DIFFERENTIABILITY OF QUADRATIC FORWARD-BACKWARD SDES WITH ROUGH DRIFTS

PETER IMKELLER, RHOSS B. LIKIBI PELLAT, AND OLIVIER MENOUKEU PAMEN

Abstract

In this paper, we consider quadratic forward-backward SDEs (QFBSDEs), when the drift in the forward equation does not satisfy the standard globally Lipschitz condition and the driver of the backward system satisfies a nonlinearity of type $f(|y|)|z|^2$, where f is any locally integrable function. We prove both the Malliavin and classical derivative of the QFBSDE and provide representations of these processes. We study a numerical approximation of this system in the sense of [19] in which the authors assumes that the drift is Lipschitz and the driver of the BSDE is quadratic in the traditional sense (i.e., f is a positive constant). We show that the rate of convergence is the same as in [19].

1. Introduction

In this paper, we address the problem of the Malliavin and the classical differentiability of a class of quadratic forward-backward SDEs (FBSDEs) with rough drift. FBSDEs have attracted a lot of interest in the last four decades due to their applications to optimal control, financial/insurance mathematics and the theory of PDE via the non-linear Feynman-Kac formula. Of particular interest is the class of BSDEs whose drivers grow quadratically in the control variable Z. Such a BSDE appears for example in exponential utility maximisation or in the Epstein-Zin utility maximisation problems. To the best of our knowledge, the first result on existence and uniqueness of BSDEs with quadratic drivers and bounded terminal value is due to Kobylansky ([27]). This result was extended to the case of unbounded terminal value in [9, 10] and in other different ways authors in [18, 33, 12, 5].

Recently the authors in [3] studied a new class of unbounded quadratic BSDE when the generator g has the following growth condition $|g(t, y, z)| \leq C(1 + f(y)|z|^2)$, where f is an integrable function. Their approach is based on an exponential transformation and an Itô-Krylov formula for BSDEs. This result was extended in [2]to the case of locally integrable function f by using the so called *domination method*.

Another question of importance that arises in the study of BSDEs is the characterisation of the control process Z. When the coefficients of the FBSDEs are smooth enough, Z is given as the "derivative", either in the sense of the non linear Feynman-Kac formula or in the Malliavin sense via the Clark-Ocone-Hausmann formula. Moreover, $(Z_t)_{t\in[0,T]}$ has a continuous version given by the Malliavin derivative of the backward equation $(D_tY_t)_{t\in[0,T]}$. The later turns out to be a crucial concept when one deals with novel discretization of BSDEs implemented with deep learning regressions (see for instance [34]). When the parameters are not smooth, the existence of Malliavin and classical differentiable solutions to FBSDEs remain a challenging question.

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Smoothness of solutions of FBSDEs, in both the classical and Malliavin sense when the coefficients are smooth enough were established in [36, 24]. In the quadratic case, the first results were derived in by the authors in [1] assuming that the driver is of the form $g(t,x,y,z) = \ell(t,x,y,z) + \alpha|z|^2$ with $\ell \in C^1$ and Lipschitz continuous in (x,y,z). This work was then extended to non-linear quadratic generator in [20]. Their method for establishing the classical differentiability relies on differences of difference quotient together with the completeness of the vector space. In order to prove the Malliavin differentiability, they proposed an approximation procedure via a family of truncated BSDEs and a compactness criterion argument. Using a technique based on Kunita's method (solving an abstract BSDE with stochastic Lipschitz conditions), the authors in [7] proved a differentiability result in the classical sense for solutions to quadratic FBSDEs. Other related results include the work [22] for differentiability of FBSDE driven by continuous martingale with quadratic growth and the work [17] for classical differentiability of FBSDEs with polynomial growth. For instance, the results obtained in the aforementioned papers do not cover the case of non uniformly Lipschitz drift coefficients. In [37], the authors established a result on Malliavin derivative solution to coupled FBSDEs with discontinuous coefficients. Their method exploits the regularization effect of the Brownian motion when the diffusion coefficient is a constant and the regularity of the *weak decoupling field* combined with a compactness criterion argument. However, the authors did provide a representation satisfied by the Malliavin derivatives in terms of BSDEs. This is mainly due to the very mild assumptions that were considered there.

