THE UNIVERSAL THETA DIVISOR OVER THE MODULI SPACE OF CURVES

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The universal theta divisor over the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g, is the divisor $\mathbf{\Theta}_g$ inside the universal abelian variety $\mathcal{X}_g$ over $\mathcal{A}_g$, characterized by two properties: (i) $\mathbf{\Theta}_{g|[A,\Theta]} = \Theta$, for every principally polarized abelian variety $[A,\Theta] \in \mathcal{A}_g$, and (ii) the restriction $s^*(\mathbf{\Theta}_g)$ along the zero section $s : \mathcal{A}_g \rightarrow \mathcal{X}_g$ is trivial on $\mathcal{A}_g$. The study of the geometry of $\mathbf{\Theta}_g$ closely mirrors that of $\mathcal{A}_g$ itself. Thus it is known that $\mathbf{\Theta}_g$ is unirational for $g \leq 4$; the case $g \leq 3$ is classical, for $g = 4$, we refer to [Ve]. The geometry of $\mathbf{\Theta}_5$ will be addressed in the forthcoming paper [FV3]. Whenever $\mathcal{A}_g$ is of general type (that is, in the range $g \geq 7$, cf. [Fr], [Mum], [T]), one can use Viehweg's additivity theorem [Vi] for the fibre space $\mathbf{\Theta}_g \rightarrow \mathcal{A}_g$ whose generic fibre is a variety of general type, to conclude that $\mathbf{\Theta}_g$ is of general type as well. The Kodaira dimension of $\mathbf{\Theta}_6$ (and that of $\mathcal{A}_6$) is unknown.

The main aim of this paper is to present a complete birational classification by Kodaira dimension of the universal theta divisor

$$\mathfrak{Th}_g := \mathcal{M}_g \times_{\mathcal{A}_g} \mathbf{\Theta}_g$$

over the moduli space of curves. If $[C] \in \mathcal{M}_g$ is a smooth curve, the Abel-Jacobi map $C_{g-1} \rightarrow \mathrm{Pic}^{g-1}(C)$ provides a resolution of singularities of the theta divisor Θ_C of the Jacobian of C. Thus one may regard the degree $g-1$ universal symmetric product $\overline{\mathcal{C}}_{g,g-1} := \overline{\mathcal{M}}_{g,g-1}/\mathfrak{S}_{g-1}$ as a birational model of $\mathfrak{Th}_g$ (having only finite quotient singularities), and ask for the place of $\mathfrak{Th}_g$ in the classification of varieties. We provide a complete answer to this question. For small genus, $\mathfrak{Th}_g$ enjoys rationality properties:

Theorem 0.1. *$\mathfrak{Th}_g$ is unirational for $g \leq 9$ and uniruled for $g \leq 11$.*

The first part of the theorem, is a consequence of Mukai's work [M1], [M2] on representing canonical curves with general moduli as linear sections of certain homogeneous varieties. When $g \leq 9$, there exists a Fano variety $V_g \subset \mathbf{P}^{N_g}$ of dimension $n_g := N_g - g + 2$ and index $n_g - 2$, such that general 1-dimensional complete intersections of V_g are canonical curves $[C] \in \mathcal{M}_g$ having general moduli. The correspondence

$$\Sigma := \big\{\big((x_1,\ldots,x_{g-1}),\Lambda\big) \in V_g^{g-1} \times G(g, N_g+1) : x_i \in \Lambda, \text{ for } i = 1,\ldots,g-1\big\}$$

maps dominantly onto $\mathfrak{Th}_g$ via the map $\big((x_1,\ldots,x_{g-1}),\Lambda\big) \mapsto [V_g \cap \Lambda, x_1 + \cdots + x_{g-1}]$. Since Σ is a Grassmann bundle over the rational variety V_g^{g-1}, it follows that $\mathfrak{Th}_g$ is unirational in the range $g \leq 9$. The cases $g = 10, 11$ are settled by the observation that in this range the space $\overline{\mathcal{M}}_{g,g-1}$ is uniruled, see [FP], [FV2].

For the remaining genera, we prove the following classification result:

Theorem 0.2. *The universal theta divisor $\mathfrak{Th}_g$ is a variety of general type for $g \geq 12$.*

We also have a birational classification theorem for the universal degree n symmetric product $\overline{\mathcal{C}}_{g,n} := \overline{\mathcal{M}}_{g,n}/\mathfrak{S}_n$ for all $1 \leq n \leq g-2$, and refer to Section 3 for details.

Our results are complete in degree $g-2$ and less precise as n decreases. Similarly to Theorem 0.2, the nature of $\overline{\mathcal{C}}_{g,g-2}$ changes when $g=12$:

Theorem 0.3. *The universal degree $g-2$ symmetric product $\overline{\mathcal{C}}_{g,g-2}$ is uniruled for $g<12$ and a variety of general type for $g\geq 12$.*

The proofs of Theorems 0.2 and 0.3 rely on two ingredients. First, we use our result [FV2], stating that for $g\geq 4$, the singularities of $\overline{\mathcal{C}}_{g,n}$ impose no *adjoint conditions*, that is, pluricanonical forms defined on the smooth locus of $\overline{\mathcal{C}}_{g,n}$ extend to a smooth model of the symmetric product. Precisely, if $\epsilon:\widetilde{\mathcal{C}}_{g,n}\to\overline{\mathcal{C}}_{g,n}$ denotes any resolution of singularities, then for any $l\geq 0$, there is a group isomorphism

$$\epsilon^*: H^0\big((\overline{\mathcal{C}}_{g,n})_{\mathrm{reg}}, K_{\overline{\mathcal{C}}_{g,n}}^{\otimes l}\big)\stackrel{\cong}{\to} H^0\big(\widetilde{\mathcal{C}}_{g,n}, K_{\widetilde{\mathcal{C}}_{g,n}}^{\otimes l}\big).$$

In particular, $\mathfrak{Th}_g$ is of general type when the canonical class $K_{\overline{\mathcal{C}}_{g,g-1}}\in\mathrm{Pic}(\overline{\mathcal{C}}_{g,g-1})$ is big. This makes the problem of understanding the effective cone of $\overline{\mathcal{C}}_{g,g-1}$ of some importance. If $\pi:\overline{\mathcal{M}}_{g,g-1}\to\overline{\mathcal{C}}_{g,g-1}$ is the quotient map, the Hurwitz formula gives that

$$\pi^*(K_{\overline{\mathcal{C}}_{g,g-1}})\equiv K_{\overline{\mathcal{M}}_{g,g-1}}-\delta_{0:2}\in\mathrm{Pic}(\overline{\mathcal{M}}_{g,g-1}). \tag{1}$$

The sum $\sum_{i=1}^{g-1}\psi_i\in\mathrm{Pic}(\overline{\mathcal{M}}_{g,g-1})^{\mathfrak{S}_{g-1}}$ of cotangent tautological classes descends to a big and nef class on $\overline{\mathcal{C}}_{g,g-1}$ (cf. Proposition 1.2), thus in order to conclude that $\mathfrak{Th}_g$ is of general type, it suffices to exhibit an effective divisor $\mathfrak{D}\in\mathrm{Eff}(\overline{\mathcal{C}}_{g,g-1})$, such that

$$\pi^*(K_{\overline{\mathcal{C}}_{g,g-1}})\in\mathbb{Q}_{>0}\Big\langle\lambda,\sum_{i=1}^{g-1}\psi_i\Big\rangle+\phi^*\mathrm{Eff}(\overline{\mathcal{M}}_g)+\mathbb{Q}_{\geq 0}\Big\langle\pi^*([\mathfrak{D}]),\delta_{i:c}: i\geq 0, c\geq 2\Big\rangle. \tag{2}$$

In this formula, $\phi:\overline{\mathcal{M}}_{g,g-1}\to\overline{\mathcal{M}}_g$ denotes the morphism forgetting the marked points, and refer to Section 1 for the standard notation for boundary divisor classes on $\overline{\mathcal{M}}_{g,n}$. Comparing condition (2) against the formula for $K_{\overline{\mathcal{C}}_{g,g-1}}$ given by (4), if one writes $\pi^*(\mathfrak{D})\equiv a\lambda-b_{\mathrm{irr}}\delta_{\mathrm{irr}}+c\sum_{i=1}^{g-1}\psi_i-\sum_{i,c}b_{i:c}\delta_{i:c}\in\mathrm{Pic}(\overline{\mathcal{M}}_{g,g-1})$, the following inequality

$$3c<b_{0:2} \tag{3}$$

is a necessary condition for the existence of a divisor $\mathfrak{D}$ satisfying (2). It is straightforward to unravel the geometric significance of the condition (3). If $[C]\in\mathcal{M}_g$ is a general curve, there is a rational map $u: C_{g-1}\dashrightarrow\overline{\mathcal{C}}_{g,g-1}$ given by restriction. Denoting by $x,\theta\in N^1(C_{g-1})_{\mathbb{Q}}$ the standard generators of the Néron-Severi group of the symmetric product, the inequality (3) characterizes precisely those divisors $\mathfrak{D}\in\mathrm{Pic}(\overline{\mathcal{C}}_{g,g-1})$ for which $u^*([\mathfrak{D}])$ lies in the fourth quarter of the (θ,x)-plane (see [K1] for details on the effective cone of C_{g-1}). The divisor $\mathfrak{D}\subset\overline{\mathcal{C}}_{g,g-1}$ playing this role in our case, is the residual divisor of the universal ramification locus of the Gauss map.

