p_1)f'(p_2)f'(p_3)f'(p_4))$ is equivalent with p_1, p_2, p_3, p_4, p and q being on a conic, a contradiction since $p \notin Q$.

Finally, when $d'-4 > [(g'+3)/2]$, we have to show that $\text{gon}(X) \geq \text{gon}(C)+1$. We note that $\dim G^1_{(g+3)/2}(C) = 1$ (for any curve one has the inequality $\dim G^1_{\text{gon}} \leq 1$). By taking $H \in (\mathbb{P}^3)^\vee$ general enough, we obtain that $p_1, \ldots, p_4$ do not occur in the same fibre of a $\mathfrak{g}^1_{(g+3)/2}$. □

As an application of all these results we give a totally different proof of the most difficult part of Chapter 1, namely Prop. 1.5.4:

Theorem 2.5 *The Kodaira dimension of $\mathcal{M}_{23}$ is ≥ 2.*

Proof: We apply Theorem 2.4 when $(d, g) = (18, 23)$. There exists a curve $C \subseteq \mathbb{P}^3$ of degree 18 and genus 23 such that $\text{gon}(C) = 13$ (generic). Hence $[C] \in \mathcal{M}^2_{23,17} \cap \mathcal{M}^3_{23,20}$ but $[C] \notin \mathcal{M}^1_{23,12}$, which basically proves *3*) of Chapter 1. □

2.6 Miscellany

In this section we gather several facts about the relative position of certain loci in $\mathcal{M}_g$. The Brill-Noether Theorem asserts that the general curve of genus g has no linear series with negative ρ. However, it is notoriously difficult to find smooth Brill-Noether general curves. We discuss whether various geometrically defined subvarieties of $\mathcal{M}_g$ (e.g. loci of curves which lie on certain surfaces, or admit irrational involutions) might possess Brill-Noether general curves.

Let us look first what kind of surfaces can a Brill-Noether general curve lie on. Lazarsfeld proved that a general $K3$ surface (with Picard number 1) contains Brill-Noether-Petri general curves. It seems pretty hard to obtain such results for other classes of surfaces. We have the following observation:

Proposition 2.6.1 *1. A general surface $S \subseteq \mathbb{P}^3$ of degree $d \geq 5$ does not contain non-degenerate Brill-Noether general curves.*
2. A smooth curve of genus $g \geq 29$ with generic gonality $[(g+3)/2]$ cannot lie on an Enriques surface.

Proof: By the Noether-Lefschetz Theorem, if S is a general surface of degree $d \geq 5$, $\text{Pic}(S) = \mathbb{Z}H$, with H being a plane section. Hence any curve $C \subseteq S$ is a complete intersection. For $C \sim mH$ with $m \geq 2$, we have that $2g(C) - 2 = md(m + d - 4)$, from which clearly $\rho(g, 3, md) < 0$, so C is not Brill-Noether general.

Suppose now that $C \subseteq S$ is a smooth curve of genus g, sitting on an Enriques surface. There exists $|2E|$ an elliptic pencil on S such that $C \cdot E \leq \sqrt{2g-2}$ (cf. [CD] Corollary 2.7.1). In the exact sequence

$$0 \longrightarrow H^0(S, 2E - C) \longrightarrow H^0(S, 2E) \longrightarrow H^0(C, 2E_C)$$

we have that $h^0(S, 2E) \geq 2$ and $H^0(S, 2E - C) = 0$ (because $(2E - C) \cdot E < 0$). Therefore C carries a $\mathfrak{g}^1_{2E \cdot C}$. Since for $g \geq 29$ we have that $2\sqrt{2g-2} \leq (g+1)/2$, the curve C does not have generic gonality. □

We would like to know whether curves having an irrational involution can be Brill-Noether general. We will restrict ourselves to double covers, although these considerations can be carried out for coverings of arbitrary degree. We have the following results:

Proposition 2.6.2 *1. For $g \geq 1$, the general point of the locus $\{[C] \in \mathcal{M}_{2g+1} : \exists \sigma : C \to X, \deg(\sigma) = 2, g(X) = g\}$ is Brill-Noether general.*
2. For odd $g \geq 1$, the general point of the locus $\{[C] \in \mathcal{M}_{2g-1} : \exists \sigma : C \to X$, étale, with $\deg(\sigma) = 2, g(X) = g\}$ is of generic gonality $g+1$.

Proof: 1. Using limit linear series we find a Brill-Noether general curve C of compact type and genus $2g$, having a map of degree 2 onto a curve X of compact type and genus g. Take (A, p), a general pointed curve of genus g, and R a smooth rational curve. Consider $X := A \cup_p R_1$, which is of genus g. Let (C_1, p_1) and (C_2, p_2) be two copies of (A, p) and (E, x, y) a 2-pointed elliptic curve such that $x - y \in \text{Pic}^0(E)$ is not torsion. We construct a curve of compact type of genus $2g+1$, by taking $C := C_1 \cup_{p_1 \sim x} E \cup_{y \sim p_2} C_2$. It is straightforward to construct a degree 2 map $\sigma : C \to X$: take $\sigma(C_i, p_i) = (A, p)$ and $\sigma_E : E \to R$, the double covering given by the linear system $|x+y|$ on E. This shows that C is a limit of smooth curves of genus $2g+1$ having a double cover with 4 branch points. The claim that C is Brill-Noether general (i.e. it does not admit any limit linear series with negative Brill-Noether number) is a byproduct of Propositions 1.3.2 and 1.4.1.

2. The idea is the same, to construct an unramified double cover $\sigma : C \to X$, with X and C of compact type, $g(C) = 2g-1, g(X) = g$ and C having no $\mathfrak{g}^1_g$'s. This time we take $X := A \cup_p E$, with (A, p) a general curve of genus $g-1$ and E an elliptic curve, and $C := C_1 \cup_{p_1} E' \cup_{p_2} C_2$, where (C_i, p_i) are just copies of (A, p) , E' is a copy of E and $p_1 - p_2 \in_2 \text{Pic}^0(E')$. We obtain an étale double cover $\sigma : C \to X$, by mapping C_1 and C_2 to A , E' to E, such that $\sigma(p_1) = \sigma(p_2) = p$. The proof that C has no limit $\mathfrak{g}^1_g$'s is similar to a few other such proofs in this thesis, so we omit it. Note that this construction in the unramified case can also be found in [Ber]. □

In the previous proposition, the restriction to odd g in the unramified case seems to be rather artificial. Although we believe that for a sufficiently large even g, there are Brill-Noether general curves of genus $2g-1$ mapping 2:1 to a curve of genus g, for $g = 6$ we have the rather surprising result which we regard as a one-off:

Proposition 2.6.3 *If $\sigma : \tilde{C} \to C$ is an étale double cover with $g(\tilde{C}) = 11, g(C) = 6$, then $\tilde{C}$ is 6-gonal (whereas the generic gonality on $\mathcal{M}_{11}$ is 7).*

Proof: Let us consider the moduli space $\mathcal{R}_6$ of pairs (C, η), where C is a smooth curve of genus 6 and $\eta \in_2 \text{Pic}^0(C)$. We denote by $\phi : \mathcal{R}_6 \to \mathcal{M}_{11}$ the map given by $\phi(C, \eta) := [\tilde{C}]$, where $\sigma : \tilde{C} \to C$ is the étale double cover corresponding to η, i.e. $\sigma_* \mathcal{O}_{\tilde{C}} = \mathcal{O}_C \oplus \eta$.

The key observation is the following result of Verra (cf. [Ve]): for a general point $(C, \eta) \in \mathcal{R}_6$, there exists an Enriques surface S such that $C \subseteq S, C^2 = 10$ (hence $\dim |C| = 5$), C is very ample and $\eta = K_{S|C}$. Moreover, if $(C, \eta) \in \mathcal{R}_6$ is general, the Enriques surface S can be chosen generally too in the 10-dimensional moduli space of Enriques surfaces. By general theory (cf. [CD]) S contains 10 curves of genus 1, $E_1, \ldots, E_{10}$, such that $E_i \cdot E_j = 1 - \delta_{ij}$ and $C \sim 1/3(E_1 + \cdots + E_{10})$. Let $\pi : X \to S$ be the $K3$ cover of S, $\tilde{C} = \pi^{-1}(C)$ and $F_i = \pi^{-1}(E_i)$, with $3\tilde{C} \sim F_1 + \cdots + F_{10}$. For some $1 \leq i \leq 10$, consider the exact sequence

$$0 \longrightarrow H^0(X, F_i - \tilde{C}) \longrightarrow H^0(X, F_i) \longrightarrow H^0(\tilde{C}, F_{i\ \tilde{C}}).$$

Certainly $H^0(X, F_i - \tilde{C}) = 0$ and although E_i might be isolated on S, when we pass to the $K3$ surface X, we get that $h^0(X, F_i) = 2$, so F_i gives a pencil on $\tilde{C}$ of degree $6 (= F_i \cdot \tilde{C} = 2E_i \cdot C)$. This shows that $\phi(\mathcal{R}_6) \subseteq \mathcal{M}^1_{11,6}$. □

Remark: In a similar way, we can show that any smooth curve lying on the $K3$-cover of a general Enriques surface cannot have maximal gonality, so it is Brill-Noether special.

Chapter 3

Divisors on moduli spaces of pointed curves

3.1 Introduction

For integers $g \geq 3$ and $n \geq 1$ we denote by $\mathcal{M}_{g,n}$ the moduli space of complex n-pointed curves of genus g and by $\overline{\mathcal{M}}_{g,n}$ its compactification, the moduli space of n-pointed stable curves. When $n = 1$ we will sometimes use the notation $\overline{\mathcal{C}}_g = \overline{\mathcal{M}}_{g,1}$ for the universal (stable) curve of genus g.

The loci $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}_g$ consisting of curves having a $\mathfrak{g}^r_d$ turned out to be extremely useful for understanding the birational geometry of $\mathcal{M}_g$. One can consider analogous Brill-Noether loci in $\mathcal{M}_{g,n}$ defined as follows: if $\alpha^1, \ldots, \alpha^n$ are Schubert indices of type (r, d) (that is $0 \leq \alpha^i_1 \leq \ldots \leq \alpha^i_r \leq d - r$, for $i = 1, \ldots, n$), we consider the subvariety

$$\mathcal{M}^r_{g,n,d}(\alpha^1, \ldots, \alpha^n) := \{[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n} : \exists l \in G^r_d(C) \text{ with } \alpha^l(p_i) \geq \alpha^i \text{ for all } i\}.$$

The 'strong Brill-Noether Theorem' of Eisenbud and Harris (cf. Section 1.3) asserts that for a general n-pointed curve $(C, p_1, \ldots, p_n)$ of genus g, the dimension of the variety

$$G^r_d(C, (p_1, \alpha^1), \ldots, (p_n, \alpha^n)) = \{l \in G^r_d(C) : \alpha^l(p_i) \geq \alpha^i \text{ for } i = 1, \ldots, n\}$$

is the adjusted Brill-Noether number $\rho(g, r, d, \alpha^1, \ldots, \alpha^n) = \rho(g, r, d) - \sum_{i=1}^n \sum_{j=0}^r \alpha^i_j$. When this number is -1 one expects to find divisors on $\mathcal{M}_{g,n}$. We will try to understand the geometry of such divisors when $r = 2$ and $n \in \{1, 2\}$, that is, we will look at loci of 1 or 2-pointed curves having a $\mathfrak{g}^2_d$ with prescribed ramification at the marked points. We mention that the case $r = n = 1$ has already been treated in [Lo], but as it will turn out, computations are significantly more involved in the case of 2-dimensional linear series.