In this paper, we assume that the driver is of the form $f(|y|)|z|^2$, where f is any increasing and locally integrable function and the drift is either measurable and bounded or bounded and Hölder continuous. We then study both the differentiablity in the Malliavin and classical sense of the solution to the quadratic FBSDE. We follow the method developed in [1, 20] and work under much weaker conditions. In case of the forward equation, the representation of the Malliavin derivative is given in terms of local time (case of bounded measurable drift) or in terms of Young's integral (case of Hölder drift).

Another motivation of this paper is the numerical approximation and namely the time discretisation of the solution to decouple FBSDEs. When the drift is Lipschitz, such a question was studied in [6, 11, 40]. Assuming that the driver is quadratic, [38] investigated numerical approximation of quadratic FBSDEs. The method is based on a truncation of the quadratic driver. The rate of convergence is then obtained by using the differentiability of the solution to the FBSDE. Their results were refined and generalized by in [39]. In this work, we generalized the result in [38] in two directions. We assume a more general class of driver and allow for bounded and Hölder continuous drift. One of the main difficulties is to find the bound of the supremum norm of the inverse of the first variation process of the forward equation. This difficulty is circumvented by using a Zvonkin transform of the drift. We obtain the same rate of convergence as in [38].

The remaining part of the paper is organized as follows: in Section 2, we provide basic definitions and results on BSDEs. The existence and uniqueness results as well as the main a priori estimates are given in Section 3. Section 5 is devoted to the smothness of the solution of the FBSDE whereas Section NA is concerned with rate of convergence of the numerical approximation of the solution to the BSDE.

2. General settings and Notations

2.1. **Some notation.** Throughout this paper a stochastic basis $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geqslant 0}, \mathbb{P}, \{B_t\}_{t\geq 0})$ is given. Here $\{\mathfrak{F}_t\}_{t\geqslant 0}$ is the standard filtration generated by the d-dimensional Brownian motion $\{B_t\}_{t\geq 0}$ augmented by all $\mathbb{P}$-null sets of $\mathfrak{F}$. For fixed $T>0, d\in\mathbb{N}$, $p\in[2,\infty)$, we denote by:

- $L^p(\mathbb{R}^d)$ the space of $\mathfrak{F}_T$-adapted random variables X such that $\|X\|^p_{L^p} := \mathbb{E}|X|^p < \infty$;
- $L^\infty(\mathbb{R}^d)$ the space of bounded random variables with norm $\|X\|_{L^\infty} := \operatorname{essup}_{\omega\in\Omega}|X|$;
- $\mathcal{S}^p(\mathbb{R}^d)$ the space of all adapted continuous $\mathbb{R}^d$-valued processes X such that $\|X\|^p_{\mathcal{S}^p(\mathbb{R}^d)} := \mathbb{E}\sup_{t\in[0,T]}|X_t|^p < \infty$;
- $\mathcal{H}^p(\mathbb{R}^d)$ the space of all predictable $\mathbb{R}^d$-valued processes Z such that $\|Z\|^p_{\mathcal{H}^p(\mathbb{R}^d)} := \mathbb{E}(\int_0^T |Z_s|^2\mathrm{d}s)^{p/2} < \infty$;

- $\mathcal{S}^\infty(\mathbb{R}^d)$ the space of continuous $\{\mathfrak{F}_s\}_{0\leq t\leq T}$-adapted processes $Y:\Omega\times[0,T]\to\mathbb{R}^d$ such that $\|Y\|_\infty := \text{essup}_{\omega\in\Omega}\sup_{t\in[0,T]}|Y_t| < \infty$;
- BMO($\mathbb{P}$) the space of square integrable martingales M with $M_0 = 0$ such that $\|M\|_{\text{BMO}(\mathbb{P})} = \sup_{\tau\in[0,T]}\|\mathbb{E}[\langle M\rangle_T - \langle M\rangle_\tau]/\mathfrak{F}_\tau\|_\infty^{1/2} < \infty$, the supremum is taken over all stopping times $\tau\in[0,T]$;
- $\mathcal{H}_{\text{BMO}}$ the space of $\mathbb{R}^d$- valued $\mathcal{H}^p$-integrable processes $(Z_t)_{t\in[0,T]}$ for all $p\geq 2$ such that $Z * B = \int_0 Z_s\mathrm{d}B_s \in \text{BMO}(\mathbb{P})$. We define $\|Z\|_{\mathcal{H}_{\text{BMO}}} := \|\int Z\mathrm{d}B\|_{\text{BMO}(\mathbb{P})}$;
- $L^\infty([0,T];C_b^\beta(\mathbb{R}^d;\mathbb{R}^d))$ the space of all vector fields $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ having all components in $L^\infty([0,T];C_b^\beta(\mathbb{R}^d))$ and $L^\infty([0,T];C_b^\beta(\mathbb{R}^d))$ stands for the set of all bounded Borel functions $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ such that

$$[b]_{\beta,T} = \sup_{t\in[0,T]}\sup_{x\neq y\in\mathbb{R}^d}\frac{|b(t,x)-b(t,y)|}{|x-y|^\beta} < \infty.$$