For a curve $[C]\in\mathcal{M}_g$, we denote by $\gamma: C_{g-1}\dashrightarrow\big(\mathbf{P}^{g-1}\big)^{\vee}$ the Gauss map, given by $\gamma(D):=\langle D\rangle$ for $D\in C_{g-1}-C^1_{g-1}$. The branch divisor $\mathrm{Br}_C(\gamma)\subset(\mathbf{P}^{g-1})^{\vee}$ is isomorphic to the dual of the canonical curve $C\subset\mathbf{P}^{g-1}$. The closure in C_{g-1} of the ramification divisor $\mathrm{Ram}_C(\gamma)$ is isomorphic to the diagonal $\Delta_C:=\{2p+D: p\in C,\ D\in C_{g-3}\}$, see [An]. In particular, this identification allows one to reconstruct the curve C from the theta divisor Θ_C and thus prove Torelli's theorem. Let us consider the *residual divisor* $\mathrm{Res}_C(\gamma)$, defined via the following equality of divisors on C_{g-1}

$$\gamma^*(\mathrm{Br}_C(\gamma)) = \mathrm{Res}_C(\gamma) + \mathrm{Ram}_C(\gamma).$$

Globalizing this construction over $\mathcal{M}_g$, we are lead to consider the effective divisor

$$\mathfrak{RT}_g := \{[C, x_1, \ldots, x_g] \in \mathcal{M}_{g,g-1} : \exists p \in C \text{ with } H^0\big(C, K_C(-x_1 - \cdots - x_{g-1} - 2p)\big) \neq 0\}.$$

The key ingredient in the proof of Theorem 0.2 is the calculation of the class of $\overline{\mathfrak{RT}}_g$:

Theorem 0.4. *The closure in $\overline{\mathcal{M}}_{g,g-1}$ of the locus $\mathfrak{RT}_g := \{[C, x_1, \ldots, x_{g-1}] \in \mathcal{M}_{g,g-1} : x_1 + \cdots + x_{g-1} \in \mathrm{Res}_C(\gamma)\}$ is linearly equivalent to,*

$$\overline{\mathfrak{RT}}_g \equiv -4(g-7)\lambda + 4(g-2)\sum_{i=1}^{g-1}\psi_i - 2\delta_{\mathrm{irr}} - (12g-22)\delta_{0:2} -$$

$$-\sum_{i=0}^{g}\sum_{s=0}^{i-1}\Big(2i^3 - 5i^2 - 3i + 4g - 4i^2 s + 14si - 6gs - s + 2s^2 g - 3s^2 + 2\Big)\delta_{i:s} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,g-1}).$$

In particular we note that condition (3) is satisfied. Since by construction, $\overline{\mathfrak{RT}}_g$ is $\mathfrak{S}_{g-1}$-invariant, it descends to an effective divisor $\widetilde{\mathfrak{RT}}_g$ on $\overline{\mathcal{C}}_{g,g-1}$ which, as it turns out, spans an extremal ray of the cone $\mathrm{Eff}(\overline{\mathcal{C}}_{g,g-1})$. Indeed, the universal theta divisor comes equipped with the rational involution $\tau : \overline{\mathcal{C}}_{g,g-1} \dashrightarrow \overline{\mathcal{C}}_{g,g-1}$ given by

$$\tau\big([C, x_1 + \cdots + x_{g-1}]\big) := [C, y_1 + \cdots + y_{g-1}],$$

where $\mathcal{O}_C(y_1 + \cdots + y_{g-1} + x_1 + \cdots + x_{g-1}) = K_C$. Then $\widetilde{\mathfrak{RT}}_g$ is the pull-back of the boundary divisor $\widetilde{\Delta}_{0:2} \subset \overline{\mathcal{C}}_{g,g-1}$ under this map. Since the extremality of $\widetilde{\Delta}_{0:2}$ is easy to establish, the following result comes naturally:

Theorem 0.5. *The effective divisor $\widetilde{\mathfrak{RT}}_g$ is covered by irreducible curves $\Gamma_g \subset \overline{\mathcal{C}}_{g,g-1}$ such that $\Gamma_g \cdot \widetilde{\mathfrak{RT}}_g < 0$. In particular $\widetilde{\mathfrak{RT}}_g \in \mathrm{Eff}(\overline{\mathcal{C}}_{g,g-1})$ is a non-movable extremal effective divisor.*

The curves Γ_g have a simple modular construction. One fixes a general linear series $A \in W^2_{g+1}(C)$, in particular A is complete and has only ordinary ramification points. The general point of Γ_g corresponds to an element $[C, D] \in \overline{\mathcal{C}}_{g,g-1}$, where $D \in C_{g-1}$ is an effective divisor such that $H^0\big(C, A \otimes \mathcal{O}_C(-2p - D)\big) \neq 0$, for some point $p \in C$, that is, D is the residual divisor cut out by a tangent line to the degree $g+1$ plane model of C given by A. Once more we refer to Section 2 for details.

We explain briefly how Theorem 0.4 implies the statement about the Kodaira dimension of $\overline{\mathcal{C}}_{g,g-1}$. We choose an effective divisor $D \equiv a\lambda - \sum_{i=0}^{[g/2]} b_i\delta_i \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ on the moduli space of curves, with $a, b_i \geq 0$, having slope $s(D) := \frac{a}{\min_i b_i}$ as small as possible. Then note that the following linear combination

$$\pi^*(K_{\overline{\mathcal{C}}_{g,g-1}}) - \frac{1}{6g-11}\Big(\frac{3}{2}[\overline{\mathfrak{RT}}_g] - (12g-25)\phi^*(D) - \sum_{i=1}^{g-1}\psi_i - \big((84g-185) - (12g-25)s(D)\big)\lambda\Big)$$

is expressible as a positive combination of boundary divisors on $\overline{\mathcal{M}}_{g,g-1}$. Since, as already pointed out, the class $\sum_{i=1}^{g-1}\psi_i \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,g-1})$ descends to a big class on $\overline{\mathcal{C}}_{g,g-1}$, one obtains the following:

Corollary 0.6. *For all g such that the slope of the moduli space of curves satisfies the inequality*

$$s(\overline{\mathcal{M}}_g) := \inf_{D\in \mathrm{Eff}(\overline{\mathcal{M}}_g)} s(D) < \frac{84g-185}{12g-25},$$

the universal theta divisor $\mathfrak{Th}_g$ is of general type.

The bound appearing in Corollary 0.6 holds precisely when $g \geq 12$; for g such that $g+1$ is composite, the inequality $s(\overline{\mathcal{M}}_g) \leq 6 + 12/(g+1)$ is well-known, and D can be chosen to be a Brill-Noether divisor $\overline{\mathcal{M}}^r_{g,d}$ corresponding to curves with a $\mathfrak{g}^r_d$ when the Brill-Noether number $\rho(g,r,d) = -1$, cf. [EH1]. When $g+1$ is prime and $g \neq 12$, then in practice $g = 2k-2 \geq 16$, and D can be chosen to be the Gieseker-Petri $\overline{\mathcal{GP}}^1_{g,k}$ consisting of curves C possessing a pencil $A \in W^1_k(C)$ such that the Petri map $\mu_0(C,A) : H^0(C,A) \otimes H^0(C, K_C \otimes A^{\vee}) \to H^0(C, K_C)$ is not an isomorphism. When $g = 12$, one has to use the divisor constructed on $\overline{\mathcal{M}}_{12}$ in [FV1]. Finally, when $g \leq 11$ it is known that $s(\overline{\mathcal{M}}_g) \geq 6 + 12/(g+1)$ and inequality (0.6) is not satisfied. In fact, as already pointed out $\kappa(\mathfrak{Th}_g) = -\infty$ in this range.

The proof of Theorem 0.3 proceeds along similar lines, and relies on finding an explicit $\mathfrak{S}_{g-2}$-invariant extremal ray of the cone of effective divisors on $\overline{\mathcal{M}}_{g,g-2}$. A representative of this ray is characterized by the geometric condition that the marked points appear in the same fibre of a pencil of degree $g-1$. One can construct such divisors on all moduli spaces $\overline{\mathcal{M}}_{g,n}$ with $1 \leq n \leq g-2$, cf. Section 3.

Theorem 0.7. *The closure inside $\overline{\mathcal{M}}_{g,g-2}$ of the locus*

$$\mathcal{F}_{g,1} := \{[C, x_1, \ldots, x_{g-2}] \in \mathcal{M}_{g,g-2} : \exists A \in W^1_{g-1}(C) \ \textit{with} \ H^0\big(C, A(-\sum_{i=1}^{g-2} x_i)\big) \neq 0\}$$

is a non-movable, extremal effective divisor on $\overline{\mathcal{M}}_{g,g-2}$. Its class is given by the formula:

$$\overline{\mathcal{F}}_{g,1} \equiv -(g-12)\lambda + (g-3)\sum_{i=1}^{g-2}\psi_i - \delta_{\mathrm{irr}} - \frac{1}{2}\sum_{s=2}^{g-2} s(g-4+sg-2s)\, \delta_{0:s} - \cdots \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,g-2}).$$

Note that again, inequality (3) is satisfied, hence $\overline{\mathcal{F}}_{g,1}$ can be used to prove that $K_{\overline{\mathcal{C}}_{g,g-2}}$ is big. Moreover, $\overline{\mathcal{F}}_{g,1}$ descends to an extremal divisor $\widetilde{\mathcal{F}}_{g,1} \in \mathrm{Eff}(\overline{\mathcal{C}}_{g,g-2})$. In fact, we shall show that $\widetilde{\mathcal{F}}_{g,1}$ is swept by curves intersecting its class negatively.

Divisors similar to those considered in Theorems 0.4 and 0.7 can be constructed on other moduli spaces. On $\overline{\mathcal{M}}_{g,g-3}$ we construct an extremal divisor using a somewhat similar construction. If $D \in C_{g-3}$ is a general effective divisor of degree $g-3$ on a curve $[C] \in \mathcal{M}_g$, we observe that $K_C \otimes \mathcal{O}_C(-D) \in W^2_{g+1}(C)$. A natural codimension one condition on $\overline{\mathcal{M}}_{g,g-3}$ is that this plane model have a triple point (a similar construction requiring instead that $K_C \otimes \mathcal{O}_C(-D)$ have a cusp, produces a "less extremal" divisor):

Theorem 0.8. *The closure inside $\overline{\mathcal{M}}_{g,g-3}$ of the locus*

$$\mathcal{D}_g := \{[C, x_1, \ldots, x_{g-3}] \in \mathcal{M}_{g,g-3} : \exists L \in W^2_g(C) \ \textit{with} \ H^0\big(C, L(-\sum_{i=1}^{g-3} x_i)\big) \neq 0\}$$

is an effective divisor. Its class in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,g-3})$ *is equal to*

$$\overline{\mathcal{D}}_g \equiv -\frac{2(g-17)}{3}\binom{g-3}{2}\lambda+\frac{2g-3}{3}\binom{g-4}{2}\sum_{i=1}^{g-3}\psi_i-\binom{g-3}{2}\delta_{\mathrm{irr}}-(g^2-5g+5)(g-5)\delta_{0:2}-\cdots.$$

1. Cones of divisors on universal symmetric products

The aim of this section is to establish certain facts about boundary divisors on $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{C}}_{g,n}$, see [AC] for a standard reference. We follow the convention set in [FV2], that is, if $\mathbf{M}$ is a Deligne-Mumford stack, we denote by $\mathcal{M}$ its coarse moduli space.