Experience shows that on $\mathcal{M}_g$ the most interesting divisors defined in terms of linear series are those consisting of curves with certain $\mathfrak{g}^r_d$'s having ramification as ordinary as possible: in general the more ramification one imposes on a linear series, the higher the slope of the resulting divisor on $\overline{\mathcal{M}}_g$ will be, hence it will be less relevant for understanding the birational geometry of $\mathcal{M}_g$. It is natural to expect the same for Brill-Noether divisors on $\overline{\mathcal{M}}_{g,n}$, which means that we will be mainly interested in the case when the α^i's are minimal.

For an integer $g \equiv 1 \mod 3$ and ≥ 4, we set $d := (2g+7)/3$, so that $\rho(g,2,d) = 1$. There are two ways to get Brill-Noether divisors with minimal ramification on $\mathcal{M}_{g,1}$. Either we take $\alpha = (0,1,1)$ and then we consider the divisor of curves with a marked point that is a *cusp*, i.e.

$$CU := \mathcal{M}^2_{g,1,d}((0,1,1)) = \{[C,p] \in \mathcal{M}_{g,1} : \exists \mathfrak{g}^2_d \text{ on } C \text{ with a cusp at } p\},$$

or we take $\alpha = (0,0,2)$ and then we get the divisor of curves with a marked point that is a *hyperflex*, i.e.

$$HF := \mathcal{M}^2_{g,1,d}((0,0,2)) = \{[C,p] \in \mathcal{M}_{g,1} : \exists \mathfrak{g}^2_d \text{ on } C \text{ with a hyperflex at } p\}.$$

We are going to compute the classes of the closures $\overline{CU}$ and $\overline{HF}$ in $\overline{\mathcal{M}}_{g,1}$.

On $\mathcal{M}_{g,2}$ there is only one way to get a Brill-Noether divisor by imposing minimal ramification, and that is by taking $\alpha^1 = \alpha^2 = (0,0,1)$ to obtain the divisor of curves with 2-marked points that are both *flexes*:

$$FL := \{[C,p_1,p_2] \in \mathcal{M}_{g,2} : \exists \mathfrak{g}^2_d \text{ on } C \text{ having flexes at } p_1 \text{ and } p_2\}.$$

We shall also compute the class $[\overline{FL}] \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,2})$.

Although I think that computations of divisor classes on $\overline{\mathcal{M}}_{g,n}$ are interesting in themselves because they enhance our understanding of families of (pointed) curves, the original motivation for studying the divisor $FL \subseteq \mathcal{M}_{g,2}$ was an attempt to prove that the moduli space $\mathcal{M}_{22,2}$ is of general type. It is proved in [Lo] that $\mathcal{M}_{22,n}$ is of general type for $n \geq 8$; since $\mathcal{M}_{23,n}$ is of general type for $n \geq 1$ and $\mathcal{M}_{21,n}$ is of non-negative Kodaira dimension for $n = 5$ and of general type for $n \geq 6$, it is natural to expect that the bound for genus 22 is some way off from being optimal. The divisor $\overline{FL}$, or some of its pullbacks to $\overline{\mathcal{M}}_{g,n}$ via the maps $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,2}$ forgetting some marked points, seemed the most likely candidate for being part of a multicanonical linear system, i.e. to have $K_{\overline{\mathcal{M}}_{22,2}} \sim a\,\overline{FL} + (\text{ effective divisor })$, for some $a \geq 0$. Unfortunately our calculation shows this not to be the case.

In Section 3.6 we compute the class of yet another divisor on the universal curve $\overline{\mathcal{M}}_{g,1}$. This time we consider a divisor which although is defined by a geometric condition in terms of linear series on curves, is different from the Brill-Noether divisors from Section 3.4, in the sense that it appears as the push-forward of a codimension 2 Brill-Noether locus in $\overline{\mathcal{M}}_{g,2}$ under the map $\pi_2 : \overline{\mathcal{M}}_{g,2} \to \overline{\mathcal{M}}_{g,1}$ forgetting the second point.

For an integer $d \geq 3$ we set $g := 2d-4$. We define the following codimension 1 locus in the universal curve

$$TR := \{[C,p] \in \mathcal{M}_{g,1} : \exists l \in G^1_d(C),\ \exists x \in C - \{p\} \text{ such that } a^l_1(p) \geq 3 \text{ and } a^l_1(x) \geq 3\},$$

that is, TR is the locus of 1-pointed curves (C,p) for which there exists a degree d map $f : C \to \mathbb{P}^1$ having triple ramification at the marked point p and at some unmarked point

$x \in C, x \neq p$. Clearly, $TR = \pi_2(\mathcal{M}^1_{g,2,d}((0,2),(0,2)))$. Since $\rho(g,1,d,(0,2),(0,2)) = -2$ the expected codimension of $\mathcal{M}^1_{g,2,d}((0,2),(0,2))$ inside $\mathcal{M}_{g,2}$ is 2 and it is easy to see that this is also the actual codimension, hence TR is a divisor on $\mathcal{M}_{g,1}$. We shall compute the class $[\overline{TR}] \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,1})$ of the closure of TR in $\overline{\mathcal{M}}_{g,1}$.

We close this chapter by proving in Section 3.7 the following

Theorem 3.1 *For $g = 11, 12, 15$ the Kodaira dimension of the universal curve C_g is $-\infty$.*

3.2 The Picard group of the moduli space of pointed curves

We review a few facts about the Picard groups of the moduli spaces $\overline{\mathcal{M}}_{g,n}$, when $g \geq 3$ and $n \geq 1$. The main references are [AC3] and [Lo] (but also [Mod], for a very comprehensible discussion on divisor classes on moduli stacks). All Picard groups we consider are with rational coefficients: in particular we have isomorphisms $\mathrm{Pic}_{fun}(\overline{\mathcal{M}}_{g,n}) \simeq \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n}) \simeq A^1_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$. Here by Pic_{fun} we understand the Picard group of the moduli stack (functor). From now on we denote $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ by $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$.

For $1 \leq i \leq n$ let us denote by $\pi_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ the morphism which forgets the i-th point. We denote by $\psi_i \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ the class on the moduli stack which associates to every n-pointed family of curves $(f : \mathcal{C} \to B, \sigma_1, \ldots, \sigma_n : B \to \mathcal{C})$ the class of the line bundle $\sigma_i^*(\omega_f)$ on B, where $\omega_f = \omega_{\mathcal{C}/B}$ is the relative dualizing sheaf of f.

For $0 \leq i \leq [g/2]$ and $A \subseteq \{1, 2, \ldots, n\}$, we denote by $\Delta_{i:A}$ the irreducible divisor on $\overline{\mathcal{M}}_{g,n}$ whose general point is a reducible curve of two components, one of genus i, the other of genus $g-i$, meeting transversally at a point and such that the genus i component contains precisely the marked points corresponding to A. The index set A is subject to the obvious conditions $\mathrm{card}(A) \geq 2$ if $i = 0$ and $1 \in A$ for $i = g/2$. We denote by $\delta_{i:A}$ the class on the moduli stack associated to the divisor $\Delta_{i:A}$. We also write $\delta_{i:A} = \delta_{g-i:A'}$, where $A' := \{1, \ldots, n\} - A$, as well as $\delta_i = \delta_{g-i:\emptyset}$. The key result is the following:

Proposition 3.2.1 (Harer, Arbarello-Cornalba) *For $g \geq 3$ and $n \geq 1$ the group $\mathrm{Pic}(\overline{\mathcal{M}}_{\mathrm{g,n}})$ is freely generated by the class of the Hodge line bundle λ and by the classes ψ_i for $1 \leq i \leq n$ and the boundary classes $\delta_{i:A}$, where $0 \leq i \leq [g/2]$ and $A \subseteq \{1, \ldots, n\}$.*

We briefly discuss now the cases that are of interest to us, i.e. when $n \in \{1, 2\}$. If $n = 1$ we denote by $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ the natural projection. Clearly $\psi = \omega = c_1(\omega_\pi)$; also

$$\pi^*(\lambda) = \lambda, \quad \pi^*(\delta_0) = \delta_0 \quad \text{and } \pi^*(\delta_i) = \delta_i + \delta_{g-i}, \text{ for } 1 \leq i < [g/2].$$

If $i = [g/2]$ we have that $\pi^*(\delta_{[g/2]}) = \delta_{[g/2]}$.

When $n = 2$ we look at the maps $\pi_i : \overline{\mathcal{M}}_{g,2} \to \overline{\mathcal{M}}_{g,1}$ forgetting one of the points. One has that $\psi_i = c_1(\omega_{\pi_i}) + \delta_{0:\{1,2\}}$. As for the pullbacks, we will need the formulas (cf. [Lo])

$$\pi_2^*(\lambda) = \lambda, \quad \pi_2^*(\delta_0) = \delta_0, \quad \pi_2^*(\psi) = \psi_1 - \delta_{0:\{1,2\}} \text{ and } \pi_2^*(\delta_{i:A}) = \delta_{i:A} + \delta_{i:A\cup\{2\}}.$$

3.3 Counting linear series on curves via Schubert calculus

In order to compute the class of Brill-Noether divisors on $\overline{\mathcal{M}}_{g,n}$, one has to figure out the intersection numbers of such divisors with various curves in $\overline{\mathcal{M}}_{g,n}$. Typically, we have to answer questions like:

Given g, r, d and $\alpha^1, \ldots, \alpha^s$ Schubert indices of type (r, d) such that $\rho(g, r, d, \alpha^1, \ldots, \alpha^s) = 0$, how many $\mathfrak{g}^r_d$'s with prescribed ramification at the s marked points does a general s-pointed curve of genus g have? In other instances one has to compute the number of $\mathfrak{g}^r_d$'s having prescribed ramification at unprescribed points.

In each case we will solve such problems using Schubert calculus: we let our curves degenerate to curves of compact type that are unions of $\mathbb{P}^1$'s and have elliptic tails and no other components. Computing the number of limit $\mathfrak{g}^r_d$'s with special properties on such curves boils down to computations in the cohomology rings of certain Grassmannians.

We now outline the method. We use as references [GH] and [EH5], while [F] is a reference for general properties of Schubert cycles in Grassmannians. Let C be an algebraic curve of genus g, let $p \in C$ be a point and $l = (\mathcal{L}, V) \in G^r_d(C)$. To this data we associate the following strictly decreasing flag in $H^0(C, \mathcal{L}/\mathcal{L}(-(d+1)p)) \simeq \mathbb{C}^{d+1}$:

$$\mathcal{F}(p) : H^0(C, \mathcal{L}/\mathcal{L}(-(d+1)p)) = W_0 \supset W_1 \supset \ldots \supset W_d \supset W_{d+1} = 0,$$

where $W_i = H^0(C, \mathcal{L}(-ip)/\mathcal{L}(-(d+1)p))$. If $\alpha = (\alpha_0, \ldots, \alpha_r)$ is a Schubert index of type (r, d), the condition $\alpha^l(p) \geq \alpha$ is equivalent with V belonging to the Schubert cycle $\sigma_\alpha \in \mathbb{G}(r, H^0(C, \mathcal{L}/\mathcal{L}(-(d+1)p))) = \mathbb{G}(r, d)$ defined w.r.t. the flag $\mathcal{F}(p)$ (see Section 1.3 for our way of denoting Schubert cycles which basically coincides with that from [GriffHa] except that we write the indices in reversed order). For instance, p is a ramification point of l if and only if $V \in \sigma_{(0,\ldots,0,1)}$.