Below, we briefly introduce the spaces of Malliavin differentiable random variables $\mathbb{D}^{k,p}$. For more information on Malliavin calculus we refer the reader to [13, 35]. Let $\mathcal{S}$ be the space of random variable ξ of the form

$$\xi = F\Big(\big(\int_0^T h_s^{1,i}\mathrm{d}W_s^1\big)_{1\leq i\leq n},\cdots,\big(\int_0^T h_s^{d,i}\mathrm{d}W_s^d\big)_{1\leq i\leq n}\Big),$$

where $F\in C_b^\infty(\mathbb{R}^{n\times d}), h^1,\ldots,h^n\in L^2([0,T];\mathbb{R}^d)$ and $n\in\mathbb{N}$. For simplicity, we assume that all h^j are written as row vectors. For $\xi\in\mathcal{S}$, we define $D=(D^1,\cdots,D^d):\mathcal{S}\to L^2(\Omega\times[0,T])^d$ by

$$D_\theta^i\xi := \sum_{j=1}^n\frac{\partial F}{\partial x_{i,j}}\Big(\int_0^T h_t^1\mathrm{d}W_t,\cdots,\int_0^T h_t^n\mathrm{d}W_t\Big)h_\theta^{i,j}, 0\leq\theta\leq T, 1\leq i\leq d,$$

and for $k\in\mathbb{N}$ and $\theta=(\theta_1,\cdots,\theta_k)\in[0,T]^k$ its k-fold iteration as

$$D^{(k)} = D^{i_1}\cdots D^{i_k}_{1\leq i_1,\cdots,i_k\leq d}.$$

For $k\in\mathbb{N}$ and $p\geq 1$, let $\mathbb{D}^{k,p}$ be the closure of $\mathcal{S}$ with respect to the norm

$$\|\xi\|_{k,p}^p = \|\xi\|_{L^p}^p + \sum_{i=1}^k|||D^{(i)}\xi|||^p_{(\mathcal{H}^p)^i}.$$

The operator $D^{(k)}$ is a closed linear operator on the space $\mathbb{D}^{k,p}$. Observe that if $\xi\in\mathbb{D}^{1,2}$ is $\mathfrak{F}_t$-measurable then $D_\theta\xi = 0$ for $\theta\in(t,T]$. Further denote $\mathbb{D}^{k,\infty} = \cap_{p>1}\mathbb{D}^{k,p}$.

For $k\in\mathbb{N}, p\geq 1$, denote by $\mathbb{L}_{k,p}(\mathbb{R}^m)$ the set of $\mathbb{R}^m$-valued progressively measurable processes $u=(u^1,\cdots,u^m)$ on $[0,T]\times\Omega$ such that

(i) For Lebesgue a.a. $t\in[0,T], u(t,\cdot)\in(\mathbb{D}^{k,p})^m$;
(ii) $(t,\omega)\to D_\theta^k u(t,\omega)\in(L^2([0,T]^{1+k}))^{d\times m}$ admits a progressively measurable version;
(iii) $\|u\|_{k,p}^p = |||u|||^p_{\mathcal{H}^p} + \sum_{i=1}^k|||D^{(i)}u|||^p_{(\mathcal{H}^p)^{1+i}} < \infty$.

For example if a process $\zeta\in\mathbb{L}_{2,2}(\mathbb{R})$, we have

$$\|\zeta\|^2_{\mathbb{L}_{1,2}} = \mathbb{E}\Big[\int_0^T|\zeta_t|^2\mathrm{d}t + \int_0^T\int_0^T|D_\theta\zeta_t|^2\mathrm{d}\theta\mathrm{d}t\Big],$$
$$\|\zeta\|^2_{\mathbb{L}_{2,2}} = \|\zeta\|^2_{\mathbb{L}_{1,2}} + \mathbb{E}\Big[\int_0^T\int_0^T\int_0^T|D_{\theta_1}D_{\theta_2}\zeta_t|^2\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}t\Big].$$

2.2. **Some preliminary results.** Consider the following BSDE

$$Y_t = \xi + \int_t^T g(s,\omega,Y_s,Z_s)\mathrm{d}s - \int_t^T Z_s\mathrm{d}B_s \tag{2.1}$$

and recall the ffollowing result on the Malliavin differentiablity of solution to the BSDE (2.1) in the Lipschitz framework (see for example [25]).