For an integer $0 \leq i \leq [g/2]$ and a subset $T \subset \{1,\ldots,n\}$, we denote by $\Delta_{i:T}$ the closure in $\overline{\mathcal{M}}_{g,n}$ of the locus of n-pointed curves $[C_1 \cup C_2, x_1,\ldots,x_n]$, where C_1 and C_2 are smooth curves of genera i and $g-i$ respectively meeting transversally in one point, and the marked points lying on C_1 are precisely those indexed by T. We define $\delta_{i:T} := [\Delta_{i:T}]_{\mathbb{Q}} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$. For $0 \leq i \leq [g/2]$ and $0 \leq s \leq g$, we set

$$\Delta_{i:s} := \sum_{\#(T)=s}\delta_{i:T}, \quad \delta_{i:s} := [\Delta_{i:s}]_{\mathbb{Q}} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n}).$$

By convention, $\delta_{0:s} := \emptyset$, for $s < 2$, and $\delta_{i:s} := \delta_{g-i:n-s}$. If $\phi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ is the morphism forgetting the marked points, we set $\lambda := \phi^*(\lambda)$ and $\delta_{\mathrm{irr}} := \phi^*(\delta_{\mathrm{irr}})$, where $\delta_{\mathrm{irr}} := [\Delta_{\mathrm{irr}}] \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ denotes the class of the locus of irreducible nodal curves. Furthermore, $\psi_1,\ldots,\psi_n \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ are the cotangent classes corresponding to the marked points. The canonical class of $\overline{\mathcal{M}}_{g,n}$ is computed via Kodaira-Spencer theory:

$$K_{\overline{\mathcal{M}}_{g,n}} \equiv 13\lambda - 2\delta_{\mathrm{irr}} + \sum_{i=1}^{n}\psi_i - 2\sum_{\substack{T\subset\{1,\ldots,n\}\\ i\geq 0}}\delta_{i:T} - \delta_{1:\emptyset} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n}). \tag{4}$$

Let $\overline{\mathcal{C}}_{g,n} := \overline{\mathcal{M}}_{g,n}/\mathfrak{S}_n$ be the universal symmetric product and $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$ (respectively $\varphi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_g$) the projection (respectively the forgetful map), so that $\phi = \varphi \circ \pi$. We denote by $\widetilde{\lambda}, \widetilde{\delta}_{\mathrm{irr}}, \widetilde{\delta}_{i:c} := [\widetilde{\Delta}_{i:c}] \in \mathrm{Pic}(\overline{\mathcal{C}}_{g,n})$ the divisor classes on the symmetric product pulling-back to the same symbols on $\overline{\mathcal{M}}_{g,n}$. Clearly, $\pi^*(\widetilde{\lambda}) = \lambda$, $\pi^*(\widetilde{\delta}_{\mathrm{irr}}) = \delta_{\mathrm{irr}}$, $\pi^*(\widetilde{\delta}_{i:c}) = \delta_{i:c}$; in the case $i = 0, c = 2$, this reflects the branching of the map π along the divisor $\widetilde{\Delta}_{0:2} \subset \overline{\mathcal{C}}_{g,n}$. Following [FV2], let $\mathbb{L}$ denote the line bundle on $\overline{\mathcal{C}}_{g,n}$, having fibre

$$\mathbb{L}[C, x_1 + \cdots + x_n] := T^{\vee}_{x_1}(C) \otimes \cdots \otimes T^{\vee}_{x_n}(C),$$

over a point $[C, x_1 + \cdots + x_n] := \pi([C, x_1,\ldots,x_n]) \in \overline{\mathcal{C}}_{g,n}$. We set $\widetilde{\psi} := c_1(\mathbb{L})$, and note:

$$\pi^*(\widetilde{\psi}) = \sum_{i=1}^{n}\Big(\psi_i - \sum_{i\in T\subset\{1,\ldots,n\}}\delta_{0:T}\Big) = \sum_{i=1}^{n}\psi_i - \sum_{s=2}^{n} s\,\delta_{0:s} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n}). \tag{5}$$

Proposition 1.1. *For $g \geq 3$ and $n \geq 0$, the morphism $\pi^* : \mathrm{Pic}(\overline{\mathcal{C}}_{g,n})_{\mathbb{Q}} \to \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}}$ is injective. Furthermore, there is an isomorphism of groups $\mathrm{Pic}(\overline{\mathcal{C}}_{g,n})_{\mathbb{Q}} \stackrel{\cong}{\to} N^1(\overline{\mathcal{C}}_{g,n})_{\mathbb{Q}}$.*

Proof. The first assertion is an immediate consequence of the existence of the norm morphism $\mathrm{Nm}_\pi : \mathrm{Pic}(\overline{\mathcal{M}}_{g,n}) \to \mathrm{Pic}(\overline{\mathcal{C}}_{g,n})$, such that $\mathrm{Nm}_\pi(\pi^*(L)) = L^{\otimes \deg(\pi)}$, for every $L \in \mathrm{Pic}(\overline{\mathcal{C}}_{g,n})$. The second part comes from the isomorphism $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}} \stackrel{\cong}{\to} N^1(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}}$, coupled with the commutativity of the obvious diagrams relating the Picard and Néron-Severi groups of $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{C}}_{g,n}$ respectively. □

One may thus identify $\mathrm{Pic}(\overline{\mathcal{C}}_{g,n})_{\mathbb{Q}} \cong \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}}^{\mathfrak{S}_n}$. The Riemann-Hurwitz formula applied to the branched covering $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$ yields,

$$\pi^*(K_{\overline{\mathcal{C}}_{g,n}}) = K_{\overline{\mathcal{M}}_{g,n}} - \delta_{0:2} \equiv 13\lambda + \sum_{i=1}^n \psi_i - 2\delta_{\mathrm{irr}} - 3\delta_{0:2} - 2\sum_{s=3}^n \delta_{0:s} - \cdots .$$

As expected, the sum of cotangent classes descends to a big line bundle on $\overline{\mathcal{C}}_{g,n}$.

Proposition 1.2. *The divisor class* $N_{g,n} := \widetilde{\psi} + \sum_{s=2}^n s\widetilde{\delta}_{0:s} \in \mathrm{Eff}(\overline{\mathcal{C}}_{g,n})$ *is big and nef.*

Proof. The class $N_{g,n}$ is characterized by the property that $\pi^*(N_{g,n}) = \sum_{i=1}^n \psi_i$. This is a nef class on $\overline{\mathcal{M}}_{g,n}$, in particular, $N_{g,n}$ is nef on $\overline{\mathcal{C}}_{g,n}$. To establish that $N_{g,n}$ is big, we express it as a combination of effective classes and the class $\widetilde{\kappa}_1 \in \mathrm{Pic}(\overline{\mathcal{C}}_{g,n})$, where

$$\pi^*(\widetilde{\kappa}_1) = \kappa_1 = 12\lambda + \sum_{i=1}^n \psi_i - \delta_{\mathrm{irr}} - \sum_{i=0}^{[g/2]} \sum_{s\geq 0} \delta_{i:s} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n}).$$

Since $\pi^*(\widetilde{\kappa}_1)$ is ample on $\overline{\mathcal{M}}_{g,n}$, it follows that $\widetilde{\kappa}_1$ is ample as well. To finish the proof, we exhibit a suitable effective class on $\overline{\mathcal{M}}_{g,n}$ having negative λ-coefficient. For that purpose, we choose $\mathcal{W}_{g,n} \subset \mathcal{C}_{g,n}$ to be the locus of effective divisors having a Weierstrass point in their support. For $i = 1, \ldots, n$, we denote by $\sigma_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,1}$ the morphism forgetting all but the i-th point, and let

$$\overline{\mathcal{W}} \equiv -\lambda + \binom{g+1}{2}\psi - \sum_{i=1}^{g-1} \binom{g-i+1}{2}\delta_{i:1} \in \mathrm{Eff}(\overline{\mathcal{M}}_{g,1}),$$

be the class of the divisor of Weierstrass points on the universal curve. Then one finds

$$\pi^*(\overline{\mathcal{W}}_{g,n}) \equiv \sum_{i=1}^n \sigma_i^*(\overline{\mathcal{W}}) = -n\lambda + \binom{g+1}{2}\sum_{i=1}^n \psi_i - \binom{g+1}{2}\sum_{s=2}^n s\delta_{0:s} - \cdots \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n}),$$

and $\overline{\mathcal{W}}_{g,n} \equiv -g\widetilde{\lambda} + \binom{g+1}{2}\widetilde{\psi} - \sum_{i=1}^{[g/2]}\sum_{s\geq 0} b_{i:s}\widetilde{\delta}_{i:s}$, where $b_{i:s} > 0$. One checks that $N_{g,n}$ can be written as a $\mathbb{Q}$-combination with positive coefficients of the ample class $\widetilde{\kappa}_1$, the effective class $[\overline{\mathcal{W}}_{g,n}]$ and other boundary divisor classes. In particular, $N_{g,n}$ is big. □

2. The universal ramification locus of the Gauss map

We begin the calculation of the divisor $\overline{\mathfrak{RT}}_g$, and for a start we consider its restriction $\mathfrak{RT}_g$ to $\mathcal{M}_{g,g-1}$. Recall that $\mathfrak{RT}_g$ is defined as the closure of the locus of pointed curves $[C, x_1, \ldots, x_{g-1}] \in \mathcal{M}_{g,g-1}$, such that there exists a holomorphic form on C vanishing at $x_1, \ldots, x_{g-1}$ and having an unspecified double zero.

Let $u : \mathbf{M}_{g,g-1}^{(1)} \to \mathbf{M}_{g,g-1}$ be the universal curve over the stack of $(g-1)$-pointed smooth curves and we denote by $\big([C, x_1, \ldots, x_{g-1}], p\big) \in \mathcal{M}_{g,g-1}^{(1)}$ a general point, where $[C, x_1, \ldots, x_{g-1}] \in \mathcal{M}_{g,g-1}$ and $p \in C$ is an arbitrary point. For $i = 1, \ldots, g-1$, let $\Delta_{ip} \subset \mathcal{M}_{g,g-1}^{(1)}$ be the diagonal divisor given by the equation $p = x_i$. Furthermore, for $i = 1, \ldots, g-1$ we consider as before the projections $\sigma_i : \mathbf{M}_{g,g-1}^{(1)} \to \mathbf{M}_{g,1}$ (respectively

$\sigma_p : \mathbf{M}_{g,g-1}^{(1)} \to \mathbf{M}_{g,1}$), obtained by forgetting all marked points except x_i (respectively p), and then set $K_i := \sigma_i^*(\omega_\phi) \in \mathrm{Pic}(\mathbf{M}_{g,g-1}^{(1)})$ and $K_p := \sigma_p^*(\omega_\phi) \in \mathrm{Pic}(\mathbf{M}_{g,g-1}^{(1)})$. We consider the following cartesian diagram of stacks

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{q} & \mathbf{M}_{g,g-1}^{(1)} \\ \downarrow f & & \downarrow \\ \mathbf{M}_{g,1} & \xrightarrow{\phi} & \mathbf{M}_g \end{array}$$

in which all the morphisms are smooth and ϕ (hence also q) is proper. For $1 \le i \le g-1$ there are tautological sections $r_i : \mathbf{M}_{g,g-1}^{(1)} \to \mathcal{X}$ as well as $r_p : \mathbf{M}_{g,g-1}^{(1)} \to \mathcal{X}$, and set $E_i := \mathrm{Im}(r_i)$, $E_p := \mathrm{Im}(r_p)$. Thus $\{E_i\}_{i=1}^{g-1}$ and E_p are relative divisors over q.