Let us now consider the special case $C = \mathbb{P}^1$. There is only one line bundle of degree d on $\mathbb{P}^1$, namely $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(d)$. Making once and for all the identification $\mathbb{C}^{d+1} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$, the variety $G^r_d(\mathbb{P}^1)$ is just the Grassmannian $\mathbb{G}(r, d)$ of projective r-planes in $\mathbb{P}^d = \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)))$. For each point $p \in \mathbb{P}^1$ we have the flag $\mathcal{F}(p)$ with respect to which we can define Schubert cycles in $\mathbb{G}(r, d)$.

Let $p_1, \ldots, p_s \in \mathbb{P}^1$ be distinct general points and $\alpha^1, \ldots, \alpha^s$ Schubert indices of type (r, d). For each $1 \leq i \leq s$ we consider the Schubert cycle $\sigma_{\alpha^i} \in \mathbb{G}(r, d)$ defined in terms of $\mathcal{F}(p_i)$. It is proved in [EH5] that $\sigma_{\alpha^1}, \ldots, \sigma_{\alpha^m}$ are *dimensionally transverse*: every component of $\bigcap_{i=1}^m \sigma_{\alpha^i}$ has codimension

$$\sum_{i=1}^{m} \operatorname{codim}(\sigma_{\alpha^i}, \mathbb{G}(r, d)) = \sum_{i=1}^{m} \alpha^i,$$

and in particular $\bigcap_{i=1}^m \sigma_{\alpha^i} = \emptyset$ if and only if $\sigma_{\alpha^1} \cdots \sigma_{\alpha^m} = 0$ in $H^*(\mathbb{G}(r, d), \mathbb{Z})$. This shows that there exists a $\mathfrak{g}^r_d$ on $\mathbb{P}^1$ having ramification $\geq \alpha^i$ at p_i, for $i = 1, \ldots, s$, if and only if $\sigma_{\alpha^1} \cdots \sigma_{\alpha^s} \neq 0$.

Proposition 3.3.1 *Let $\alpha^1, \ldots, \alpha^s$ be Schubert indices of type (r, d) such that*

$$\sum_{i=1}^{s}\sum_{j=0}^{r}\alpha_j^i + rg = (r+1)(d-r).$$

and $p_1, \ldots, p_g, x_1, \ldots, x_s \in \mathbb{P}^1$ distinct general points. Then, the variety of $\mathfrak{g}_d^r$'s on $\mathbb{P}^1$ having cusps at the points $p_1, \ldots, p_g$ and ramification $\geq \alpha^i$ at x_i for $i = 1, \ldots, s$ is reduced, 0-dimensional and consists of $\sigma_{(0,1,\ldots,1)}^g \sigma_{\alpha^1} \cdots \sigma_{\alpha^s}$ points.

Proof: This has been basically settled in [EH5]. To be precise, Eisenbud and Harris proved a similar statement for the Schubert cycles of the form $\sigma_{(0,\ldots,0,m)}$ rather than $\sigma_{(0,1,\ldots,1)}$, but by duality we obtain the claimed statement from theirs. Just use that the dual of the cycle $\sigma_{(0,1,\ldots,1)}$ in $\mathbb{G}(r, d)$ is the cycle $\sigma_{(0,\ldots,0,r)}$ in $\mathbb{G}(d-r-1, d)$. □

We will repeatedly use the following formula (cf. [GH, page 269]): if $\alpha = (\alpha_0, \ldots, \alpha_r)$ is a Schubert index of type (r, d) such that $\sum_{i=0}^r \alpha_i + rg = (r+1)(d-r)$ we have that

$$\sigma_{(\alpha_0,\ldots,\alpha_r)}\ \sigma_{(0,1,\ldots,1)}^g = g!\frac{\prod_{i<j}(\alpha_j - \alpha_i + j - i)}{\prod_{i=0}^r (g-d+i+\alpha_i+r)!}. \tag{3.1}$$

Another formula that we will find quite useful is the classical *Plücker formula* (see [Mod, page 257]): If C is a smooth curve of genus g and l a $\mathfrak{g}_d^r$ on C, then

$$\sum_{p\in C} w^l(p) = (r+1)d + (r+1)r(g-1). \tag{3.2}$$

where $w^l(p)$ is the weight of p in l (cf. Chapter 1).

3.4 Divisors on $\overline{\mathcal{M}}_{g,1}$

In this section we compute the classes of the divisors $\overline{CU}$ and $\overline{HF}$ of curves with a marked points that is either a cusp or a hyperflex. We will use Theorem 4.1 from [EH2] which gives informations about the subspace of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$ generated by the classes of the Brill-Noether divisors.

Fix $g \geq 3$ and let us denote by $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ the natural projection. Inside $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$ one can look at the subspace generated by the classes of all divisors $\overline{\mathcal{M}}_{g,d}^r(\alpha)$, where $\alpha = (\alpha_0, \ldots, \alpha_r)$ is a Schubert index of type (r, d) such that $\rho(g, r, d) - \sum_{i=0}^r \alpha_i = -1$. We note that the Brill-Noether loci we consider on $\mathcal{M}_{g,1}$ will have exactly one divisorial component (cf. [EH2] Theorem 1.2) and possibly some other lower dimensional components. It is not known whether the Brill-Noether divisors on $\overline{\mathcal{M}}_{g,n}$, with $n \geq 2$ are irreducible or not.

One distinguished Brill-Noether divisor on $\mathcal{M}_{g,1}$ is the locus of Weierstrass points

$$\mathcal{W} := \{[C, p] \in \mathcal{M}_{g,1} : p \text{ is a Weierstrass point of } C\}.$$

Clearly $W = \mathcal{M}^1_{g,1,g}((0, g-1))$, that is, the locus of those (C, p) for which there is a $\mathfrak{g}^1_g$ with total ramification at p. The class of the closure $\overline{W}$ in $\overline{\mathcal{M}}_{g,1}$ has been computed (cf. [Cuk]):

$$[\overline{W}] = -\lambda + \frac{g(g+1)}{2}\psi - \sum_{i=1}^{g-1} \frac{(g-i)(g-i+1)}{2}\delta_i.$$

Another divisor class we consider on $\overline{\mathcal{M}}_{g,1}$ is

$$BN := (g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{g-1} i(g-i)\delta_i.$$

Here, BN is (modulo multiplication by a positive rational constant) the class of the pullback of any Brill-Noether divisor $\overline{\mathcal{M}}^r_{g,d}$ on $\overline{\mathcal{M}}_g$, when $\rho(g,r,d) = -1$. The class BN is effective when there are Brill-Noether divisors on $\overline{\mathcal{M}}_g$ and this happens precisely when $g+1$ is composite. When $g+1$ is prime, it is not clear whether BN is effective (as a matter of fact, the slope conjecture (see end of Chapter 1) predicts it is not). For $g+1$ prime (in particular we can then write $g = 2k-2$), the effective divisor on $\overline{\mathcal{M}}_g$ having the largest slope known to this date, is the closure of the locus

$$E^1_k := \{[C] \in \mathcal{M}_g : \exists \mathfrak{g}^1_k \text{ on } C \text{ such that } 2\mathfrak{g}^1_k \text{ is special }\}.$$

The slope of $\overline{E}^1_k$ is $(6k^2+k-6)/k(k-1)$ (cf. [EH3]). A general curve of genus $2k-2$ has $(2k-2)!/(k!(k-1)!)$ linear systems $\mathfrak{g}^1_k$, and E^1_k is the locus of curves C for which the scheme $G^1_k(C)$ is nonreduced. This happens when two $\mathfrak{g}^1_k$'s come together, so that we get a $\mathfrak{g}^1_k$ with dim $2\mathfrak{g}^1_k \geq 3$, or equivalently by Riemann-Roch, $2\mathfrak{g}^1_k$ is special.

We have the following remarkable result of Eisenbud and Harris (cf. [EH2]):

Proposition 3.4.1 *The Brill-Noether subspace in* $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$ *is two-dimensional, generated by the classes* $[\overline{W}]$ *and* BN.

Remark: More generally, Logan has proved in [Lo] that for $n \geq 1$ the subspace of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ generated by the classes of the Brill-Noether divisors on $\overline{\mathcal{M}}_{g,n}$ has dimension $1 + \binom{n+1}{2}$.

We now compute the classes of the divisors $\overline{HF}$ and $\overline{CU}$.

Theorem 3.2 *Let* $g \equiv 1$ *mod* 3 *be an integer* ≥ 4 *and set* $d := (2g+7)/3$. *We have the following relations in* $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$:

1. $[\overline{HF}] = c\,(a\,\lambda + b\,\psi - e_0\,\delta_0 - \sum_{i=1}^{g-1} e_i\,\delta_i)$,

   where

   $a = 2(g^3+7g^2-10g-166)$, $b = 15g(g-2)(g-d+6)$, $e_0 = (g^2+9g+23)(g-d+1)$,

$\epsilon_i = (g-i)(5g^2 + 2ig^2 + 3ig + 45g - 110 - 74 - i)$ *and*

$c = 8(g-3)!/\big((g-d+6)!\ (g-d+3)!\ (g-d+1)!\big)$.

2. $[\overline{CU}] = c'\Big((g+4)\ \lambda + g\ \psi - \frac{g-2}{6}\ \delta_0 - \sum_{i=1}^{g-1}(i+1)(g-i)\ \delta_i\Big)$,

where $c' = 24(g-2)!/\big((g-d+5)!\ (g-d+3)!\ (g-d+1)!\big)$.

Remark: Let us try to understand the meaning of our formulas in the simplest case, $g = 4, d = 5$. By substitution we get

$$[\overline{HF}] = -2\lambda + 20\psi - 12\delta_1 - 6\delta_2 - 2\delta_3.$$

On the other hand, on a curve C of genus 4 every $\mathbf{g}^2_5$ is of the form $|K_C - x|$, for some $x \in C$. The marked point $p \in C$ is a hyperflex of $|K_C - x|$ if and only if $h^0(C, 4p + x) \geq 3$. This implies that p is a Weierstrass point of C and x is one of the 2 points in the effective divisor $K_C - 4p$. Therefore $\overline{HF} = 2\ \overline{\mathcal{W}}$, and this can also be seen by comparing our formula to Cukierman's (cf. [Cuk]):

$$[\overline{\mathcal{W}}] = -\lambda + 10\psi + -6\delta_1 - 3\delta_2 - \delta_3.$$

As for the other divisor, when $g = 4$, one finds that

$$CU = \{(C, p) \in \mathcal{C}_4 :\ \text{there exists } x \in C \text{ such that } h^0(C, 2p + x) \geq 2\}.$$

Our formula gives in this case $[\overline{CU}] = 8\lambda + 4\psi - 6\delta_1 - 6\delta_2 - 4\delta_3$. Note that the class of the divisor CU on $\mathcal{C}_4$ already appears in [Fa] page 423 (that is the $\{\lambda, \psi\}$ part in our formula).

Proof: 1) We start by computing $[\overline{HF}]$ which is technically a bit more difficult than computing $[\overline{CU}]$.

Since $\overline{HF} = \overline{\mathcal{M}}^2_{g,1,d}((0,0,2))$ is a Brill-Noether divisor, by applying Prop. 4.1 it follows that there are rational constants ν, μ such that

$$[\overline{HF}] = \mu\ BN + \nu\ [\overline{\mathcal{W}}]. \qquad (3.3)$$

and we just have to determine the coefficients μ and ν. We use the method of test curves, i.e. we intersects both sides of (3) with curves in $\overline{\mathcal{M}}_{g,1}$: we write down 1-dimensional families of 1-pointed curves of genus g and compute the degrees of λ, ψ and the δ's on that curve as well as the degree of $\overline{HF}$. We need two test curves in $\overline{\mathcal{M}}_{g,1}$ which will provide two linear equations in μ and ν. Since it is pretty difficult to write down explicit families of curves of genus g with smooth general member, most of the test curves we use, will be entirely contained in the boundary $\overline{\mathcal{M}}_{g,1} - \mathcal{M}_{g,1}$.