Theorem 2.1 (Malliavin differentiability)**.** *Suppose $\xi \in \mathbb{D}^{1,2}$ and $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable in (y,z), with uniformly bounded derivatives. Suppose for each $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, the process $g(\cdot,y,z)$ belongs to $\mathbb{L}_{1,2}(\mathbb{R}^d)$ with Malliavin derivative denoted by $D_\theta g(t,y,z)$. Let (Y,Z) be the solution of the associated BSDE (2.1) and suppose in addition*

(i) $(g(t,0,0))_{t\in[0,T]} \in \mathcal{H}^4(\mathbb{R})$ *and* $\xi \in L^4(\mathbb{R})$,

(ii) $\int_0^T \mathbb{E}[|D_\theta \xi|^2]\mathrm{d}\theta < \infty, \int_0^T \|(D_\theta g)(t,Y,Z)\|^2_{\mathcal{H}^2}\mathrm{d}\theta < \infty$ *a.s. and for a.a.* $\theta \in [0,T]$ *and any* $t \in [0,T], (y^1,z^1),(y^1,z^1) \in \mathbb{R}\times\mathbb{R}^d$
$$|D_\theta g(t,y^1,z^1) - D_\theta g(t,y^2,z^2)| \leq K_\theta(t)(|y^1-y^2| + |z^1-z^2|) \text{ a.s.},$$
with $\{K_\theta(t), 0 \leq \theta, t \leq T\}$ *a positive real valued adapted process satisfying* $\int_0^T \|K_\theta\|^4_{\mathcal{H}^4}\mathrm{d}\theta < \infty$.

Then $(Y,Z) \in L^2(\mathbb{D}^{1,2} \times (\mathbb{D}^{1,2})^d)$. Furthermore for each $1 \leq i \leq d$ a version of $\{(D^i_\theta Y_t, D^i_\theta Z_t); 0 \leq \theta, t \leq T\}$ is given by

$$D^i_\theta Y_t = 0, \quad D_\theta Z_t = 0, \qquad 0 \leq t < \theta \leq T;$$

$$D^i_\theta Y_t = D^i_\theta \xi + \int_t^T [(D^i_\theta g)(s,Y_s,Z_s) + \langle(\nabla g)(s,Y_s,Z_s),(D^i_\theta Y_s, D^i_\theta Z_s)\rangle]\mathrm{d}s - \int_t^T D^i_\theta Z_s \mathrm{d}B_s, \qquad \theta \leq t \leq T.$$

Moreover $\{D_\theta Y_t; 0 \leq t \leq T\}$ defined by the solution to the above BSDE is a version of $\{Z_t; 0 \leq t \leq T\}$.

We end this section by recalling the following result (see [35, Theorem 1.2.3])

Lemma 2.2. *Let $(F_n)_{n\geq 1}$ be a sequence of random variables in $\mathbb{D}^{1,2}$ that converges to F in $L^2(\Omega)$ and such that*
$$\sup_{n\geq 1} \mathbb{E}\left[\|DF_n\|_{L^2}\right] < \infty.$$
Then, F belongs to $\mathbb{D}^{1,2}$, and the sequence of derivatives $(DF_n)_{n\geq 1}$ converges to DF in the weak topology of $L^2(\Omega \times [0,T])$.