For a point $\big([C, x_1, \ldots, x_{g-1}], p\big) \in \mathcal{M}_{g,g-1}^{(1)}$, we denote $D := \sum_{i=1}^{g-1} x_i + 2p \in C_{g+1}$, and have the following exact sequence:

$$0 \to \frac{H^0\big(\mathcal{O}_C(D)\big)}{H^0(\mathcal{O}_C)} \to H^0\big(\mathcal{O}_D(D)\big) \stackrel{\alpha_D}{\to} H^1(\mathcal{O}_C) \to H^1\big(\mathcal{O}_C(D)\big) \to 0.$$

In particular, the morphisms α_D globalize to a morphism of vector bundles over $\mathbf{M}_{g,g-1}^{(1)}$

$$\alpha : \mathcal{A} := q_*\Big(\mathcal{O}_\mathcal{X}(\sum_{i=1}^{g-1} E_i + 2E_p)/\mathcal{O}_\mathcal{X}\Big) \to R^1 q_* \mathcal{O}_\mathcal{X}.$$

The subvariety $\mathcal{Z} := \big\{\big([C, x_1, \ldots, x_{g-1}], p\big) \in \mathcal{M}_{g,g-1}^{(1)} : H^0\big(K_C(-2p - \sum_{i=1}^{g-1} x_i)\big) \neq 0\big\}$ is the non-surjectivity locus of α and $\mathfrak{RT}_g := u_*(\mathcal{Z}) \subset \mathcal{M}_{g,g-1}$. The class of $\mathcal{Z}$ is equal to

$$[\mathcal{Z}] = c_2\Big(\mathcal{A}^\vee - \big(R^1 q_* \mathcal{O}_\mathcal{X}\big)^\vee\Big) = c_2\Big(-q_! \mathcal{O}_\mathcal{X}(\sum_{i=1}^{g-1} E_i + 2E_p)\Big) \in A^2(\mathbf{M}_{g,g-1}^{(1)}),$$

where the last term can be computed by Grothendieck-Riemann-Roch:

$$\mathrm{ch}\Big(q_! \mathcal{O}_\mathcal{X}(\sum_{i=1}^{g-1} E_i + 2E_p)\Big) = q_*\Big[\Big(\sum_{k \ge 0} \frac{(\sum_{i=1}^{g-1} E_i + 2E_p)^k}{k!}\Big) \cdot \Big(1 - \frac{c_1(\omega_q)}{2} + \frac{c_1^2(\omega_q)}{12} + \cdots\Big)\Big],$$

and we are interested in evaluating the terms of degree 1 and 2 in this expression. The result of applying GRR to the morphism q, can be summarized as follows:

Lemma 2.1. *One has the following relations in* $A^*(\mathbf{M}_{g,g-1}^{(1)})$*:*
(i)

$$\mathrm{ch}_1\Big(q_*\big(\mathcal{O}_\mathcal{X}(\sum_{i=1}^{g-1} E_i + 2E_p)\big)\Big) = \lambda - \sum_{i=1}^{g-1} K_i - 3K_p + 2\sum_{i=1}^{g-1} \Delta_{ip}.$$

(ii)

$$\mathrm{ch}_2\Big(q_*\big(\mathcal{O}_\mathcal{X}(\sum_{i=1}^{g-1} E_i + 2E_p)\big)\Big) = \frac{5}{2} K_p^2 + \frac{1}{2}\sum_{i=1}^{g-1} K_i^2 - 2\sum_{i=1}^{g-1} (K_i + K_p) \cdot \Delta_{ip}.$$

Proof. We apply systematically the push-pull formula and the following identities:

$$E_i^2 = -E_i \cdot q^*(K_i),\ E_p^2 = -E_p \cdot q^*(K_p),\ E_i \cdot c_1(\omega_q) = E_i \cdot q^*(K_i),\ E_p \cdot c_1(\omega_q) = E_p \cdot q^*(K_p),$$
$$E_i \cdot E_j = 0 \text{ for } i \neq j,\ E_i \cdot E_p = E_i \cdot q^*(\Delta_{ip}), \text{ and } q_*(c_1^2(\omega_q)) = 12\lambda.$$

□

Proposition 2.2. *The formula* $\mathfrak{RT}_g \equiv -4(g-7)\lambda + (4g-8)\sum_{i=1}^{g-1} \psi_i \in \mathrm{Pic}(\mathbf{M}_{g,g-1})$ *holds.*

Proof. We apply the results of Lemma 2.1, as well as the formulas from [HM] p. 55, in order to estimate the push-forward under u of the degree 2 monomials in tautological classes. Setting $\mathcal{F} := q_*\big(\mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{g-1} E_i + 2E_p)\big)$, we obtain that

$$u_*\big(\mathrm{ch}_1^2(\mathcal{F})\big) = -(8g-116)\lambda + (8g-24)\sum_{i=1}^{g-1}\psi_i, \text{ and } u_*\big(\mathrm{ch}_2(\mathcal{F})\big) = 30\lambda - 4\sum_{i=1}^{g-1}\psi_i,$$

hence $[\mathfrak{RT}_g] = u_*\big(\mathrm{ch}_1^2(\mathcal{F}) - 2\mathrm{ch}_2(\mathcal{F})\big)/2$, and the claimed formula follows at once. □

We proceed now towards proving Theorem 0.4 and expand the divisor class $[\overline{\mathfrak{RT}}_g] \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,g-1})$ in the standard basis of the Picard group, that is,

$$\overline{\mathfrak{RT}}_g \equiv a\lambda + c\sum_{i=1}^{g-1}\psi_i - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_{i=0}^{g}\sum_{s=0}^{i-1} b_{i:s}\delta_{i:s}.$$

We have just computed $a = -4(g-7)$ and $c = 4(g-2)$. The remaining coefficients are determined by intersecting $\overline{\mathfrak{RT}}_g$ with curves lying in the boundary of $\overline{\mathcal{M}}_{g,g-1}$ and understanding how $\overline{\mathfrak{RT}}_g$ degenerates. We begin with the coefficient $b_{0:2}$:

Proposition 2.3. *One has the relation* $(4g-6)c - (g-2)b_{0:2} = (4g-2)(g-2)$. *It follows that* $b_{0:2} = 12g - 22$.

Proof. We fix a general pointed curve $[C, x_1, \ldots, x_{g-2}] \in \mathcal{M}_{g,g-2}$ and consider the family

$$C_{x_{g-1}} := \big\{[C, x_1, \ldots, x_{g-2}, x_{g-1}] : x_{g-1} \in C\big\} \subset \overline{\mathcal{M}}_{g,g-1}.$$

The curve $C_{x_{g-1}}$ is the fibre over $[C, x_1, \ldots, x_{g-2}]$ of the morphism $\overline{\mathcal{M}}_{g,g-1} \to \overline{\mathcal{M}}_{g,g-2}$ forgetting the point labeled by x_{g-1}. Note that $C_{x_{g-1}} \cdot \psi_i = 1$ for $i = 1, \ldots, g-2$ and $C_{x_{g-1}} \cdot \psi_{g-1} = 3g-4 = 2g-2+(g-2)$. Obviously $C_{x_i} \cdot \delta_{0:2} = g-2$ and the points in the intersection correspond to the case when x_{g-1} collides with one of the fixed points $x_1, \ldots, x_{g-2}$. The intersection of C_{x_i} with the remaining generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,g-1})$ is equal to zero. We set $A := K_C \otimes \mathcal{O}_C(-x_1 - \cdots - x_{g-2}) \in W^1_g(C)$. By the generality assumption, $h^0(C, A) = 2$, and all ramification points of A are simple. Pointed curves in the intersection $C_{x_{g-1}} \cdot \overline{\mathfrak{RT}}_g$ correspond to points $x_{g-1} \in C$, such that there exists a (ramification) point $p \in C$ with $H^0\big(C, A \otimes \mathcal{O}_C(-2p - x_{g-1})\big) \neq 0$. The pencil A carries $4g-2$ ramification points. For each of them there are $g-2$ possibilities of choosing $x_{g-1} \in C$ in the same fibre as the ramification point, hence the conclusion follows. □

Next we determine the coefficient b_{irr}. First we note that the relation

$$a - 12b_{\mathrm{irr}} + b_{1:0} = 0 \qquad (6)$$

holds. Indeed, the divisor $\overline{\mathfrak{RT}}_g$ is disjoint from the curve in $\Delta_{1:0} \subset \overline{\mathcal{M}}_{g,g-1}$, obtained from a fixed pointed curve $[C, x_1, \ldots, x_{g-1}, q] \in \overline{\mathcal{M}}_{g-1,g}$, by attaching at the point q a pencil of plane cubics along a section of the pencil induced by one of the 9 base points.

Proposition 2.4. *One has the relation* $b_{\mathrm{irr}} = 2$.

Proof. We fix a general curve $[C, q, x_1, \ldots, x_{g-1}] \in \overline{\mathcal{M}}_{g-1,g}$, and we define the family

$$C_{\mathrm{irr}} := \big\{[C/t \sim q, x_1, \ldots, x_{g-1}] : t \in C\big\} \subset \Delta_{\mathrm{irr}} \subset \overline{\mathcal{M}}_{g,g-1}.$$

Then $C_{\mathrm{irr}} \cdot \psi_i = 1$ for $i = 1, \ldots, g-1$, $C_{\mathrm{irr}} \cdot \delta_{\mathrm{irr}} = -(\deg(K_C) + 2) = -2g + 2$, and finally $C_{\mathrm{irr}} \cdot \delta_{1:0} = 1$. All other intersection numbers with generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,g-1})$ equal zero.