We obtain the first test curve as follows: Take a general curve B of genus $g-1$ and a general 2-pointed elliptic curve $(E, 0, p)$. We get a 1-dimensional family in $\Delta_1 \subseteq \overline{\mathcal{M}}_{g,1}$ by identifying the fixed point $0 \in E$ with a variable point $q \in B$, the marked point being the fixed point $p \in E$. Let us denote by $\{X_q := E \cup_q B, p \in E\}_{q \in B}$ the resulting family.

The degrees of the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$ on the family we have just constructed, are as follows:

$$\deg(\lambda) = 0, \quad \deg(\psi) = 0, \quad \deg(\delta_1) = -\deg(K_B) = 4 - 2g,$$

while δ_0 and δ_i for $2 \leq i \leq g-1$, all vanish. Next we evaluate $\deg(\overline{HF})$, that is the degree of the divisor $\overline{HF}$ on the curve in $\overline{\mathcal{M}}_{g,1}$ we have written down.

Let us take $(X_q = E \cup_q B, p \in E)$ a member of our family. Then $[X_q, p] \in \overline{HF}$ if and only if there exists a (smoothable) limit $\mathfrak{g}^2_d$ on X_q, say l, with vanishing $\geq (0, 1, 4)$ at p. Using the additivity of the Brill-Noether number we have that:

$$-1 \geq \rho(l, \alpha^l(q)) \geq \rho(l_B, \alpha^{l_B}(q)) + \rho(l_E, \alpha^{l_E}(q), \alpha^{l_E}(p)).$$

Since $\rho(l_E, \alpha^{l_E}(q), \alpha^{l_E}(p)) \geq 0$ (if we assume $p - q \in \mathrm{Pic}^0(E)$ not to be a torsion class) and also $\rho(l_B, \alpha^{l_B}(q)) \geq -1$ (because $[B] \in \mathcal{M}_{g-1}$ is general), it follows that $\rho(l_B, \alpha^{l_B}(q)) = -1$, that $\rho(l_E, \alpha^{l_E}(q), \alpha^{l_E}(p)) = 0$ and $a^{l_E}(p) = (0, 1, 4)$. By using Prop. 1.4.1 we have that $d - 1 \leq a_i^{l_E}(p) + a_{2-i}^{l_E}(q) \leq d$ for $i = 0, 1, 2$ and there are two cases two consider:
1st case: $a^{l_E}(q) = (d-4, d-2, d-1)$ from which $a^{l_B}(q) = (1, 2, 4)$. By 'The Regeneration Theorem' (cf. Chapter 1), all these linear series are smoothable. In order to compute the contribution to $\deg(\overline{HF})$ in this case, we have to count how many points $q \in B$ there are such that there is a $\mathfrak{g}^2_{d-1}$ on B with ordinary ramification at q.

At this point, one might worry about the multiplicities with which we count such linear series. It turns out that all multiplicities we encounter during this proof are equal to 1. The reason is that if we denote by $(f : \mathcal{X} \to B, \tilde{p} : B \to \mathcal{X})$ the versal deformation space of (X_q, p), then in a similar way to the proof of Lemma 3.4 from [EH2], one can show that the variety $\mathcal{G}^2_d(\mathcal{X}/B, (\tilde{p}, (0, 0, 2)))$ of $\mathfrak{g}^2_d$'s with hyperflexes on 1-pointed curves nearby (X_q, p), is transversal to our test curve.

By standard Brill-Noether theory, B possesses $2(g-1)!/((g-d+2)!(g-d+3)!(g-d+4)!)$ linear series $\mathfrak{g}^2_{d-1}$. By Plücker's formula (3.2) each such $\mathfrak{g}^2_{d-1}$ has $3d + 6g - 15$ ramification points (all ordinary, since B is general). We thus get a contribution of

$$\frac{2(3d + 6g - 15)(g - 1)!}{(g - d + 4)!\,(g - d + 3)!\,(g - d + 2)!}. \tag{3.4}$$

2nd case: $a^{l_E}(q) = (d - 5, d - 2, d)$, hence $a^{l_B}(q) = (0, 2, 5)$. Once again, all these linear series are smoothable and the contribution to $\deg(\overline{HF})$ we get in this case, is the number of $\mathfrak{g}^2_d$'s on a general curve of genus $g - 1$ with vanishing sequence $(0, 2, 5)$ at an unspecified point. We now determine this number.

Let the curve B degenerate in a 1-dimensional family $\{B_t\}$ having smooth generic fibre and as special fibre, a curve of compact type $B_0 := \mathbb{P}^1 \cup E_1 \cup \ldots \cup E_{g-1}$, where E_i are general elliptic curves, $\{p_i\} = E_i \cap \mathbb{P}^1$ and $p_1, \ldots, p_{g-1} \in \mathbb{P}^1$ are general points. We count the number of limit $\mathfrak{g}^2_d$'s on B_0 with ramification $(0, 1, 3)$ at some unspecified point

$x \in B_0$.

Note that in principle we could have $x = p_i$ for some i, that is, there exists $x_t \in B_t$ and and a family of 2-dimensional linear series $l_t \in G^2_d(B_t, (x_t, (0,1,3)))$ for $t \neq 0$, such that $\lim_{t\to 0} x_t = p_i$, (i.e. the hyperflexes x_t specialize to a node of the central fibre). In this case however, by making a finite base-change, blowing-up sufficiently often the nodes of B_0 and resolving the resulting singularities, we obtain a new generically smooth family (B'_t, x'_t) and linear series $l'_t \in G^2_d(B'_t, (x'_t, (0,1,3)))$ for $t \neq 0$, such that no ramification point of l'_t specializes to a node of B'_0. The central fibre B'_0 is derived from B_0 by inserting chains of $\mathbb{P}^1$'s at the nodes of B_0. Since this operation (explained in [EH1]) does not change the Brill-Noether theory of the central fibre, we may assume from the beginning that all ramification has been swerved away from the nodes.

Conversely, using semicontinuity of fibre dimension for the space of (limit) $\mathfrak{g}^2_d$'s with hyperflexes on curves nearby B_0, we find that all limit $\mathfrak{g}^2_d$ on B_0 with vanishing $\geq (0,1,4)$ at an unmarked point, are smoothable to every nearby curve in a way that maintains the hyperflex. This shows that the number of limit $\mathfrak{g}^2_d$'s on B_0 having a hyperflex, is the same as the number of (honest) $\mathfrak{g}^2_d$'s with a hyperflex on a general curve of genus $g-1$.

Let l be a limit $\mathfrak{g}^2_d$ on B_0 with ramification $\geq (0,1,3)$ at a smooth point x. In general, from the Plücker formula it follows that for any linear series $\mathfrak{g}^r_d$ on $\mathbb{P}^1$ and for any number of points $y_1, \ldots, y_m \in \mathbb{P}^1$, the inequality $\rho(\mathfrak{g}^r_d, \alpha(y_1), \ldots, \alpha(y_m)) \geq 0$ holds. Using this observation and that on B_0 we have $\rho(l, \alpha^l(x)) = -1$, by additivity it follows that x must be on one of the elliptic tails, say $x \in E_1$. Then we must have $\rho(l_{E_1}, \alpha^{l_{E_1}}(x), \alpha^{l_{E_1}}(p_1)) = -1$, $\rho(l_{E_i}, \alpha^{l_{E_i}}(p_i)) = 0$ for $2 \leq i \leq g-1$ and $\rho(l_{\mathbb{P}^1}, \alpha^{l_{\mathbb{P}^1}}(p_1), \ldots, \alpha^{l_{\mathbb{P}^1}}(p_{g-1})) = 0$. This means that the aspect $l_{\mathbb{P}^1}$ has cusps at the points $p_2, \ldots, p_{g-1}$. As for the E_1-aspect of l there are three possibilities:

- $a^{l_{E_1}}(p_1) = (d-5, d-2, d-1)$, so $a^{l_{\mathbb{P}^1}}(p_1) = (1,2,5)$. Then clearly $3p_1 + 2x \sim 5x$, so $3x \sim 3p_1$ on E_1. On $\mathbb{P}^1$ we have (after subtracting the base point p_1) a $\mathfrak{g}^2_{d-1}$ with ramification $(0,0,2)$ at p_1 and cusps at $p_2, \ldots, p_{g-2}$. According to Section 3.3 the number of such linear series is $\sigma_{(0,0,2)}\sigma^{g-2}_{(0,1,1)}$ (the product is taken in $H^{top}(\mathbb{G}(2, d-1), \mathbb{Z})$). Since there are 8 choices for $x \in E_1$ and x can lie on any of the $g-1$ elliptic tails, using formula (3.1), we get a total contribution of

$$8(g-1)\ \sigma_{(0,0,2)}\ \sigma^{g-2}_{(0,1,1)} = \frac{96(g-1)!}{(g-d+5)!\ (g-d+2)!\ (g-d+1)!}\,. \qquad (3.5)$$

- $a^{l_{E_1}}(p_1) = (d-6, d-2, d)$, so $a^{l_{\mathbb{P}^1}}(p_1) = (0,2,6)$. Then $2x \sim 2p_1$ on E_1, and in this case we obtain a contribution of

$$3(g-1)\ \sigma_{(0,1,4)}\ \sigma^{g-2}_{(0,1,1)} = \frac{144(g-1)!}{(g-d+6)!\ (g-d+2)!\ (g-d)!}\,. \qquad (3.6)$$

- $a^{l_{E_1}}(p_1) = (d-5, d-3, d)$, hence $a^{l_{\mathbb{P}^1}}(p_1) = (0, 3, 5)$. Then $5x \sim 5p_1$ on E_1 and the contribution to $\deg(\overline{HF})$ is

$$24(g-1)\ \sigma_{(0,2,3)}\ \sigma_{(0,1,1)}^{g-2} = \frac{720(g-1)!}{(g-d+5)!\ (g-d+3)!\ (g-d)!}. \tag{3.7}$$

By adding (3.4),(3.5),(3.6) and (3.7), we obtain that

$$\deg(\overline{HF}) = \frac{16(g-1)!\ (7g^2+48g-184)}{(g-d+6)!\ (g-d+3)!\ (g-d+2)!}. \tag{3.8}$$

Since from (3.3) we have that $\deg(\overline{HF}) = 2(g-1)(g-2)\mu + g(g-1)(g-2)\nu$, the equation (3.8) provides one linear relation between μ and ν.

In order to obtain a second relation, we use as test curve a general fibre of the map $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$. We fix C, a general curve of genus g and let $p \in C$ vary. For this family of course $\deg(\omega) = 2g-2$, while λ and all the δ's vanish. We also need $\deg(\overline{HF})$ which is just the number of $\mathfrak{g}^2_d$'s on a general curve of genus g having a hyperflex at an unspecified point.

To compute this number we let C degenerate to $C_0 := \mathbb{P}^1 \cup E_1 \cup \ldots \cup E_g$, where E_i are general elliptic curves, $\{p_i\} = E_i \cap \mathbb{P}^1$ and $p_1, \ldots, p_g \in \mathbb{P}^1$ are general points. We count limit $\mathfrak{g}^2_d$'s on C_0 with vanishing $\geq (0,1,4)$ at some point $x \in C_0$. As before, it turns out that all these $\mathfrak{g}^2_d$'s are smoothable and no two $\mathfrak{g}^2_d$ on smooth curves nearby C_0 coalesce. Plücker's formula forces the point x to sit on an elliptic tail.