3. BSDEs with quadratic drivers

3.1. **Standing assumptions and solvability.** Our main aim in this section is the well posedness of the BSDE (2.1) when the parameters ξ and g satisfy the following assumptions:

Assumption 3.1. *ξ is an $\mathfrak{F}_T$-measurable uniformly bounded random variable, i.e., $\|\xi\|_{L^\infty} < \infty$;*

Assumption 3.2. *The function $g : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ is $\mathfrak{F}$-predictable and continuous in its space variables. There exist $\Lambda_0, \Lambda_y, \Lambda_z > 0$, and a locally bounded function $f : \mathbb{R} \mapsto \mathbb{R}_+$; $f \in L^1_{loc}(\mathbb{R}^+)$ such that for all $(t,\omega,y,z),(t,\omega,y',z') \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d$, $\alpha \in [0,1)$ $\|g(t,0,0)\|_{L^\infty} \leq \Lambda_0$ and*
$$|g(t,\cdot,y,z)-g(t,\cdot,y',z')| \leq \Lambda_y\Big(1+|z|^\alpha+|z'|^\alpha\Big)|y-y'|+\Lambda_z\Big(1+(f(|y|)+f(|y'|))(|z|+|z'|)\Big)|z-z'| \text{ a.s.}$$

Remark 3.3. *It is readily seen that under Assumptions 3.1 and 3.2, the generator g is necessary of the following form:*
$$|g(t,\cdot,y,z)| \leqslant \Lambda_0 + \Lambda_y|y| + \Lambda_z(|z| + f(|y|)|z|^2) \text{ a.s.} \tag{3.1}$$
Indeed,
$$\begin{aligned}|g(t,y,z)| &\leq |g(t,0,0)| + |g(t,y,0) - g(t,0,0)| + |g(t,y,z) - g(t,y,0)|\\ &\leq \Lambda_0 + \Lambda_y|y| + \Lambda_z|z| + 2\Lambda_z f(|y|)|z|^2 \text{ a.s.}\end{aligned}$$

Unless otherwise stated, in this paper φ stand for the smallest continuous and increasing function such that $f(x) \leq \varphi(x)$ for all $x \in \mathbb{R}$ (compare with [2]).

The next result concerns the existence and uniqueness of solution to the BSDE (2.1) under Assumptions 3.1 and 3.2.

Theorem 3.4 (Existence and uniqueness)**.** *Under Assumptions 3.1 and 3.2, the BSDE (2.1) has a unique strong solution $(Y,Z) \in \mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$.*

Proof. **Existence:** Since the generator g is of quadratic type, using [2, Theorem 3.1 and Corollary 3.2], the equation (2.1) has a maximal solution $(Y,Z)\in\mathcal{S}^\infty\times\mathcal{H}_{\mathrm{BMO}}$.
Uniqueness: Let $t\in[0,T]$ and let (Y,Z) and $(U,V)\in\mathcal{S}^\infty\times\mathcal{H}_{\mathrm{BMO}}$ be two solutions to BSDE (2.1). The $\mathcal{S}^\infty$-norms of the processes Y and U are uniformly bounded by the constant $\Upsilon^{(1)}$ that only depends on $\|\xi\|_{L^\infty}$, Λ_y and T (see the Lemma below). Now, set $\delta Y=Y-U$, $\delta Z=Z-V$ then the dynamics of $(\delta Y)_t$ is given by

$$\delta Y_t=\int_t^T\big(g(s,Y_s,Z_s)-g(s,U_s,Z_s)\big)\mathrm{d}s-\int_t^T\delta Z_s\mathrm{d}B_s.$$

Let $\tilde\Gamma_t,\tilde e_t$ and $\tilde\Pi_t$ be defined by

$$\tilde\Gamma_t:=\frac{g(t,Y_t,Z_t)-g(t,U_t,Z_t)}{Y_t-U_t}1_{\{Y_t-U_t\neq0\}},\quad \tilde e_t:=\exp\Big(\int_0^t\tilde\Gamma_s\mathrm{d}s\Big),\tag{3.2}$$

$$\tilde\Pi_t:=\frac{g(t,U_t,Z_t)-g(t,U_t,V_t)}{|Z_t-V_t|^2}(Z_t-V_t)1_{\{|Z_t-V_t|\neq0\}}.\tag{3.3}$$