We fix an effective divisor $D \in C_e$ of degree $e \geq g$ (for instance $D = q + \sum_{i=1}^{g-1} x_i$). For each pair of points $(t, p) \in C \times C$, there is an exact sequence on C

$$0 \to H^0\big(C, K_C(q + t - 2p - \sum_{i=1}^{g-1} x_i)\big) \to H^0\big(C, K_C(D + q + t - 2p - \sum_{i=1}^{g-1} x_i)\big) \stackrel{\beta_{t,p}}{\to}$$

$$H^0\big(D, K_C(D + q + t - 2p - \sum_{i=1}^{g-1} x_i)\big) \to H^1\big(C, K_C(q + t - 2p - \sum_{i=1}^{g-1} x_i)\big) \to 0.$$

The intersection $C_{\mathrm{irr}} \cdot \overline{\mathfrak{RT}}_g$ corresponds to the locus of pairs $(t, p) \in C \times C$ such that the map $\beta_{t,p}$ is not injective. On the triple product of C, we consider two of the projections $f : C \times C \times C \to C \times C$ and $p_1 : C \times C \times C \to C$ given by $f(x, t, p) = (t, p)$ and $p_1(x, t, p) = x$, then set $A := K_C(q - \sum_{i=1}^{g-1} x_i) \in \mathrm{Pic}^{g-2}(C)$. We denote by $\Delta_{12}, \Delta_{13} \subset C \times C \times C$ the corresponding diagonals, and finally, introduce the line bundle on $C \times C \times C$

$$\mathcal{F} := p_1^*(A) \otimes \mathcal{O}_{C \times C \times C}(\Delta_{12} - 2\Delta_{13}).$$

Applying the Porteous formula, one can write

$$C_{\mathrm{irr}} \cdot \overline{\mathfrak{RT}}_g = c_2(R^1 f_* \mathcal{F} - R^0 f_* \mathcal{F}) = \frac{\mathrm{ch}_1^2(f_! \mathcal{F}) + 2\mathrm{ch}_2(f_! \mathcal{F})}{2} \in A^2(C \times C).$$

We evaluate $\mathrm{ch}_i(f_! \mathcal{F})$ using GRR applied to the morphism f, that is,

$$\mathrm{ch}(f_! \mathcal{F}) = f_* \Big[\Big(\sum_{a \geq 0} \frac{\big(p_1^*(A) + \Delta_{12} - 2\Delta_{13}\big)^a}{a!}\Big) \cdot \Big(1 - \frac{1}{2} p_1^*(K_C)\Big)\Big].$$

Denoting by $F_1, F_2 \in H^2(C \times C)$ the class of the fibres, after calculations one finds that

$$\mathrm{ch}_1(f_* \mathcal{F}) = -(g-2)F_1 - 4(g-2)F_2 - 2\Delta_C \in H^2(C \times C, \mathbb{Q}),$$

$$\mathrm{ch}_2(f^* \mathcal{F}) = -2(g-2) \in H^4(C \times C, \mathbb{Q}),$$

that is, $c_2(R^1 f_* \mathcal{F} - R^0 f_* \mathcal{F}) = 4(g-2)(g-1)$. Coupled with (6), this yields $b_{\mathrm{irr}} = 2$. $\square$

We are left with the task of determining the coefficient of $\delta_{i:s}$ in the expansion of $[\overline{\mathfrak{RT}}_g]$. This requires solving a number of enumerative geometry problems in the spirit of de Jonquières' formula. We fix integers $0 \leq i \leq g$ and $s \leq i - 1$ as well as general pointed curves $[C, x_1, \ldots, x_s] \in \overline{\mathcal{M}}_{i,s}$ and $[D, q, x_{s+1}, \ldots, x_{g-1}] \in \overline{\mathcal{M}}_{g-i,g-s}$, then construct a pencil of stable curves of genus g, by identifying the fixed point $q \in D$ with a variable point, also denoted by q, on the component C:

$$C_{i:s} := \big\{[C \cup_q D, x_1, \ldots, x_s, x_{s+1}, \ldots, x_{g-1}] : q \in C\big\} \subset \Delta_{i:s} \subset \overline{\mathcal{M}}_{g,g-1}.$$

We summarize the non-zero intersection numbers of $C_{i:s}$ with generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,g-1})$:

$$C_{i:s} \cdot \psi_1 = \cdots = C_{i:s} \cdot \psi_s = 1, \;\; C_{i:s} \cdot \delta_{i:s-1} = i, \; C_{i:s} \cdot \delta_{i:s} = 2i - 2 + s.$$

Theorem 2.5. *We fix integers $0 \leq i \leq g$ and $0 \leq s \leq i-1$. Then, the following formula holds:*

$$b_{i:s} = 2i^3 - 5i^2 - 3i + 4g - 4i^2 s + 14si - 6gs - s + 2s^2 g - 3s^2 + 2.$$

In the proof an essential role is played by the following calculation:

Proposition 2.6. *Let i, s be integers such that $0 \leq s \leq i-1$, and $[C, x_1, \ldots, x_s] \in \mathcal{M}_{i,s}$ a general pointed curve. The number of pairs $(q,p) \in C \times C$ such that*

$$H^0\big(C, K_C \otimes \mathcal{O}_C(-x_1 - \cdots - x_s - (i-s-1)q - 2p)\big) \neq 0,$$

is equal to $a(i,s) := 2(i-s-1)(2i^3 - 5i^2 + i + 2 - 2i^2 s + 3is)$.

Remark 2.7. By specializing, one recovers well-known formulas in enumerative geometry. For instance, $a(3,0) = 56$ is twice the number of bitangents of a smooth plane quartic, whereas $a(4,0) = 324$ equals the number of canonical divisors of type $3q + 2p + x \in |K_C|$, where $[C] \in \mathcal{M}_4$. This matches de Jonquières' formula, cf. [ACGH] p.359.

Proof of Theorem 2.5. We fix a general point $[C \cup_q D, x_1, \ldots, x_{g-1}] \in C_{i:s} \cdot \overline{\mathfrak{RT}}_g$, corresponding to a point $q \in C$. We shall show that q is not one of the marked points $x_1, \ldots, x_s$ on C, then give a geometric characterization of such points and count their number. Let

$$\omega_D \in H^0\big(D, K_D \otimes \mathcal{O}_D(2iq)\big) \quad \text{and} \quad \omega_C \in H^0\big(C, K_C \otimes \mathcal{O}_C(2g-2i)q\big)$$

be the aspects of the section of the limit canonical series on $C \cup_q D$, which vanishes doubly at an unspecified point $p \in C \cup D$ as well as along the divisor $x_1 + \cdots + x_{g-1}$. The condition $\mathrm{ord}_q(\omega_C) + \mathrm{ord}_q(\omega_D) \geq 2g-2$, comes from the definition of a limit linear series. We distinguish two cases depending on the position of the point p. If $p \in D$ then,

$$\mathrm{div}(\omega_C) \geq x_1 + \cdots + x_s, \quad \mathrm{div}(\omega_D) \geq x_{s+1} + \cdots + x_{g-1} + 2p.$$

Since the points $q, x_{s+1}, \ldots, x_{g-1} \in D$ are general, we find that $\mathrm{ord}_q(\omega_D) \leq i + s - 2$. Moreover, $K_D \otimes \mathcal{O}_D((i-s+2)q - x_{s+1} - \cdots - x_{g-1}) \in W^1_{g-i+1}(D)$ is a pencil, and $p \in D$ is one of its (simple) ramification points. The Hurwitz formula gives $4(g-i)$ choices for such $p \in D$.

By compatibility, $\mathrm{ord}_q(\omega_C) \geq 2g - i - s$. A parameter count implies that equality must hold. The condition $H^0\big(C, K_C \otimes \mathcal{O}_C(-x_1 - \cdots - x_s - (i-s)q\big) \neq 0$, is equivalent to asking that $q \in C$ be a ramification point of $K_C \otimes \mathcal{O}_C(-\sum_{j=1}^s x_j) \in W^{i-s-1}_{2i-2-s}(C)$. Since the points $x_1, \ldots, x_s \in C$ are chosen to be general, all ramification points of this linear series are simple and occur away from the marked points. From Plücker's formula, the number of ramification points equals $(i-s)(i^2-1-is)$. Multiplying this with the number of choices for $p \in D$, we obtain a total contribution of $4(g-i)(i-s)(i^2-is-1)$ to the intersection $C_{i:s} \cdot \overline{\mathfrak{RT}}_g$, stemming from the case when $p \in D$. The proof that each of these points of intersection is to be counted with multiplicity 1 is standard and proceeds along the lines of [EH2] Lemma 3.4.

We assume now that $p \in C$. Keeping the notation from above, it follows that $\mathrm{ord}_q(\omega_D) = i + s - 1$ and $\mathrm{ord}_q(\omega_C) = 2g - i - s - 1$, therefore

$$0 \neq \sigma_C \in H^0\big(C, K_C \otimes \mathcal{O}_C(-\sum_{j=1}^{s} x_j - (i-s-1)q - 2p)\big).$$

The section ω_D is uniquely determined up to multiplication by scalars, whereas there are $a(i,s)$ choices on the side of C, each counted with multiplicity 1.

In principle, the double zero of the limit holomorphic form could specialize to the point of attachment $q \in C \cap D$, and we prove that this would contradict our generality hypothesis. One considers the semistable curve $X := C \cup_{q_1} E \cup_{q_2} D$, obtained from $C \cup D$ by inserting a smooth rational component E at q, where $\{q_1\} := C \cap E$ and $\{q_2\} := D \cap E$. There also exist non-zero sections

$$\omega_D \in H^0\big(D, K_D(2iq_2)\big),\ \omega_E \in H^0\big(E, \mathcal{O}_E(2g-2)\big),\quad \omega_C \in H^0\big(C, K_C((2g-2i)q_1)\big),$$

satisfying $\mathrm{ord}_{q_1}(\omega_C) + \mathrm{ord}_{q_1}(\omega_E) \geq 2g-2$ and $\mathrm{ord}_{q_2}(\omega_E) + \mathrm{ord}_{q_2}(\omega_D) \geq 2g-2$. Furthermore, ω_E vanishes doubly at a point $p \in \{q_1, q_2\}^c$. Since ω_C (respectively ω_D) also vanishes along the divisor $x_1 + \cdots + x_s$ (respectively $x_{s+1} + \cdots + x_{g-1}$), it follows that $\mathrm{ord}_{q_1}(\omega_C) \leq 2g - i - s$ and $\mathrm{ord}_{q_2}(\omega_D) \leq i + s - 1$, hence by compatibility, $\mathrm{ord}_{q_1}(\omega_E) + \mathrm{ord}_{q_2}(\omega_E) \geq 2g-3$. This rules out the possibility of a further double zero and shows that this case does not occur.

To summarize, keeping in mind that the ψ-coefficient of $[\overline{\mathfrak{RT}}_g]$ is equal to $4g-8$, we find the relation

$$(2i-2+s)b_{i:s} - sb_{i:s-1} + s(4g-8) = 4(g-i)(s-1)(si-2i+2) + a(i,s). \tag{7}$$

For $s = 0$, we have by convention $b_{i:-1} = 0$, which gives $b_{i:0} = 2i^3 - 5i^2 - 3i + 4g + 1$. By induction, we find using recursion (7) the claimed formula for $b_{i:s}$. □

As already explained, having calculated the class $[\overline{\mathfrak{RT}}_g] \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,g-1})$ and using known bound on the slope $s(\overline{\mathcal{M}}_g)$, one derives that $\mathfrak{Th}_g$ is of general type when $g \geq 12$. We discuss the last cases in Theorem 0.1 and thus complete the birational classification of $\mathfrak{Th}_g$:

End of proof of Theorem 0.1. We noted in the Introduction that for $g \leq 9$ the space $\mathfrak{Th}_g$ is unirational, being the image of a variety which is birational to a Grassmann bundle over the rational *Mukai variety* V_g^{g-1}. When $g \in \{10, 11\}$, the space $\overline{\mathcal{M}}_{g,g-1}$ is uniruled [FP]. This implies the uniruledness of $\mathfrak{Th}_g$ as well. □

3. The Kodaira dimension of $\overline{\mathcal{C}}_{g,n}$

In this section we provide results concerning the Kodaira dimension of the symmetric product $\overline{\mathcal{C}}_{g,n}$, where $n \leq g-2$. There are two cases depending on the parity of the difference $g-n$. When $g-n$ is even, we introduce a subvariety inside $\mathcal{C}_{g,n}$, consisting of divisors $D \in C_n$ which appear in a fibre of a pencil of degree $(g+n)/2$ on a curve $[C] \in \mathcal{M}_g$. We set integers $g \geq 1$ and $1 \leq m \leq g/2$, then consider the locus

$$\mathcal{F}_{g,m} := \{[C, x_1, \ldots, x_{g-2m}] \in \mathcal{M}_{g,g-2m} : \exists A \in W^1_{g-m}(C) \text{ with } H^0\big(C, A(-\sum_{j=1}^{g-2m} x_j)\big) \neq 0\}.$$

A parameter count shows that $\mathcal{F}_{g,m}$ is expected to be an effective divisor on $\overline{\mathcal{M}}_{g,g-2m}$. We shall prove this, then compute the class of its closure in $\overline{\mathcal{M}}_{g,g-2m}$.