Take l a limit $\mathfrak{g}^2_d$ on C_0 with a hyperflex at a point x and assume that $x \in E_1$. It is straightforward to see that $a^{l_{E_1}}(x) = (0,2,4)$ and $a^{l_{E_1}}(p_1) = (d-4, d-2, d)$, from which $4p_1 \sim 4x$, which gives 15 choices for $x \in E_1$. On the spine $\mathbb{P}^1$ we have to count $\mathfrak{g}^2_d$'s with vanishing sequence $(0,2,4)$ at p_1 and cusps at $p_2, \ldots, p_g$. The number of such linear series is $\sigma_{(0,1,2)}\ \sigma_{(0,1,1)}^{g-1}$, the product being computed in $H^{top}(\mathbb{G}(2,d), \mathbb{Z})$. Since x can sit on any of the tails $E_1, \ldots, E_g$, we get that

$$\deg(\overline{HF}) = 15g\ \sigma_{(0,1,2)}\ \sigma_{(0,1,1)}^{g-1} = \frac{240g!}{(g-d+5)!\ (g-d+3)!\ (g-d+1)!}. \tag{3.9}$$

which immediately gives

$$\nu = \frac{240(g-2)!}{(g+1)\ (g-d+5)!\ (g-d+3)!\ (g-d+1)!}.$$

By plugging in we obtain μ as well, hence $[\overline{HF}]$ too.

2) In order to compute $[\overline{CU}]$ we could use exactly the same test curves employed for the computation of $[\overline{HF}]$, but there is a shorter way of doing things in this case. Thankfully, our results do not depend on which method we choose.

We fix a general elliptic curve E and consider the map $j : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_{g+1}$ given by

$j([B,p]) := [B \cup_p E]$ (attaching an elliptic tail). Then $\overline{CU} = j^*(\overline{\mathcal{M}}^2_{g+1,d})$. As already pointed out in Chap.1, we have

$$[\overline{\mathcal{M}}^2_{g+1,d}] = f\big((g+4)\ \lambda - (g+2)/6\ \delta_0 - \sum_{i=1}^{[(g+1)/2]} i(g+1-i)\ \delta_i\big).$$

where $f = 3a/(2g-2)$, with a being the number of $\mathfrak{g}^2_d$'s on a 1-pointed curve of genus $g-1$ and having ramification $(0,1,2)$ at the marked point. By degenerating the 1-pointed curve of genus $g-1$ to $(\mathbb{P}^1 \cup E_1 \cup \ldots \cup E_g\ , p \in \mathbb{P}^1)$, where the E_i are elliptic, $\{p_i\} = \mathbb{P}^1 \cap E_i$ for $i = 1, \ldots, g$ and $p, p_1, \ldots, p_g \in \mathbb{P}^1$ are general points, we see that $a = \sigma_{(0,1,2)}\ \sigma^{g-1}_{(0,1,1)}$ $(\in H^{top}(\mathbb{G}(2,d),\mathbb{Z}))$, from which we obtain

$$f = \frac{24(g-2)!}{(g-d+5)!\ (g-d+3)!\ (g-d+1)!}\ .$$

The pullback j^* acts on the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{g+1})$ as follows:

$j^*(\delta_0) = \delta_0$, $j^*(\lambda) = \lambda$, $j^*(\delta_1) = -\psi + \delta_{g-1}$ (by adjunction), $j^*(\delta_i) = \delta_{g-i} + \delta_{i-1}$ for $i \geq 2$.

We get immediately the stated formula for $[\overline{CU}]$. □

3.5 A divisor on $\overline{\mathcal{M}}_{g,2}$

Using results from the previous section we compute the class of the divisor of curves with 2 marked points that are flexes of a 2-dimensional linear series on the curve:

Theorem 3.3 *Let $g \equiv 1$ mod 3 be an integer ≥ 4 and set $d := (2g+7)/3$. We have the following formula in* $\mathrm{Pic}(\overline{\mathcal{M}}_{g,2})$:

$$[\overline{FL}] = c''\big(A\ \lambda + B\ (\psi_1 + \psi_2) - C\ \delta_0 - D\ \delta_{0:\{1,2\}} - \sum_{i=1}^{g-1} a_i\ \delta_{i:\{1\}} - \sum_{i=1}^{g-1} b_i\ \delta_{i:\{1,2\}}\big),$$

where

$$A = 6(g^3 + 9g^2 - 2g - 140),\quad B = 6(g+1)(g+11)(g-2),\quad C = g^3 + 7g^2 - 10g - 76,$$

$$D = 12g(g-2)(g+11),\quad b_i = 6(g-i)\big((g^2+4g)(i+2) - (32i+44)\big),$$
$$a_i = 6\big(g^3(i+1) - g^2(i^2-4i-10) - g(4i^2+32i+13) + (32i^2-22)\big) \quad \textit{and}$$
$$c'' = 4(g-3)!/\big((g-d+6)!\ (g-d+3)!\ (g-d+1)!\big).$$

Remark: When $g = 4, d = 5$ we have that

$FL = \{[C,p_1,p_2] \in \mathcal{M}_{4,2} :$ there exists $x \in C$ such that $h^0(C, x + 3p_i) \geq 2$ for $i = 1,2\}$.

Our formula gives in this case

$$[\overline{FL}] = 6\lambda + 15(\psi_1+\psi_2) - \delta_0 - 24\delta_{0:\{1,2\}} - 15(\delta_{1:\{1\}}+\delta_{2:\{1\}}+\delta_{3:\{1\}}) - 18\delta_{1:\{1,2\}} - 12\delta_{2:\{1,2\}} - 6\delta_{3:\{1,2\}}.$$

Proof of Theorem 3.3: We determine the coefficients in the expression of $[\overline{FL}]$ in three steps. First, we consider the map $\pi_2 : \overline{\mathcal{M}}_{g,2} \to \overline{\mathcal{M}}_{g,1}$ which forgets the second point. We claim that

$$(\pi_2)_*([\overline{FL}] \cdot \delta_{0:\{1,2\}}) = [\overline{HF}] + [\overline{CU}] . \tag{3.10}$$

This is almost obvious: If $[X = C \cup_q \mathbb{P}^1, p_1, p_2]$, with $p_1, p_2 \in \mathbb{P}^1$, is a general point in $\overline{FL} \cap \Delta_{0:\{1,2\}}$, then there exists l, a limit $\mathfrak{g}^2_d$ on X having flexes at p_1 and p_2. Using the additivity of the Brill-Noether number on X we have that such a linear series is refined, that $\rho(l_{\mathbb{P}^1}, \alpha^l(p_1), \alpha^l(p_2), \alpha^l(q)) = 0$ and $\rho(l_C, \alpha^{l_C}(q)) = -1$. It follows that $w^{l_C}(q) = 2$ and this happens if either $a^{l_C}(q) = (0,1,4)$ (and then $[C,q] \in \overline{HF}$), or $a^{l_C}(q) = (0,2,3)$ (and then $[C,q] \in \overline{CU}$). On $\mathbb{P}^1$ on the other hand, there is precisely one $\mathfrak{g}^2_d$ with flexes at both p_1 and p_2 and vanishing $(d-4, d-1, d)$ (resp. $(d-3, d-2, d)$) at the point q, so indeed $(\pi_2)_*([\overline{FL}] \cdot \delta_{0:\{1,2\}}) = [\overline{HF}] + [\overline{CU}]$. We use (3.10) together with Theorem 3.2 to determine a few coefficients in the expression of $[\overline{FL}]$.

Let us write $[\overline{FL}] = A\ \lambda + B\ (\psi_1 + \psi_2) - C\ \delta_0 - D\ \delta_{0:\{1,2\}} - \sum_{i=1}^{g-1} a_i\ \delta_{i:\{1\}} - \sum_{i=1}^{g-1} b_i\ \delta_{i:\{1,2\}}$.

Since $(\pi_2)_*(\lambda \cdot \delta_{0:\{1,2\}}) = \lambda$, $(\pi_2)_*(\delta^2_{0:\{1,2\}}) = -\psi$, $(\pi_2)_*(\delta_0 \cdot \delta_{0:\{1,2\}}) = \delta_0$, $(\pi_2)_*(\psi_i \cdot \delta_{0:\{1,2\}}) = 0$ for $i = 1, 2$, $(\pi_2)_*(\delta_{i:\{1\}} \cdot \delta_{0:\{1,2\}}) = 0$ and $(\pi_2)_*(\delta_{i:\{1,2\}} \cdot \delta_{0:\{1,2\}}) = \delta_{i:\{1\}}$ for $1 \le i \le g-1$, from (3.10) and Theorem 3.2 we obtain the coefficients A, C, D and b_i for $1 \le i \le g-1$.

In order to get the coefficients of ψ_1 and ψ_2, we use the following test curve: fix C a general curve of genus g, let $p_1 \in C$ be a general fixed point and $p_2 \in C$ a variable point. When p_2 hits p_1, by blowing-up we insert a $\mathbb{P}^1$ at $p_1 \in C$, therefore $\deg(\delta_{0:\{1,2\}}) = 1$ for this family. Moreover, $\deg(\psi_1) = 1$ (the restriction of ψ_1 to this family is $\mathcal{O}_C(p_1)$), $\deg(\psi_2) = 2g-1$ (because the restriction of ψ_2 to this family is the line bundle $\Omega_C(p_1)$) and finally λ and all the other δ's vanish.

We now compute $\deg(\overline{FL})$. By the Schubert calculus we have already employed a number of times, we see that C has $\sigma_{(0,0,1)}\ \sigma^g_{(0,1,1)}$ $(\in H^{top}(\mathbb{G}(2,d),\mathbb{Z}))$ linear series $\mathfrak{g}^2_d$ with a flex at a fixed point p_1. By Plücker, each of these linear series has $3d+6g-7$ ramification points different from p_1, thus we get that

$$\deg(\overline{FL}) = (3d+6g-7)\sigma_{(0,0,1)}\ \sigma^g_{(0,1,1)} = \frac{(3d+6g-7)\ g!}{(g-d+5)!\ (g-d+3)!\ (g-d+2)!} .$$

We have in this way the relation $2gB - D = \deg(\overline{FL})$, which allows us to determine B.

We are left with the task of determining the coefficients a_i of $\delta_{i:\{1\}}$, when $1 \le i \le g-1$. For this purpose, we use a new test curve. Take (B, q) a general 1-pointed curve of genus i and (C, q, p_2) a general 2-pointed curve of genus $g-i$. We take $X := B \cup_q C$ and consider as marked points a moving point $p_1 \in B$ and the fixed point $p_2 \in C$. For this family we have that

$$\deg(\psi_1) = 2i-1,\ \deg(\psi_2) = 0,\ \deg(\delta_{g-i:\{1,2\}}) = 1,\ \deg(\delta_{i:\{1\}}) = -1.$$

while λ and the remaining δ's vanish. Thus we have the equation

$$a_i = -(2i-1)B + b_{g-i} + \deg(\overline{FL}). \tag{3.11}$$

where the only unknown in the left-hand side is $\deg(\overline{FL})$ which we now determine.

We have to solve the following problem: let (Y, p) a general 1-pointed curve of genus g degenerating to $(X = B \cup_q C, p_2)$. There are $\sigma_{(0,0,1)}\ \sigma^g_{(0,1,1)}$ linear series $\mathfrak{g}^2_d$ on Y having a flex at p. Fix one of them. How many of its ramification points will end up on the genus i component B as Y specializes to X?