From Assumption 3.2, we obtain $|\tilde\Pi|\le\Lambda_z(1+2f(|U|)(|Z|+|V|)$, from which we have $\|\tilde\Pi\|_{\mathcal{H}_{\mathrm{BMO}}}\le\tilde\Lambda:=\Lambda_z(\sqrt{T}+\varphi(\|U\|_{\mathcal{S}^\infty})(\|Z\|_{\mathcal{H}_{\mathrm{BMO}}}+\|V\|_{\mathcal{H}_{\mathrm{BMO}}}))$, where $\varphi(\|U\|_{\mathcal{S}^\infty}):=\sup_{0\le y\le\|U\|_{\mathcal{S}^\infty}}\varphi(y)<\infty$. Thus $\tilde\Pi*B$ is a BMO martingale since $Z,V\in\mathcal{H}_{\mathrm{BMO}}$. Hence the probability measure $\tilde{\mathbb{Q}}$ with Radon-Nykodim density $\mathrm{d}\tilde{\mathbb{Q}}/\mathrm{d}\mathbb{P}=\mathcal{E}(\int_0^\cdot\tilde\Pi.\mathrm{d}B.)$ is well defined and the process $B^{\tilde{\mathbb{Q}}}_\cdot=B.-\int_0^\cdot\tilde\Pi_s\mathrm{d}s$ is a $\tilde{\mathbb{Q}}$-Brownian motion. Moreover for $r>1$, we have $\mathcal{E}(\tilde\Pi)\in L^r$ (see Lemma A.1). On the other hand, using Assumption 3.2 once more, we deduce that $|\tilde\Gamma|\le\Lambda(1+2|Z|^\alpha)$. This implies that $\tilde\Gamma\in\mathcal{H}_{\mathrm{BMO}}$. Thus, the process $\tilde e$ is integrable (see (P4) in Lemma A.1) i.e., there is $p\ge1$ and $\varepsilon\in(0,2)$ such that $\mathbb{E}\Big[\exp\Big(p\int_0^T|\tilde\Gamma_t|^\varepsilon\mathrm{d}t\Big)\Big]<\infty$. In addition for all $p\ge1$, $\tilde e\in\mathcal{S}^p(\mathbb{R})$. The Girsanov theorem and Hölder's inequality yield: for every $r>1$

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big[\int_0^T|\tilde e_s|^2|\delta Z_s|^2\mathrm{d}s\Big]&\le\mathbb{E}\Big[\mathcal{E}\Big(\int_0^T\tilde\Pi_s\mathrm{d}B_s\Big)\sup_{0\le t\le T}|\tilde e_t|^2\int_0^T|\delta Z_s|^2\mathrm{d}s\Big]\\&\le\mathbb{E}\Big[\mathcal{E}\Big(\int_0^T\tilde\Pi_s\mathrm{d}B_s\Big)^r\Big]^{\frac1r}\mathbb{E}\Big[\sup_{0\le t\le T}|\tilde e_t|^{2q}\Big(\int_0^T|\delta Z_s|^2\mathrm{d}s\Big)^q\Big]^{\frac1q}<\infty,\end{aligned}$$

where q is the Hölder conjugate of r. Thus, the stochastic integral $\int_0^\cdot\tilde e_s\delta Z_s\mathrm{d}B^{\tilde{\mathbb{Q}}}_s$ defines a true $\tilde{\mathbb{Q}}$-martingale. By applying Itô's formula to the semimartingale $(\tilde e\delta Y)_t$ under $\tilde{\mathbb{Q}}$, we obtain that

$$\tilde e_t\delta Y_t+\int_t^T\tilde e_s\delta Z_s\mathrm{d}B^{\tilde{\mathbb{Q}}}_s=\int_t^T\tilde e_s[-\tilde\Gamma_s\delta Y_s+g(s,Y_s,Z_s)-g(s,U_s,Z_s)]\mathrm{d}s=0.\tag{3.4}$$

It follows that $\tilde e_t\delta Y_t=0$ $\tilde{\mathbb{Q}}$-a.s. for all $t\in[0,T]$. Thus, $\tilde e_t\delta Y_t=0$ $\mathbb{P}$-a.s. for all $t\in[0,T]$ (since $\tilde{\mathbb{Q}}$ and $\mathbb{P}$ are equivalents). Consequently $\delta Y_t=0$ $\mathbb{P}$-a.s. for all $t\in[0,T]$ provided that $\tilde e_t(\omega)>0$ for all ω outside a $\mathbb{P}$-negligible set A. The later is satisfied due to the continuity of the process $\tilde e_t$ for all $t\in[0,T]$. Hence, $Y_t=U_t$ $\mathbb{P}$-a.s., $\forall t\in[t,T]$. Using (3.4) and the fact that $\delta Y=0$, we deduce from the Itô's isometry that $\mathbb{E}^{\tilde{\mathbb{Q}}}\int_0^T|\tilde e_s|^2|\delta Z_s|^2\mathrm{d}s=0$, the latter implies that $Z_t=V_t$ $\mathrm{d}t\otimes\mathbb{P}$-a.s. □