Theorem 3.1. *Fix integers $g \geq 1$ and $1 \leq m \leq g/2$, then set $n := g-2m$ and $d := g-m$. The class of the compactification inside $\overline{\mathcal{M}}_{g,g-2m}$ of the divisor $\mathcal{F}_{g,m}$ is given by the formula:*

$$\overline{\mathcal{F}}_{g,m} \equiv \Big(\frac{10n}{g-2}\binom{g-2}{d-1} - \frac{n}{g}\binom{g}{d}\Big)\lambda + \frac{n-1}{g-1}\binom{g-1}{d-1}\sum_{j=1}^{n}\psi_j - \frac{n}{g-2}\binom{g-2}{d-1}\delta_{\mathrm{irr}} -$$

$$-\sum_{s=2}^{n}\frac{s(n^2-g+sgn-sn)}{2(g-1)(g-d)}\binom{g-1}{d}\delta_{0:s}-\cdots\in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n}).$$

Proof. We fix a general curve $[C]\in\mathcal{M}_g$ and consider the incidence correspondence

$$\Sigma:=\{(D,A)\in C_{g-2m}\times W^1_{g-m}(C): H^0(C,A\otimes\mathcal{O}_C(-D))\neq 0\},$$

together with the projection $\pi_1:\Sigma\to C_{g-2m}$. It follows from [F1] Theorem 0.5, that Σ is pure of dimension $g-2m-1\big(=\rho(g,1,g-m)+1\big)$. To conclude that $\overline{\mathcal{F}}_{g,m}$ is a divisor inside $\overline{\mathcal{M}}_{g,g-2m}$, it suffices to show that the general fibre of the map π_1 is finite, which implies that $\phi^{-1}([C])\cap\overline{\mathcal{F}}_{g,m}$ is a divisor in $\phi^{-1}([C])$; we also note that the fibre $\phi^{-1}([C])$ is isomorphic to the n-th Fulton-Macpherson configuration space of C. We specialize to the case $D=(g-2m)\cdot p$, where $p\in C$. One needs to show that for a general curve $[C]\in\mathcal{M}_g$, there exist finitely many pencils $A\in W^1_{g-m}(C)$ with $h^0\big(C,A\otimes\mathcal{O}_C(-(g-m)p)\big)\geq 1$, for some point $p\in C$. This follows from [HM] Theorem B, or alternatively, by letting C specialize to a flag curve consisting of a rational spine and g elliptic tails, in which case the point p specializes to a $(g-2m)$-torsion points on one of the elliptic tails (in particular it can not specialize to a point on the spine). For each of these points, the pencils in question are in bijective correspondence to points in a transverse intersection of Schubert cycles in $G(2,g-m+1)$. In particular their number is finite.

In order to compute the class $[\overline{\mathcal{F}}_{g,m}]$, we expand it in the usual basis of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$

$$\overline{\mathcal{F}}_{g,m}\equiv a\lambda+c\sum_{i=1}^{g-2m}\psi_i-b_{\mathrm{irr}}\delta_{\mathrm{irr}}-\sum_{i,s\geq 0}b_{i:s}\delta_{i:s},$$

then note that the coefficients a,c and b_{irr} respectively, have been computed in [F2] Theorem 4.9. The coefficient $b_{0:2}$ is determined by intersecting $\overline{\mathcal{F}}_{g,m}$ with a fibral curve

$$C_{x_n}:=\{[C,x_1,\ldots,x_{n-1},x_n]:x_n\in C\}\subset\overline{\mathcal{M}}_{g,n},$$

corresponding to a general $(n-1)$-pointed curve $[C,x_1,\ldots,x_{n-1}]\in\overline{\mathcal{M}}_{g,n-1}$. By letting the points $x_1,\ldots,x_{n-1}\in C$ coalesce to a point $q\in C$, points in the intersection $C_{x_n}\cdot\overline{\mathcal{F}}_{g,m}$ are in $1:1$ correspondence with points $x_n\in C$, such that $h^0\big(C,A(-(n-1)q-x_n)\big)\geq 1$. This number equals $(g-2m-1)\binom{g}{m}$, see [HM] Theorem A, that is,

$$(2g+2n-4)c-(n-1)b_{0:2}=C_{x_n}\cdot\overline{\mathcal{F}}_{g,m}=$$

$$(m+1)\ \#\Big\{A\in W^1_{g-m}(C): h^0\big(C,A\otimes\mathcal{O}_C(-(g-2m-1)q)\big)\geq 1\Big\}=(g-2m-1)\binom{g}{m},$$

which determines $b_{0:2}$. The coefficients $b_{0:s}$ are computed recursively, by exhibiting an explicit test curve $\Gamma_{0:s}\subset\Delta_{0:s}$ which is disjoint from $\overline{\mathcal{F}}_{g,m}$. We fix a general element $[C,q,x_{s+1},\ldots,x_n]\in\overline{\mathcal{M}}_{g,n+1-s}$ and a general s-pointed rational curve $[\mathbf{P}^1,x_1,\ldots,x_s]\in\overline{\mathcal{M}}_{0,s}$. We glue these curves along a moving point q lying on the rational component:

$$\Gamma_{0:s}:=\{[\mathbf{P}^1\cup_q C,x_1,\ldots,x_s,x_{s+1},\ldots,x_n]:q\in\mathbf{P}^1\}\subset\Delta_{0:s}\subset\overline{\mathcal{M}}_{g,n}.$$

Clearly, $\Gamma_{0:s}\cdot\overline{\mathcal{F}}_{g,m}=s\,c-(s-2)\,b_{0:s}+s\,b_{0:s-1}$. We claim $\Gamma_{0:s}\cap\overline{\mathcal{F}}_{g,m}=\emptyset$. Assume that on the contrary, one can find a point $q\in\mathbf{P}^1$ and a limit linear series $\mathfrak{g}^1_d$ on $\mathbf{P}^1\cup_q C$,

$$l=\big((A,V_C),(\mathcal{O}_{\mathbf{P}^1}(d),V_{\mathbf{P}^1})\big)\in G^1_d(C)\times G^1_d(\mathbf{P}^1),$$

together with sections $\sigma_C \in V_C$ and $\sigma_{\mathbf{P}^1} \in V_{\mathbf{P}^1}$, satisfying $\mathrm{ord}_q(\sigma_C) + \mathrm{ord}_q(\sigma_{\mathbf{P}^1}) \geq d$ and

$$\mathrm{div}(\sigma_C) \geq x_{s+1} + \cdots + x_n, \quad \mathrm{div}(\sigma_{\mathbf{P}^1}) \geq x_1 + \cdots + x_s.$$

Since $\sigma_{\mathbf{P}^1} \neq 0$, one finds that $\mathrm{ord}_q(\sigma_{\mathbf{P}^1}) \leq g-m-s$, hence by compatibility, $\mathrm{ord}_q(\sigma_C) \geq s$. We claim that this is impossible, that is, $H^0\big(C, A \otimes \mathcal{O}_C(-sq-x_1-\cdots-x_n)\big) \neq 0$, for every $A \in W^1_{g-m}(C)$. Indeed, by letting all points $x_{s+1}, \ldots, x_n, q \in C$ coalesce, the statement $H^0\big(C, A \otimes \mathcal{O}_C(-(g-2m) \cdot q)\big) = 0$, for a general $[C, q] \in \overline{\mathcal{M}}_{g,1}$ is a consequence of the "pointed" Brill-Noether theorem as proved in [EH1] Theorem 1.1. This shows that

$$0 = \Gamma_{0:s} \cdot \overline{\mathcal{F}}_{g,m} = sc + (s-2)b_{0:s} - sb_{0:s-1},$$

for $3 \leq s \leq n$, which determines recursively all coefficients $b_{0:s}$. The remaining coefficients $b_{i:s}$ with $1 \leq i \leq [g/2]$ can be determined via similar test curve calculations, but we skip these details. □

Keeping the notation from the proof of Theorem 3.1, a direct consequence is the calculation of the class of the divisor $\mathcal{F}_{g,m}[C] := \pi_1(\Sigma)$ inside C_{g-2m}. This offers an alternative proof of [Mus] Proposition III; furthermore the proof of Theorem 3.1, answers in the affirmative the question raised in loc.cit., concerning whether the cycle $\mathcal{F}_{g,m}[C]$ has expected dimension, and thus, it is a divisor on C_{g-2m}.

We denote by $\theta \in H^2(C_{g-2m}, \mathbb{Q})$ the class of the pull-back of the theta divisor, and by $x \in H^2(C_{g-2m}, \mathbb{Q})$ the class of the locus $\{p_0 + D : D \in C_{g-2m-1}\}$ of effective divisors containing a fixed point $p_0 \in C$. For a very general curve $[C] \in \mathcal{M}_g$, the group $N^1(C_{g-2m})_{\mathbb{Q}}$ is generated by x and θ, see [ACGH].

Let $\widetilde{\mathcal{F}}_{g,m}$ be the effective divisor on $\overline{\mathcal{C}}_{g,g-2m}$ to which $\overline{\mathcal{F}}_{g,m}$ descends, that is, $\pi^*(\widetilde{\mathcal{F}}_{g,m}) = \overline{\mathcal{F}}_{g,m}$. The class of $\widetilde{\mathcal{F}}_{g,m}$ is completely determined by Theorem 3.1.

Corollary 3.2. *Let $[C] \in \mathcal{M}_g$ be a general curve. The cohomology class of the divisor*

$$\mathcal{F}_{g,m}[C] := \{D \in C_{g-2m} : \exists A \in W^1_{g-m}(C) \text{ such that } H^0(C, A \otimes \mathcal{O}_C(-D)) \neq 0\}$$

is equal to $(1 - \frac{2m}{g})\binom{g}{m}\big(\theta - \frac{g}{g-2m}x\big)$. In particular, the class $\theta - \frac{g}{g-2m}x \in N^1(C_{g-2m})_{\mathbb{Q}}$ is effective.