To answer this we let X further degenerate to $X' := E_1 \cup \ldots \cup E_g$, a string of g elliptic curves, the marked point, call it r_0 being on E_1. We assume that the component C of X degenerates to $\cup_{j=1}^{g-i} E_j$, whereas B degenerates to $\cup_{j=g-i+1}^{g} E_j$. If $\{r_i\} = E_i \cap E_{i+1}$, assume $r_i - r_{i-1} \in \mathrm{Pic}^0(E_i)$ is not a torsion class. Pick l one of the limit $\mathfrak{g}^2_d$'s on X' that have a flex at r_0. Because of our assumptions, $\rho(l_{E_i}, \alpha^{l_{E_i}}(r_{i-1}), \alpha^{l_{E_i}}(r_i)) = 0$, for $1 \le i \le g$. By Plücker, the aspect l_{E_i} has 8 flexes which are smooth points of X', which means that there will be $8i$ flexes on the components $E_{g-i+1}, \ldots, E_g$, hence finally,

$$\deg(\overline{FL}) = 8i\sigma_{(0,0,1)}\ \sigma^g_{(0,1,1)} = \frac{48ig!}{(g-d+5)!\ (g-d+3)!\ (g-d+2)!}.$$

Substituting in (3.11) we have the coefficients a_i as well. We have determined all terms in the expression of $[\overline{FL}]$. $\square$

Remark: We discuss now the case $g = 22, d = 17$, which as we already pointed out, was the initial motivation for computing $[\overline{FL}]$.
One tries to show that the Kodaira dimension of the moduli space $\mathcal{M}_{22,2}$ is ≥ 0 by exhibiting an explicit effective multicanonical divisor. Recall (cf. Chap. 1) that on $\overline{\mathcal{M}}_{23}$ the Brill-Noether divisor $\overline{\mathcal{M}}^2_{23,17}$ of curves with a $\mathfrak{g}^2_{17}$, is multicanonical (modulo a positive combination of the classes δ_i, for $i \ge 1$). Therefore, it seemed possible that on $\overline{\mathcal{M}}_{22,2}$, the divisor $\overline{FL}$ of 2-pointed curves with a $\mathfrak{g}^2_{17}$ having flexes at both marked points, would be multicanonical as well.

The canonical class $K_{\overline{\mathcal{M}}_{g,2}}$ can be computed easily (see also [Lo]): if $\pi_2 : \overline{\mathcal{M}}_{g,2} \to \overline{\mathcal{M}}_{g,1}$ and $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ are the natural maps, then $K_{\overline{\mathcal{M}}_{g,2}} = \pi_2^*(K_{\overline{\mathcal{M}}_{g,1}}) + c_1(\omega_{\pi_2})$ and $K_{\overline{\mathcal{M}}_{g,1}} = \pi^*(K_{\overline{\mathcal{M}}_g}) + \psi$, which gives,

$$K_{\overline{\mathcal{C}}_g} = 13\lambda + \psi - 2\delta_0 - 3(\delta_1 + \delta_{g-1}) - 2\sum_{i=2}^{g-2}\delta_i, \quad \text{and}$$

$$K_{\overline{\mathcal{M}}_{g,2}} = 13\lambda + \psi_1 + \psi_2 - 2\delta_0 - 2\delta_{0:\{1,2\}} - 3\sum_{A}\delta_{1:A} - 2\sum_{i\ge 2,A}\delta_{i:A}.$$

To make computations easier to handle, we introduce the following notation: for D_1 and D_2 divisor classes on $\overline{\mathcal{M}}_{g,n}$, we write

$D_1 \ge_\delta D_2 \iff D_1 - D_2$ is a non-negative combination of the classes $\delta_{i:A}$, where $i \ge 1$.

Our Theorem 3.3 shows that in the case $g = 22$, we have that

$$\frac{247}{2}\ \lambda + \frac{253}{2}\ (\psi_1 + \psi_2) - 242\ \delta_{0:\{1,2\}} - \frac{229}{12}\ \delta_0 \geq_\delta 0,$$

(the left-hand side is modulo boundary classes a rational multiple of $[\overline{FL}]$). Other known effective divisor classes on $\overline{\mathcal{M}}_{22,2}$ are the following:

- The class of the Weierstrass divisor Wei $:= \pi_1^*(\overline{\mathcal{W}}) + \pi_2^*(\overline{\mathcal{W}})$, that is the closure of the locus of those $[C, p_1, p_2]$ for which either p_1 or p_2 is a Weierstrass point of C. One has that $[\text{Wei}] \leq_\delta -2\ \lambda + 253\ (\psi_1 + \psi_2) - 506\ \delta_0$.

- The pullback of the divisor $\overline{E}^1_{12}$ from $\overline{\mathcal{M}}_{22}$, that is, the closure of the locus of those $[C, p_1, p_2]$ for which C has a $\mathfrak{g}^1_{12}$ such that dim $2\mathfrak{g}^1_{12} \geq 3$. One has that that $[\overline{E}^1_{12}] \leq_\delta 870\ \lambda - 132\ \delta_0$.

- The closure of the locus $D := \{[C, p_1, p_2] \in \mathcal{M}_{22,2} : h^0(C, 11p_1 + 11p_2) \geq 2\}$. One has (cf. [Lo]) that $[\overline{D}] \leq_\delta -\lambda + 66\ (\psi_1 + \psi_2) - 253\ \delta_{0:\{1,2\}}$.

One can then show that $[\overline{FL}]$ is not expressible as a positive linear combination of $[\text{Wei}], [\overline{E}^1_{12}]$ and $[\overline{D}]$, so by knowing $[\overline{FL}]$ we really extend the knowledge of the effective cone on $\text{Pic}(\overline{\mathcal{M}}_{22,2})$.

On the other hand, one sees that $K_{\overline{\mathcal{M}}_{22,2}}$ cannot be expressed as a positive combination of the four effective classes mentioned before. Although our computation of $[\overline{FL}]$ provides a new effective divisor class on $\overline{\mathcal{M}}_{22,2}$, this enlargement is not big enough to include the canonical class. In the spirit of the slope conjecture that predicts that on $\overline{\mathcal{M}}_g$ the effective divisors of lowest slope are the Brill-Noether divisors, since $K_{\overline{\mathcal{M}}_{22,2}}$ lies outside the Brill-Noether subspace in $\text{Pic}(\overline{\mathcal{M}}_{22,2})$, we make the following:

Conjecture 2 *The Kodaira dimension of $\mathcal{M}_{22,2}$ (and hence that of $\mathcal{M}_{22,1}$) is $-\infty$.*

3.6 The divisor of curves with two triple ramification points

In this section we compute the class of the divisor $\overline{TR}$ of 1-pointed curves that admit a map to $\mathbb{P}^1$ having both the marked point and some unspecified point as triple ramification points.

Let us fix an integer $d \geq 3$ and we set $g := 2d - 4$. For a general 1-pointed curve (C, p) of genus g the variety of pencils $G^1_d(C)$ is an irreducible smooth surface. Among the ∞^2 pencils of degree d there are finitely many $l \in G^1_d(C)$ for which p is a triple ramification point, that is, $a^l_1(p) \geq 3$. Moreover, all linear series l satisfying this condition are complete, base point free and all ramification apart from p is ordinary. Imposing the condition that there exists a degree d map $f : C \to \mathbb{P}^1$ with two triple ramification points, one of which is marked, we obtain a codimension 1 condition on $\overline{\mathcal{M}}_{g,1}$. We have the following:

Theorem 3.4 *Let $d \geq 3$ be an integer and set $g := 2d-4$. We have the following relation in* $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$:

$$[\overline{TR}] = m\ (a\ \lambda + b\ \psi - \sum_{i=0}^{g-1} c_i\ \delta_i).$$

where

$a = 2(18d^3 - 39d^2 - 120d + 290)$, $b = 8(6d^2 - 28d + 35)(2d - 4)$,

$c_0 = 6d^3 - 24d^2 + 13d + 30$,

$c_i = 2(2d - 4 - i)(24d^2 + 9id^2 - 112d - 42id + 140 + 50i)$ *for* $i \geq 1$ *and*

$m = 6(2d - 6)!/(d!\ (d - 3)!)$.

Remark: In the simplest case $d = 3, g = 2$ our formula gives the relation

$$[\overline{TR}] = 80\psi + 10\delta_0 - 120\lambda. \tag{3.12}$$

where λ is a boundary class on $\overline{\mathcal{M}}_{2,1}$, namely $\lambda = \delta_0/10 + \delta_1/5$ (cf. [EH3]). For genus 2 one has the following interpretation for our divisor

$$TR = \{[C,p] \in \mathcal{M}_{2,1} : \exists x \in C, x \neq p, \text{ such that } 3p \sim 3x\ \}.$$

During the proof of Theorem 3.4 we will need the result for the particular case $g = 2$, hence we will settle this case independently.

3.6.1 Counting pencils with two triple points

In order to determine the intersection multiplicities of $\overline{TR}$ and various test curves in $\overline{\mathcal{M}}_{g,1}$ we will need certain enumerative results contained in the following result:

Proposition 3.6.1 *1) Let (C,p,q) be a general 2-pointed curve of genus $2d - 6$ with $d \geq 3$. The number of pencils $\mathfrak{g}^1_d$ on C having triple points at both p and q is*

$$F(d) = (2d - 6)!\ \Big(\frac{1}{(d-3)!^2} - \frac{1}{d!\ (d-6)!}\Big).$$

2) Let C be a general curve of genus $2d - 4$ with $d \geq 3$. The number of pencils $\mathfrak{g}^1_d$ on C having triple ramification at some distinct points $x, y \in C$ is

$$N(d) = \frac{48(6d^2 - 28d + 35)\ (2d - 4)!}{d!\ (d-3)!}\ .$$

Remarks: 1. In the expression of $F(d)$ we make the convention $1/n! = 0$ for $n < 0$.
2. For $d = 3$ our formula gives $N(3) = 80$, that is, for a general curve C of genus 2 there are $160 = 2 \cdot 80$ pairs of points $(x, y) \in C \times C$, $x \neq y$, such that $3x \sim 3y$. This can also be seen directly by considering the map $\psi : C \times C \to \mathrm{Pic}^0(C)$ given by $\psi(x, y) = \mathcal{O}_C(3x - 3y)$. Then $\psi^*(0) = \frac{1}{2}\int_{C\times C} \psi^*(\omega \wedge \omega) = 2 \cdot 3^2 \cdot 3^2 = 162$, where ω is a

differential form representing θ. To get the answer to our enumerative question we have to subtract from 162 the contribution of the diagonal $\Delta \subseteq C \times C$. This excess intersection contribution is equal to 2 (cf. [Di]), so in the end we get $160 = 162 - 2$ pairs of distinct points $(x, y) \in C \times C$ with $3x \sim 3y$.