Below we provide a more precise bounds for both the $\mathcal{S}^\infty$ and $\mathcal{H}_{\mathrm{BMO}}$ norms of the processes Y and Z respectively

Lemma 3.5. *Under Assumptions 3.1 and 3.2, the solution (Y,Z) to the BSDE (2.1) satisfies the following bounds:*

$$\|Y\|_{\mathcal{S}^\infty}\le\Upsilon^{(1)}:=(\|\xi\|_{L^\infty}+\Lambda_0T)e^{\Lambda_yT},\tag{3.5}$$

$$\|Z*B\|_{BMO}\le\Upsilon^{(2)}:=2\Upsilon^{(1)}\Big(\Upsilon^{(1)}+T(\Lambda_0+\Lambda_z+\Lambda_y\Upsilon^{(1)})\Big)\exp(4\|(1+\Lambda_zf)\|_{L^1[0,\Upsilon^{(1)}]}).\tag{3.6}$$

Proof. Let Π_t be defined by: $|Z_t|^2\Pi_t:=(g(t,Y_t,Z_t)-g(t,Y_t,0))Z_t1_{\{Z_t\neq0\}}$. By using the same observations as in the proof of the previous theorem, we have that $\mathcal{E}(\Pi*B)$ is uniformly integrable and the process $B^{\mathbb{Q}}_\cdot = B_\cdot - \int_0^\cdot \Pi_s \mathrm{d}s$ is a $\mathbb{Q}$-Brownian motion with the measure $\mathbb{Q}$ given by $\mathrm{d}\mathbb{Q} = \mathcal{E}(\Pi * B)\mathrm{d}\mathbb{P}$. Then,

$$|Y_t| \leq \mathbb{E}^{\mathbb{Q}}\Big(|\xi| + \int_t^T |g(s,Y_s,0)|\mathrm{d}s/\mathfrak{F}_t\Big) \leq \|\xi\|_{L^\infty} + \Lambda_0 T + \Lambda_y \mathbb{E}^{\mathbb{Q}}\Big(\int_t^T |Y_s|\mathrm{d}s/\mathfrak{F}_t\Big).$$

Therefore, the Gronwall's lemma yields : $|Y_t| \leq (\|\xi\|_{L^\infty} + \Lambda_0 T)e^{\Lambda_y T}$.

On the other hand for any locally integrable function f_1, we define:

$$K(y) := \int_0^y \exp\Big(-2\int_0^z f_1(u)\mathrm{d}u\Big)\mathrm{d}z$$
$$v(x) := \int_0^x K(y)\exp\Big(2\int_0^y f_1(u)\mathrm{d}u\Big)\mathrm{d}y.$$

It's readily seen that $v \in W^{1,2}_{\text{loc}}(\mathbb{R})$ and satisfies almost everywhere: $1/2v''(x) - f_1(x)v'(x) = 1/2$ (see [2]). Moreover for any $R > 0$ such that $|x| \leq R$, we have $|v(x)| \leq R^2\exp(4\|f_1\|_{L^1[0,\Upsilon^{(1)}]})$ and $|v'(x)| \leq R\exp(4\|f_1\|_{L^1[0,\Upsilon^{(1)}]})$. Recall from Remark 3.3 that the driver g satisfies $|g(t,y,z)| \leq \Lambda + \Lambda_y|y| + f_1(|y|)|z|^2$, where $\Lambda = \Lambda_0 + \Lambda_z$ and $f_1(|y|) := \Lambda_z(1 + f(|y|))$. Then, using the Itô-Krylov formula for BSDE (see [3, Theorem 2.1]) we obtain that

$$\begin{aligned} v(|Y_\tau|) &= v(|Y_T|) + \int_\tau^T \operatorname{sgn}(Y_r)v'(|Y_u|)g(u,Y_u,Z_u)\mathrm{d}u - \frac{1}{2}\int_\tau^T v''(|Y_u|)|Z_u|^2\mathrm{d}u + M_\tau^T \\ &\leq v(|Y_T|) - \frac{1}{2}\int_\tau^T |Z_u|^2\mathrm{d}u + \int_\tau^T (\Lambda + \Lambda_y|Y_u|)v'(|Y_u|)\mathrm{d}u + M_\tau^T \end{aligned}$$

for any stopping time τ. Here M_τ^T represents the martingale part. By taking the conditional expectation with respect to $\mathfrak{F}_\tau$, we deduce that

$$\frac{1}{2}\mathbb{E}\Big(\int_\tau^T |Z_u|^2\mathrm{d}u/\mathfrak{F}_\tau\Big) \leq \mathbb{E}\Big(v(|Y_T|) + \int_\tau^T (\Lambda + \Lambda_y|Y_u|)v'(|Y_u|)\mathrm{d}u/\mathfrak{F}_\tau\Big).$$

The result is obtained from the bounds of Y, v and v'. This completes the proof. □

Remark 3.6.