Proof. Let $u : C_{g-2m} \dashrightarrow \overline{\mathcal{C}}_{g,g-2m}$ be the rational map given by

$$u(x_1 + \cdots + x_{g-2m}) = [C, x_1 + \cdots + x_{g-2m}].$$

Note that u is well-defined outside the codimension 2 locus of effective divisors with support of length at most $g-2m-2$. We have that $u^*(\widetilde{\delta}_{0:2}) = \delta_C$, where $\delta_C := [\Delta_C]/2$ is the reduced diagonal. Its class is given by the MacDonald formula, cf. [K1] Lemma 7:

$$\delta_C \equiv -\theta + (g+d-1)x \equiv -\theta + (2g-2m-1)x.$$

Furthermore, $u^*(\widetilde{\psi}) \equiv \theta + \delta_C + (g-n-1)x$, see [K2] Proposition 2.7. Thus $\mathcal{F}_{g,m}[C] \equiv u^*([\widetilde{\mathcal{F}}_{g,m}])$, and the conclusion follows after some calculations. □

The divisor $\widetilde{\mathcal{F}}_{g,m}$ is defined in terms of a correspondence between pencils and effective divisors on curves, and it is fibred in curves as follows: We fix a complete pencil $A \in W^1_{g-m}(C)$ with only simple ramification points. The variety of secant divisors

$$V^1_{g-2m}(A) := \{D \in C_{g-2m} : H^0(C, A \otimes \mathcal{O}_C(-D)) \neq 0\}$$

is a curve (see [F1]), disjoint from the indeterminacy locus of the rational map $u : C_{g-2m} \dashrightarrow \overline{\mathcal{C}}_{g,g-2m}$. We set $\Gamma_{g-2m}(A) := u(V^1_{g-2m}(A)) \subset \overline{\mathcal{C}}_{g,g-2m}$. By varying $[C] \in \mathcal{M}_g$ and $A \in W^1_{g-m}(C)$, the curves $\Gamma_{g-2m}(A)$ fill-up the divisor $\widetilde{\mathcal{F}}_{g,m}$. It is natural to test the extremality of $\widetilde{\mathcal{F}}_{g,m}$ by computing the intersection number $\Gamma_{g-2m}(A) \cdot \widetilde{\mathcal{F}}_{g,m}$. To state the next result in a unified form, we adopt the convention $\binom{a}{b} := 0$, whenever $b < 0$.

Proposition 3.3. *For all integers $1 \leq m < g/2$, we have the formula:*

$$\Gamma_{g-2m}(A) \cdot \widetilde{\mathcal{F}}_{g,m} = (m-1)\binom{g-m-2}{m}\binom{g}{m}.$$

In particular, $\Gamma_{g-2}(A) \cdot \widetilde{\mathcal{F}}_{g,1} = 0$, and the divisor $\widetilde{\mathcal{F}}_{g,1} \in \mathrm{Eff}(\overline{\mathcal{C}}_{g,g-2})$ is extremal.

Proof. This is an immediate application of Corollary 3.2. The class $[V^1_{g-2m}(A)]$ can be computed using Porteous' formula, see [ACGH] p.342:

$$[V^1_{g-2m}(A)] \equiv \sum_{j=0}^{g-2m-1} \binom{-m-1}{j} \frac{x^j \cdot \theta^{g-2m-j-1}}{(g-2m-1-j)!} \in H^{2(g-2m-1)}(C_{g-2m}, \mathbb{Q}).$$

Using the push-pull formula, we write $\Gamma_{g-2m}(A) \cdot \widetilde{\mathcal{F}}_{g,m} = \mathcal{F}_{g,m}[C] \cdot [V^1_{g-2m}(A)]$, then estimate the product using the identity $x^k\theta^{g-2m-k} = g!/(2m+k)! \in H^{2(g-2m)}(C_{g-2m}, \mathbb{Q})$ for $0 \leq k \leq g-2m$. For $m = 1$, observe that $\Gamma_{g-2}(A) \cdot \widetilde{\mathcal{F}}_{g,1} = 0$. Since the curves of type $\Gamma_{g-2}(A)$ cover $\widetilde{\mathcal{F}}_{g,1}$, this implies that $[\widetilde{\mathcal{F}}_{g,1}] \in \mathrm{Eff}(\overline{\mathcal{C}}_{g,g-2})$ generates an extremal ray. □

We can use Theorem 3.1 to describe the birational type of $\overline{\mathcal{C}}_{g,n}$ when $12 \leq g \leq 21$ and $1 \leq n \leq g-2$. We recall that when $g \leq 9$, the space $\overline{\mathcal{C}}_{g,n}$ is uniruled for all values of n. The transition cases $g = 10, 11$, as well as the case of the universal Jacobian $\overline{\mathcal{C}}_{g,g}$, are discussed in detail in [FV2]. Furthermore $\overline{\mathcal{C}}_{g,n}$ is uniruled when $n \geq g+1$; in this case the symmetric product C_n of any curve $[C] \in \mathcal{M}_g$ is birational to a $\mathbf{P}^{n-g}$-bundle over the Jacobian $\mathrm{Pic}^n(C)$. Our main result is that, in the range described above, $\overline{\mathcal{C}}_{g,n}$ is of general type in all the cases when $\overline{\mathcal{M}}_{g,n}$ is known to be of general type, see [Log], [F2]. We note however that the divisors $\overline{\mathcal{F}}_{g,m}$ only carry one a certain distance towards a full solution. The classification of $\overline{\mathcal{C}}_{g,n}$ is complete only when $n \in \{g-1, g-2, g\}$.

Theorem 3.4. *For integers $g = 12, \ldots, 21$, the universal symmetric product $\overline{\mathcal{C}}_{g,n}$ is of general type for all $f(g) \leq n \leq g-1$, where $f(g)$ is described in the following table.*

g	12	13	14	15	16	17	18	19	20	21
$f(g)$	10	11	10	10	9	9	9	7	6	4

Proof. The strategy described in the Introduction to prove that $K_{\overline{\mathcal{C}}_{g,g-1}}$ is big, applies to the other spaces $\overline{\mathcal{C}}_{g,n}$, with $1 \leq n \leq g-2$ as well. To show that $\overline{\mathcal{C}}_{g,n}$ is of general type, it suffices to produce an effective class on $\overline{\mathcal{C}}_{g,n}$ which pulls back via π to $a\lambda + c\sum_{i=1}^n \psi_i - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_{i,s} b_{i:s}\delta_{i:s} \in \mathrm{Eff}(\overline{\mathcal{M}}_{g,n})^{\mathfrak{S}_n}$, such that the following conditions are fulfilled:

$$\frac{a + s(\overline{\mathcal{M}}_g)(2c - b_{\mathrm{irr}})}{13c} < 1 \quad \text{and} \quad \frac{b_{0:2}}{3c} > 1. \tag{8}$$

When $g - n$ is even, we write $g - n = 2m$, and for all entries in the table above one can express $K_{\overline{\mathcal{C}}_{g,n}}$ as a positive combination of $\sum_{i=1}^n \psi_i$, $[\overline{\mathcal{F}}_{g,m}]$, $\varphi^*(D)$, where $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$, and other boundary classes.

If $g-n=2m+1$ with $m\in\mathbb{Z}_{\geq 0}$, for each integer $1\leq j\leq n+1$, we denote by $\phi_j:\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ the projection forgetting the j-th marked point and consider the effective $\mathfrak{S}_n$-invariant effective $\mathbb{Q}$-divisor on $\overline{\mathcal{M}}_{g,n}$

$$E:=\frac{1}{n+1}\sum_{j=1}^{n+1}(\phi_j)_*\big(\overline{\mathcal{F}}_{g,m}\cdot\delta_{0:\{j,n+1\}}\big)\in\mathrm{Eff}(\overline{\mathcal{M}}_{g,n}).$$

Using Theorem 3.1 as well as elementary properties of push-forwards of tautological classes, $K_{\overline{\mathcal{C}}_{g,n}}$ is expressible as a positive $\mathbb{Q}$-combination of boundaries, $[E]$, a pull-back of an effective divisor on $\overline{\mathcal{M}}_g$, and the big and nef class $\sum_{i=1}^n\psi_i$ precisely in the cases appearing in the table. □

Remark 3.5. When $g\notin\{12,16,18\}$, the bound $s(\overline{\mathcal{M}}_g)\leq 6+12/(g+1)$, emerging from the slope of the Brill-Noether divisors, has been used to verify (8). In the remaining cases, we employ the better bounds $s(\overline{\mathcal{M}}_{12})=4415/642<6+12/13$ (see [FV1]), and $s(\overline{\mathcal{M}}_{16})=407/61<6+12/17$ see [F2], coming from Koszul divisors on $\overline{\mathcal{M}}_{12}$ and $\overline{\mathcal{M}}_{16}$ respectively. On $\overline{\mathcal{M}}_{18}$, we use the estimate $s(\overline{\mathcal{M}}_{18})\leq 302/45$ given by the class of the Petri divisor $\overline{\mathcal{GP}}^1_{18,10}$, see [EH1]. Improvements on the estimate on $s(\overline{\mathcal{M}}_g)$ in the other cases, will naturally translate in improvements in the statement of Theorem 3.4.

4. An effective divisor on $\overline{\mathcal{M}}_{g,g-3}$

The aim of this section is to prove Theorem 0.8. We begin by solving the following enumerative question which comes up repeatedly in the process of computing $[\overline{\mathcal{D}}_g]$.