Proof: 1) We let (C, p, q) degenerate to the following 2-pointed curve of compact type $(C_0 := \mathbb{P}^1 \cup E_1 \cup \ldots \cup E_{2d-6}, p_0, q_0)$, where E_i are general elliptic curves, $\{p_i\} = E_i \cap \mathbb{P}^1$ and $p_1, \ldots, p_{2d-6}, p_0, q_0 \in \mathbb{P}^1$ are general points. We have to count the number of limit $\mathfrak{g}^1_d$'s on C_0 having triple ramification at p_0 and q_0. This is the same as the number of $\mathfrak{g}^1_d$'s on $\mathbb{P}^1$ having cusps (i.e. ordinary ramification) at $p_1, \ldots, p_{2d-6}$ and triple ramification at p_0 and q_0. By Prop.3.3.1 this number is $\sigma^2_{(0,2)}\ \sigma^{2d-6}_{(0,1)}$ (in $H^{top}(\mathbb{G}(1,d), \mathbb{Z})$). This product can be computed using formula (v) at the bottom of page 273 in [F] and one has that

$$\sigma^2_{(0,2)}\ \sigma^{2d-6}_{(0,1)} = (2d-6)!\ \left(\frac{1}{(d-3)!^2} - \frac{1}{d!\ (d-6)!}\right).$$

2) Once more, we let C degenerate to $C_0 = \mathbb{P}^1 \cup E_1 \cup \ldots \cup E_{2d-4}$, where E_i are general elliptic curves, $\{p_i\} = \mathbb{P}^1 \cap E_i$ and $p_1, \ldots, p_{2d-4} \in \mathbb{P}^1$ are general points. We count limit $\mathfrak{g}^1_d$'s on C_0 with vanishing $\geq (0,3)$ at two distinct points $x, y \in C_0$. Let l be such a limit $\mathfrak{g}^1_d$. By a standard argument we have already outlined before, we can assume that both x and y are smooth points of C_0 and by the additivity of the Brill-Noether number we obtain that x, y must lie on the tails E_i. Since E_i are general, we can assume that $j(E_i) \neq 0$ (that is, none of the E_i's is the Fermat cubic), hence there can be no $\mathfrak{g}^1_3$ on E_i with three triple points. There are two cases:

a) There are $1 \leq i < j \leq 2d-4$ such that $x \in E_i$ and $y \in E_j$. Then $a^{l_{E_i}}(p_i) = a^{l_{E_j}}(p_j) = (d-3, d)$, hence $3x \sim 3p_i$ on E_i and $3y \sim 3p_j$ on E_j. There are 8 choices for $x \in E_i$, 8 choices for $y \in E_j$ and $\binom{2d-4}{2}$ choices for the tails E_i and E_j containing the triple points. On $\mathbb{P}^1$ we count $\mathfrak{g}^1_d$'s with cusps at $\{p_1, \ldots, p_{2d-4}\} - \{p_i, p_j\}$ and triple points at p_i and p_j. This number is again $\sigma^2_{(0,2)}\ \sigma^{2d-6}_{(0,1)}$, so we get in this case a contribution of

$$64\binom{2d-4}{2}\sigma^2_{(0,2)}\ \sigma^{2d-6}_{(0,1)} = 32(2d-4)!\ \left(\frac{1}{(d-3)!^2} - \frac{1}{d!\ (d-6)!}\right). \qquad (3.13)$$

b) There is $1 \leq i \leq 2d-4$ such that $x, y \in E_i$. We distinguish between two cases here:

b_1) $a^{l_{E_i}}(p_i) = (d-3, d-1)$. On $\mathbb{P}^1$ we count $\mathfrak{g}^1_{d-1}$'s with cusps at $p_1, \ldots, p_{2d-4}$ and this number is $\sigma^{2d-4}_{(0,1)}$ (in $H^{top}(\mathbb{G}(1, d-1), \mathbb{Z})$). On E_i we have to compute the number of $\mathfrak{g}^1_3$'s having triple ramification at some unspecified points $x, y \in E_i - \{p_i\}$ and which also have simple ramification at p_i. Let us denote $(E_i, p_i) = (E, p)$. If we regard $p \in E$ as the origin of E, then the translation $(x, y) \mapsto (y - x, -x)$ establishes a bijection between the set of pairs $(x, y) \in E \times E - \Delta$, $x \neq p \neq y$, such that there is a $\mathfrak{g}^1_3$ in which x, y, p appear with multiplicities 3, 3 and 2 respectively, and the set of pairs $(u, v) \in E \times E - \Delta$, with $u \neq p \neq v$ such that there is a $\mathfrak{g}^1_3$ in which u, v, p appear with multiplicities 3, 2 and 3 respectively. The latter set has obviously cardinality 16, hence the number of pencils $\mathfrak{g}^1_3$

we are counting is $8 = 16/2$. All in all we have a contribution of

$$8(2d-4)\ \sigma_{(0,1)}^{2d-4} = \frac{8(2d-4)\ (2d-4)!}{(d-2)!\ (d-1)!}\ . \tag{3.14}$$

b_2) $a^{l_{E_i}}(p_i) = (d-4, d)$. This time, on $\mathbb{P}^1$ we look at $\mathfrak{g}_d^1$'s with cusps at $\{p_1, \dots, p_{2d-4}\} - \{p_i\}$ and a 4-fold point at p_1. Their number is $\sigma_{(0,3)}\ \sigma_{(0,1)}^{2d-5}$ (in $H^{top}(\mathbb{G}(1,d),\mathbb{Z})$). On E_i we shall compute the number of $\mathfrak{g}_4^1$'s for which there are distinct points $x, y \in E_i - \{p_i\}$ such that p, x, y appear with multiplicities 4, 3 and 3 respectively. Again, for simplicity we denote $(E_i, p_i) = (E, p)$ and we proceed as follows: We consider Σ the closure in $E \times E$ of the locus

$$\{(u,v) \in E \times E - \Delta : \exists l \in G_4^1(E) \text{ such that } a_1^l(p) = 4, a_1^l(u) \geq 3, a_1^l(v) \geq 2\}.$$

The class of the curve Σ can be computed readily. If F_i denotes the numerical equivalence class of a fibre of the projection $\pi_i : E \times E \to E$, for $i = 1, 2$, then

$$\Sigma \sim 10F_1 + 5F_2 - 2\Delta. \tag{3.15}$$

The coefficients in this expression are determined by intersecting Σ with Δ and the fibres of π_i. One has that $\Sigma \cap \Delta = \{(x,x) \in E \times E : x \neq p, 4p \sim 4x\}$ and $\Sigma \cap \pi_2^{-1}(p) = \{(y,p) \in E \times E : y \neq p, 3p \sim 3y\}$. It is easy to check that these intersections are transversal, hence $\Sigma \cdot \Delta = 15, \Sigma \cdot F_2 = 8$ whereas obviously $\Sigma \cdot F_1 = 3$ and these relations yield (3.15).

The number of pencils $l \subseteq |4p|$ having two extra triple points will then be equal to $1/2\ \#(\text{ramification points of } \pi_2 : \Sigma \to E) = \Sigma^2/2 = 20$. We have obtained in this case a contribution of

$$20(2d-4)\ \sigma_{(0,3)}\ \sigma_{(0,1)}^{2d-5} = 80\frac{(2d-4)!}{(d-4)!\ d!}\ . \tag{3.16}$$

Adding together (3.13),(3.14) and (3.16), we obtain the stated number $N(d)$. □

3.6.2 A divisor class on $\overline{\mathcal{M}}_{2,1}$

Here we compute the class of $\overline{TR}$ when $g = 2$. We have the following:

Proposition 3.6.2 *Let us consider the divisor*

$$TR = \{[C,p] \in \mathcal{M}_{2,1} : \exists x \in C - \{p\}, \text{ such that } 3x \sim 3p\}.$$

Then $[\overline{TR}] = 80\psi + 10\delta_0 - 120\lambda$.

Proof: There are a few ways to compute $[\overline{TR}]$. One is to consider the map $j : \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_4$ given by $j([B,p]) := [B \cup_p C_0]$, where (C_0, p) is a general 1-pointed curve of genus 2. On $\overline{\mathcal{M}}_4$ we have the divisor of curves with an abnormal Weierstrass point, that is,

$$D := \{[C] \in \mathcal{M}_4 : \exists x \in C \text{ such that } h^0(C, 3x) \geq 2\}.$$

One knows (cf. [Di]) that

$$[\overline{D}] = 264\lambda - 30\delta_0 - 96\delta_1 - 128\delta_2. \tag{3.17}$$

We claim that $j^*(\overline{D}) = \overline{TR} + 16\ \overline{\mathcal{W}}$. Indeed, let $[B, p] \in \mathcal{M}_{2,1}$ be such that $j([B,p]) \in \overline{D}$. Then there is a limit $\mathfrak{g}^1_3$ on $X = B \cup_p C_0$, say l, which has a point of total ramification at some $x \in X$. There are two cases depending on whether x lies on C_0 or on B.

If $x \in B$, then $a^{l_{C_0}}(p) = (0,3)$, hence $l_{C_0} = |3p|$ (and there is a single choice for l_{C_0}), while on B we have the linear equivalence $3p \sim 3x$, hence $[B,p] \in TR$.

If $x \in C_0$, then $a^{l_B}(p) = (1,3)$, i.e. $p \in B$ is a Weierstrass point and $l_B = p + |2p|$. On C_0 we have $a^{l_{C_0}}(p) = (0,2)$ and $a^{l_{C_0}}(x) = (0,3)$, so there exists $y \in C_0 - \{p, x\}$ such that $3x \sim 2p + y$. To compute the number of such points $x \in C_0 = C$, we intersect the curves $f_1(C)$ and $f_2(C)$ inside $\mathrm{Pic}^3(C)$, where $f_i : C \to \mathrm{Pic}^3(C)$ are given by $f_1(t) = \mathcal{O}_C(3t)$ and $f_2(t) = \mathcal{O}_C(2p+t)$ respectively. Clearly $[f_1(C)] = 9\theta$ and $[f_2(C)] = \theta$, hence $f_1(C)\cdot f_2(C) = 18$. However, we have to discard from this intersection the point $\mathcal{O}_C(3p)$ at which the condition $x \neq p$ is no longer satisfied. At this point the curves $f_i(C)$ have a common tangent line and using Lemma 6.2 or 6.4 from [Di] one gets that the intersection multiplicity at $\mathcal{O}_C(3p)$ is actually 2. The answer to our enumerative problem is thus $16 = 18 - 2$.

We have proved that $j^*(\overline{D}) = \overline{TR} + 16\ \overline{\mathcal{W}}$. From this and from (3.17) we get the expression for $[\overline{TR}]$ if we take into account that $j^*(\delta_0) = \delta_0$, $j^*(\delta_1) = \delta_1$, $j^*(\delta_2) = -\psi$ and $j^*(\lambda) = \lambda = \delta_0/10 + \delta_1/5$. □

3.6.3 The class of the divisor $\overline{TR}$

We now compute the class of the divisor $\overline{TR}$ in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$:

Proof of Theorem 3.4: By Prop. 3.2.1 there are rational constants $A, B, C_0, \ldots, C_{g-1}$ such that the following relation holds:

$$[\overline{TR}] = A\ \lambda + B\ \psi - \sum_{i=0}^{g-1} C_i\ \delta_i.$$

We first consider the map $\rho : \overline{\mathcal{M}}_{0,g+1} \to \overline{\mathcal{M}}_{g,1}$ obtained by associating to a $(g+1)$-pointed curve $(R, p_0, \ldots, p_g)$ of genus 0 the 1-pointed curve (C, p_0), with $C = R \cup E_1 \cup \ldots \cup E_g$ and $\{p_i\} = E_i \cap \mathbb{P}^1$ for $1 \le i \le g$, where E_i are general elliptic curves. We show that $\rho(\overline{\mathcal{M}}_{0,g+1}) \cap \overline{TR} = \emptyset$. Then by using Lemma 4.2 from [EH2] we obtain relations between the coefficients of $\overline{TR}$: for $i \ge 1$,

$$C_i = \frac{(g-i)(g-i-1)}{g(g-1)}\ B + \frac{i(g-i)}{g-1}\ C_{g-1}.$$

Note that Lemma 4.2 as stated in [EH2] is not applicable to the divisor $\overline{TR}$, but a brief inspection of its proof shows that its conclusions are valid for any divisor on $\overline{\mathcal{M}}_{g,1}$ whose support is disjoint from $\mathrm{Im}(\rho)$, hence for $\overline{TR}$ too.

The proof that $\mathrm{Im}(o) \cap \overline{TR} = \emptyset$ is an immediate application of Prop.1.4.1 together with Plücker's formula (3.2).