Theorem 4.1. *Let $[C,p]\in\mathcal{M}_{g,1}$ be a general pointed curve of genus g and $0\leq\gamma\leq g-3$ a fixed integer. Then there exist a finite number of pairs $(L,x)\in W^2_g(C)\times C$ such that*

$$H^0\big(C,L\otimes\mathcal{O}_C(-\gamma\,x-(g-3-\gamma)\,p)\big)\geq 1.$$

Their number is computed by the formula

$$N(g,\gamma):=\frac{g(g-1)(g-5)}{3}\gamma(\gamma g-3\gamma-1).$$

Proof. We introduce auxiliary maps $\chi:C\times C_3\to C_{\gamma+3}$ and $\iota:C_{\gamma+3}\to C_g$ given by,

$$\chi(x,D):=\gamma\cdot x+D,\ \text{ and } \iota(E):=E+(g-3-\gamma)\cdot p.$$

The number we evaluate is $N(g,\gamma):=\chi^*\iota^*\big([C^2_g]\big)$, where $C^2_g:=\{D\in C_g:\dim|D|\geq 3\}$. The cohomology class of this variety of special divisors is computed in [ACGH] p.326:

$$[C^2_g]=\frac{\theta^4}{12}-\frac{x\theta^3}{3}+\frac{x^2\theta^2}{6}\in H^8(C_{g-3},\mathbb{Q}).$$

Noting that $\iota^*(\theta)=\theta$ and $\iota^*(x)=x$, one needs to estimate the pull-backs of the tautological monomials $x^\alpha\theta^{4-\alpha}$. For this purpose, we use [ACGH] p.358:

$$\chi^*(x^\alpha\theta^{4-\alpha})=\frac{g!}{(g-4+\alpha)!}\Big[\big(1+\gamma t_1+t_2\big)^\alpha\cdot\big(1+\gamma^2t_1+t_2\big)^{4-\alpha}\Big]_{t_1t_2^3},$$

where the last symbol indicates the coefficient of the monomial $t_1t_2^3$ in the polynomial appearing on the right side of the formula. The rest follows after a routine evaluation. □

The second enumerative ingredient in the proof of Theorem 0.8 is the following result, which can be proved by degeneration using Schubert calculus:

Proposition 4.2. *For a general curve $[C]\in\mathcal{M}_{g-1}$, there exist a finite number of pairs $(L,x)\in W^2_g(C)\times C$ satisfying the conditions*

$$h^0\big(C, L\otimes\mathcal{O}_C(-2x)\big)\geq 2, \text{ and } h^0\big(C, L\otimes\mathcal{O}_C(-(g-2)x)\big)\geq 1.$$

Each pair corresponds to a complete linear series L. The number of such pairs is equal to

$$n(g-1):=(g-1)(g-2)(g-3)(g-4)^2.$$

Proof of Theorem 0.8. We expand $[\overline{\mathcal{D}}_g]\in \mathrm{Pic}(\overline{\mathcal{M}}_{g,g-3})$, and begin the calculation by determining the coefficients of $\lambda, \delta_{\mathrm{irr}}$ and $\sum_{i=1}^{g-3}\psi_i$ respectively. It is useful to observe that if $\phi_n:\overline{\mathcal{M}}_{g,n}\rightarrow\overline{\mathcal{M}}_{g,n-1}$ is the map forgetting the marked point labeled by n for some $n\geq 1$ and D is any divisor class on $\overline{\mathcal{M}}_{g,n}$, then for distinct labels $i,j\neq n$, the λ, δ_{irr} and ψ_j coefficients of the divisors D on $\overline{\mathcal{M}}_{g,n}$ and $(\phi_n)_*(D\cdot\delta_{0:in})$ on $\overline{\mathcal{M}}_{g,n-1}$ respectively, coincide. The divisor $(\phi_n)_*(D\cdot\delta_{0:in})$ can be thought of as the locus of points $[C,x_1,\ldots,x_n]\in D$ where the points x_i and x_n are allowed to come together. By iteration, the divisor $\overline{\mathcal{D}}_g^{g-3}$ on $\overline{\mathcal{M}}_{g,1}$ obtained by letting all points $x_1,\ldots,x_{g-3}$ coalesce, has the same λ and δ_{irr} coefficients as $\overline{\mathcal{D}}_g$. But obviously

$$\overline{\mathcal{D}}_g^{g-3}=\{[C,x]\in\mathcal{M}_{g,1}:\exists L\in W^2_g(C)\text{ such that }h^0\big(C,L\otimes\mathcal{O}_C(-(g-3)x)\big)\geq 1\},$$

and note that this is a "pointed Brill-Noether divisor" in the sense of Eisenbud-Harris. The cone of Brill-Noether divisors on $\overline{\mathcal{M}}_{g,1}$ is 2-dimensional, see [EH2] Theorem 4.1, and exists constants $\mu,\nu\in\mathbb{Q}$, such that $\overline{\mathcal{D}}_g^{g-3}\equiv\mu\cdot\mathfrak{BN}+\nu\cdot\overline{\mathcal{W}}$, where

$$\mathfrak{BN}:=(g+3)\lambda-\frac{g+1}{6}\delta_{\mathrm{irr}}-\sum_{j=1}^{g-1}j(g-j)\delta_{j:1}\in\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$$

is the pull-back from $\overline{\mathcal{M}}_g$ of the Brill-Noether divisor class and $\overline{\mathcal{W}}\in\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$ is the class of the Weierstrass divisor. The coefficients μ and ν are computed by intersecting both sides of the previous identity with explicit curves inside $\overline{\mathcal{M}}_{g,1}$. First we fix a genus g curve C and let the marked point vary along C. If $C_x:=\phi^{-1}([C])\subset\overline{\mathcal{M}}_{g,1}$ denotes the induced curve in moduli, then the only generator of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$ which has non-zero intersection number with C_x is ψ, and $C_x\cdot\psi=2g-2$. On the other hand $C_x\cdot\overline{\mathcal{D}}_g^{g-3}=N(g,g-3)$, that is, $\nu=N(g,g-3)/(g(g-1)(g+1))$.

To compute μ, we construct a curve inside $\Delta_{1:1}$ as follows: Fix a 2-pointed elliptic curve $[E,x,y]\in\mathcal{M}_{1,2}$ such that the class $x-y\in\mathrm{Pic}^0(E)$ is not torsion, and a general curve $[C]\in\mathcal{M}_{g-1}$. We define the family $\bar{C}_1:=\{[C\cup_y E,x]\}_{y\in C}$, obtained by varying the point of attachment along C, while keeping the marked point fixed on E. The only generator of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$ meeting $\bar{C}_1$ non-trivially is $\delta_{1:1}=\delta_{g-1:\emptyset}$, in which case $\bar{C}_1\cdot\delta_{1:1}=-2g+4$. On the other hand, $\bar{C}_1\cdot\overline{\mathcal{D}}_g^{g-3}$ is equal to the number of limit linear series $\mathfrak{g}^2_g$ on curves of type $C\cup_y E$, having vanishing sequence at least $(0,1,g-3)$ at $x\in E$. This can happen only if this linear series is refined and its C-aspect has vanishing sequence at the point of attachment $y\in C$ equal to either (i) $(1,2,g-3)$, or (ii) $(0,2,g-2)$. In both cases, the E-aspect being uniquely determined, we obtain that $\bar{C}_1\cdot\overline{\mathcal{D}}_g^{g-3}=N(g-1,g-4)+n(g-1)$. This leads to $\mu=3(g-3)(g-4)/(g+1)$.

Next, let $\overline{\mathcal{D}}_g^{g-4}$ be the divisor on $\overline{\mathcal{M}}_{g,2}$ obtained from $\overline{\mathcal{D}}_g$ by letting all marked points except one, come together. Precisely, $\overline{\mathcal{D}}_g^{g-4}$ is the closure of the locus of curves $[C,x,y]\in\mathcal{M}_{g,2}$ such that there exists $L\in W^2_g(C)$ with $h^0\big(C,L\otimes\mathcal{O}_C(-x-(g-4)y)\big)\geq 1$. We express $\overline{\mathcal{D}}_g^{g-4}\equiv c_x\psi_x+c_y\psi_y-e\delta_{0:xy}-\cdots\in \mathrm{Pic}(\overline{\mathcal{M}}_{g,2})$, and observe that c_x equals the ψ-coefficient of $\overline{\mathcal{D}}_g$, whereas the coefficient $e=\nu\binom{g+1}{2}$ has already been calculated. We fix a general curve $[C]\in\overline{\mathcal{M}}_g$ and define test curves $C_x:=\{[C,x,y]:x\in C\}\subset\overline{\mathcal{M}}_{g,2}$ and $C_y:=\{[C,x,y]:y\in C\}\subset\overline{\mathcal{M}}_{g,2}$, by fixing one general marked point on C and letting the other vary freely. By intersecting $\overline{\mathcal{D}}_g^{g-4}$ with these curves we obtain the formulas:

$$(2g-1)c_x+c_y-e=C_x\cdot\overline{\mathcal{D}}_g^{g-4}=N(g,1) \ \text{ and } \ c_x+(2g-1)c_y-e=C_y\cdot\overline{\mathcal{D}}_g^{g-4}=N(g,g-4).$$

Solving this system, determines c_x. Finally, the $\delta_{0:2}$-coefficient of $\overline{\mathcal{D}}_g$ is computed by intersecting $\overline{\mathcal{D}}_g$ with the test curve $\phi_{g-3}^{-1}([C,x_1,\ldots,x_{g-4}])\subset\overline{\mathcal{M}}_{g,g-3}$, obtained by fixing $g-4$ marked points on a general curve, and letting the remaining point vary. □

As an application, we bound the effective cone of the symmetric product of degree $g-3$ on a general curve $[C]\in\mathcal{M}_g$. As before, let $u: C_{g-3}\dashrightarrow\overline{\mathcal{C}}_{g,g-3}$ the (rational) fibre map and $\widetilde{\mathcal{D}}_g$ the effective divisor on $\overline{\mathcal{C}}_{g,g-3}$ to which $\overline{\mathcal{D}}_g$ descends. Then $\mathcal{D}_g[C]:=u^*(\widetilde{\mathcal{D}}_g)$ is an effective divisor on C_{g-3}:

Theorem 4.3. *The cohomology class of the codimension one locus inside C_{g-3}*

$$\mathcal{D}_g[C]:=\{D\in C_{g-3}:\exists L\in W^2_g(C) \ \textit{with}\ h^0\big(C,L\otimes\mathcal{O}_C(-D)\big)\geq 1\} \quad \textit{equals}$$

$$[\mathcal{D}_g(C)]=\frac{(g-5)(g-3)(g-1)}{3}\Big(\theta-\frac{g}{g-3}x\Big).$$

It is natural to wonder whether the class $\theta-\frac{g}{g-3}x$ is extremal in $\mathrm{Eff}(C_{g-3})$. If so, $\mathcal{D}_g[C]$ together with the diagonal class $\delta_C\equiv-\theta+(2g-4)x$ would generate the effective cone inside the 2-dimensional space $N^1(C_{g-3})_{\mathbb{Q}}$. We refer to [K1] Theorem 3, for a proof that δ_C spans an extremal ray, which shows that in order to compute $\mathrm{Eff}(C_{g-3})$, one only has to determine the slope of $\mathrm{Eff}(C_{g-3})$ in the fourth quadrant of the (θ,x)-plane. A similar description of the effective cone of C_{g-2} was given in [Mus]. We have a partial result in this direction, showing that all effective divisors of slope higher than $\frac{g}{g-3}$ (if any), must contain a geometric codimension one subvariety of $\mathcal{D}_g[C]$.

Proposition 4.4. *Any irreducible effective divisor on C_{g-3} with class proportional to $\theta-\alpha x\in H^2(C_{g-3},\mathbb{Q})$, where $\alpha>\frac{g}{g-3}$, contains the codimension two locus inside C_{g-3}*

$$Z_{g-3}[C]:=\{D\in C_{g-3}:\exists A\in W^1_{g-2}(C)\ \textit{with}\ H^0(C,A\otimes(-D))\neq 0\}.$$

Proof. By calculation, note that for $A\in W^1_{g-2}(C)$, the inequality $[V^1_{g-3}(A)]\cdot(\theta-\alpha x)<0$ holds, whereas $[V^1_{g-3}(A)]\cdot\mathcal{D}_g[C]=0$. □

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