Next, we take the map $j : \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_{g,1}$ sending a 1-pointed curve (B, q) of genus 2 to $(X = B \cup_q C_0, p)$, where (C_0, p, q) is a general 2-pointed curve of genus $g-2$. The pull-back j^* acts on the generators of the Picard group as follows: $j^*(\lambda) = \lambda$, $j^*(\psi) = 0$, $j^*(\delta_0) = \delta_0$, $j^*(\delta_{g-2}) = -\psi$, $j^*(\delta_{g-1}) = \delta_1$ and $j^*(\delta_i) = 0$ for $1 \le i \le g-3$. Since on $\overline{\mathcal{M}}_{2,1}$ we also have the relation $\delta_1 = 5\lambda - \delta_0/2$, we obtain

$$j^*([\overline{TR}]) = (A - 5C_{g-1})\lambda + (C_{g-1}/2 - C_0)\delta_0 + C_{g-2}\psi. \tag{3.18}$$

We now compute $j^*(\overline{TR})$. Let us take $[B, q] \in j^*(\overline{TR})$. Then there exists l, a limit $\mathfrak{g}^1_d$ on $X = B \cup_q C_0$ and a point $x \in X$ such that $a^l_1(x) \ge 3$ and $a^l_1(p) \ge 3$.

If $x \in C_0$, then $\rho(l_B, a^{l_B}(q)) = -1$ and we get that $q \in B$ is a Weierstrass point. The multiplicity with which $\overline{\mathcal{W}}$ appears in $j^*(\overline{TR})$ is the number of $\mathfrak{g}^1_d$'s on C_0 in which p, q and an unspecified point $x \ne p, q$ appear with multiplicities 3, 2 and 3 respectively. By Schubert calculus this number is

$$n_1 = 8(2d-6)\ \sigma^2_{(0,2)}\ \sigma^{2d-6}_{(0,1)} = 8(2d-6)(2d-6)!\ \left(\frac{1}{(d-3)!^2} - \frac{1}{d!\ (d-6)!}\right).$$

If $x \in B$, then we have the linear equivalence $3q \sim 3x$ on B, that is, $[B, q] \in TR_2$, where we have denoted by TR_2 the divisor TR when $g = 2$. The multiplicity with which $\overline{TR}_2$ appears in $j^*(\overline{TR})$ is just the number of $\mathfrak{g}^1_d$'s on C_0 with triple ramification at the fixed points p and q. According to Prop.3.6.1 this number is $n_2 = F(d)$.

We have thus obtained that $j^*([\overline{TR}]) = n_1\ [\overline{\mathcal{W}}] + n_2\ [\overline{TR}_2]$, which according to (3.18) provides three new relations between A and the C_i's.

Finally we determine the coefficient B. It is enough to intersect $\overline{TR}$ with a general fibre of the map $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ and to divide the intersection number by $2g-2$. The intersection number is twice the number of $\mathfrak{g}^1_d$'s on a general curve C of genus $2d-4$ having two points of triple ramification. By Prop.3.6.1 this number is $2N(d)$, hence we obtain that $B = N(d)/(2d-5)$. □

3.7 The Kodaira dimension of the universal curve

At the beginning of Chapter 1 we recounted attempts by various people to understand the birational geometry of $\mathcal{M}_g$, in particular to compute its Kodaira dimension. Similar questions can be asked about the moduli spaces $\mathcal{M}_{g,n}$. Obviously the problem is non-trivial only for $g \le 23$: since for $g \ge 24$ the moduli space $\mathcal{M}_g$ is of general type, the spaces $\mathcal{M}_{g,n}$, with $n \ge 1$ will be of general type too.

The case $g = 23$ turns out to be quite easy too: since the relative dualizing sheaf of the map $\pi : \overline{\mathcal{M}}_{23,1} \to \overline{\mathcal{M}}_{23}$ is big one only needs the effectiveness of $K_{\overline{\mathcal{M}}_{23}}$ to conclude that $K_{\overline{\mathcal{M}}_{23,1}} = \psi + \pi^*(K_{\overline{\mathcal{M}}_{23}})$ is big too, hence $\mathcal{M}_{23,n}$ is of general type for all $n \ge 1$. In the case $4 \le g \le 22$, Logan computed a number $f(g)$ such that for all $n \ge f(g)$ the moduli space $\mathcal{M}_{g,n}$ is of general type (see [Lo]).

We shall content ourselves with the case $n = 1$ which we approach from a different angle: it is known that for $g \leq 16$, $g \neq 14$, the Kodaira dimension of $\mathcal{M}_g$ is $-\infty$. Is there a similar result for the universal curve $\mathcal{C}_g = \mathcal{M}_{g,1}$ in this range?

The problem is almost trivial for $g \leq 10$: the universal curve is unirational for these genera. To see this, one can easily adapt Severi's argument about the unirationality of $\mathcal{M}_g$. This is also remarked in [Lo]. For most remaining cases we have the following:

Theorem 1 *For $g = 11, 12, 15$ the Kodaira dimension of the universal curve $\mathcal{C}_g$ is $-\infty$.*

Proof: We assume that $\kappa(\mathcal{C}_g) \geq 0$, i.e. some multiple of the canonical divisor $K_{\overline{\mathcal{C}}_g}$ is effective. We are going to reach a contradiction with some estimates for the slope s_g of $\overline{\mathcal{M}}_g$ (see end of Chapter 1 for the definition of s_g).

We denote as usual by $\pi : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ the natural projection and we have seen that $K_{\overline{\mathcal{C}}_g} = 13\lambda + \psi - 3(\delta_1 + \delta_{g-1}) - 2\sum_{i=2}^{g-2} \delta_i$.

Assume there exists $m > 0$ such that $mK_{\overline{\mathcal{C}}_g}$ is effective. We consider the divisor of Weierstrass points $\overline{\mathcal{W}} \subseteq \overline{\mathcal{C}}_g$, whose class, we recall, is

$$[\overline{\mathcal{W}}] = -\lambda + g(g+1)/2\ \psi - \sum_{i=1}^{g-1}(g-i)(g-i+1)/2\ \delta_i.$$

Clearly $mK_{\overline{\mathcal{C}}_g}$ cannot contain $\overline{\mathcal{W}}$ with arbitrarily high mutiplicity. In fact, it suffices to choose $a \in \mathbb{Z}_{\geq 1}$ such that $ag(g+1)/2 > m$ and then $a\overline{\mathcal{W}} \not\subseteq mK_{\overline{\mathcal{C}}_g}$. Indeed, otherwise the divisor $mK_{\overline{\mathcal{C}}_g} - a\overline{\mathcal{W}}$ would have negative degree on the fibres $\pi^{-1}[C]$, where $[C] \in \mathcal{M}_g$ is arbitrary and this is impossible.

After choosing such an a, we consider the push-forward $D := \pi_*(mK_{\overline{\mathcal{C}}_g} \cdot a\overline{\mathcal{W}})$, which is an effective divisor on $\overline{\mathcal{M}}_g$. In particular, we have that $s_D \geq s_g$, where by s_D we denote the slope of D. Since

$$[D] = ma\ \pi_*(K_{\overline{\mathcal{C}}_g} \cdot [\overline{\mathcal{W}}]) \geq_\delta (13g^3 + 6g^2 - 9g + 2)\lambda - 1/2\ g(g+1)(4g-3)\ \delta,$$

we obtain that

$$s_D = \frac{2(13g^3 + 6g^2 - 9g + 2)}{g(g+1)(4g-3)}.$$

On the other hand, it is known that $s_{11} = 7 (= 6 + 12/(g+1))$, $s_{12} \geq 41/6 = 6.63...$ (cf. [Tan]) and $s_{15} \geq 6.667$ (cf. [CR4]). The values of s_D are 6.62... for $g = 11$, 6.61... for $g = 12$ and 6.59... for $g = 15$, hence in each of these cases we have found an effective divisor on $\overline{\mathcal{M}}_g$ having slope $\geq s_g$, which is a contradiction.

We note that there is also a bound $s_{16} \geq 6.56$ (cf. [CR4]), but for this genus we have $s_D = 6.58...$, so we cannot conclude that $\kappa(\mathcal{C}_{16}) = -\infty$. □

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Samenvatting

In dit proefschrift bestuderen we de meetkunde van de moduliruimte $\mathcal{M}_g$ van algebraïsche krommen van geslacht g. Deze ruimte $\mathcal{M}_g$ is de universele parameterruimte van krommen van geslacht g in die zin, dat ieder punt van $\mathcal{M}_g$ correspondeert met een isomorfieklasse van krommen.

Een belangrijk probleem in de algebraïsche meetkunde is de natuur van $\mathcal{M}_g$ als algebraïsche variëteit. Het is een klassiek resultaat dat voor $g \leq 10$ the moduliruimte $\mathcal{M}_g$ unirationaal is, terwijl $\mathcal{M}_g$ van algemeen type is voor $g \geq 24$, zoals Harris, Mumford en Eisenbud hebben bewezen. Men vermoedt dat alle moduliruimten $\mathcal{M}_g$ met $g \leq 22$ gedomineerd worden door regelvariëteiten, en dit laat één geval over, namelijk de moduliruimte $\mathcal{M}_{23}$, dat dan een overgangsgeval is tussen twee uitersten: gedomineerd door een regelvariëteit tegenover van algemeen type.

In Hoofdstuk 1 bewijzen we dat de Kodairadimensie van $\mathcal{M}_{23}$ minstens 2 is. We dragen verder feiten aan die suggereren, dat de Kodairadimensie van $\mathcal{M}_{23}$ gelijk is aan 2. Ons bewijs is gebaseerd op de expliciete studie van drie Brill-Noetherdivisoren op $\mathcal{M}_{23}$ die multikanoniek blijken te zijn.

In Hoofdstuk 2 bestuderen we de geografie (relatieve positie) van verschillende Brill-Noetherloci (dat wil zeggen, van loci van krommen die zekere $\mathfrak{g}^r_d$'s bezitten). In paragraaf 2.4 tonen we het bestaan aan van een reguliere component van het Hilbertschema van krommen van bigraad (k, d) in $\mathbb{P}^1 \times \mathbb{P}^r$ voor bepaalde k en d en $r \geq 3$. In paragraaf 2.5 construeren we gladde krommen $C \subset \mathbb{P}^3$ van graad d en geslacht g die de verwachte gonaliteit $\min(d-4, [(g+3)/2])$ bezitten. Als gevolg hiervan verkrijgen we een nieuw bewijs van ons resultaat $\kappa(\mathcal{M}_{23}) \geq 2$.

In Hoofdstuk 3 berekenen we de klasse van verschillende divisoren op de moduliruimten $\overline{\mathcal{M}}_{g,n}$ voor $n = 1, 2$. Deze divisoren worden gedefinieerd door meetkundige voorwaarden voor het bestaan van lineaire reeksen op krommen. In paragraaf 3.6 berekenen we de Kodairadimensie van de universele kromme $\mathcal{C}_g$ voor g gelijk aan 11, 12 en 15.

Curriculum Vitae

I was born on the 19th of March 1973 in the city of Oradea, Romania. Between 1987 and 1991 I studied at the Emanuil Gojdu Highschool in Oradea. In September 1991 I began studying mathematics at the Babes-Bolyai University in Cluj-Napoca, Romania. I graduated from university in June 1995 and in September 1995 I came to the Netherlands for the MRI organized Master Class in K-Theory. After succesfully completing this program, in September 1996 I started as a Ph.D. student (OIO) in algebraic geometry at the University of Amsterdam. After the defense of my thesis I am going to take up a three-year Assistant Professorship at the University of Michigan.