1. *Note that the bound in (3.5) does not depend on α, Λ_z, thus not on the $\mathcal{H}_{BMO}$-norm of the control process Z. In addition, if $f \equiv 0$, i.e., the driver g is Lipschitz in z and still stochastic Lipschitz in y, the BSDE (2.1) has a unique solution $(Y,Z) \in \mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ such that the bound of Y is given by (3.5).*
2. *Assume the driver g is deterministic and Lipschitz in y and z. Assume further the terminal value ξ is uniformly bounded and Malliavin differentiable with $\|D\xi\|_{\mathcal{S}^\infty} = \sup_{0\leq t\leq T}\|D_t\xi\| < \infty$ $\mathbb{P}$-a.s. Then, the BSDE (2.1) has a unique solution $(Y,Z) \in \mathcal{S}^2 \times \mathcal{H}^2$ (see [8, Proposition 2.4]) such that:*

$$|Y_t| \leq (\|\xi\|_{L^\infty} + \Lambda_0 T)e^{\Lambda_y T} \text{ and } |Z_t| \leq e^{\Lambda_y T}\|D\xi\|_{\mathcal{S}^\infty}.$$

The above remark leads to the following result which can be seen as an extension of the lemma 2.1 in [8].

Proposition 3.7. *Let Assumptions 3.1–3.2 be in force with g deterministic and Lipschitz in $y(\alpha \equiv 0)$. Assume further that ξ has a bounded Malliavin derivative. Then, the BSDE (2.1) has a solution $(Y,Z) \in \mathcal{S}^\infty \times \mathcal{S}^\infty$.*

Proof. Let us remark that, the key bound for the proof is given by the (3.5). This bound does not depend on Λ_z nor on the BMO-norm of the martingale $Z * B = \int_0^\cdot Z_s \mathrm{d}B_s$. Then we can obtain the result by applying similar technique as in the proof of [8, Lemma 2.1]. This concludes the proof. □

3.2. **Comparison theorem and a-priori estimates.** In this subsection, we provide a comparison theorem for quadratic BSDEs whose generators satisfy Assumptions 3.1-3.2. The proofs of the following Lemmatas are deported in Appendix B.

Lemma 3.8 (Comparison theorem). *Let $(Y^i, Z^i) \in \mathcal{S}^\infty \times \mathcal{H}_{BMO}$ be the solution to the BSDE (2.1), with terminal value ξ^i and generator g^i for $i \in \{1,2\}$. satisfying Assumptions 3.1-3.2 hold. In addition, suppose*

$$\xi^1 \leq \xi^2, \text{ and } g^1(t, Y_t^2, Z_t^2) \leq g^2(t, Y_t^2, Z_t^2) \quad \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{-a.s.}.$$

Then for all $t \in [0,T]$ $Y_t^1 \leq Y_t^2$ $\mathbb{P}$-a.s.. If either $\xi^1 < \xi^2$ or $g^1(t, Y_t^2, Z_t^2) < g^2(t, Y_t^2, Z_t^2)$ in a set of positive $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-measure then, $Y_0^1 < Y_0^2$.

The next Lemma provides the main (a-priori) estimates of this paper. These bounds will be extensively used to establish both the classical and variational differentiability of solution (Y, Z) to the BSDE (5.2).

Lemma 3.9 (A priori estimates). *Let $(Y^i, Z^i) \in \mathcal{S}^\infty \times \mathcal{H}_{BMO}$ be the solution to BSDE(2.1), with terminal value ξ^i and generator g^i for $i \in \{1,2\}$ satisfying Assumptions (3.1) and (3.2). Then for $p > 1$, there exists $q \in (1, \infty)$ only depending on, T, r and the BMO norm of $Z * B$ such that*

$$\|Y^1 - Y^2\|_{\mathcal{S}^{2p}}^{2p} + \|Z^1 - Z^2\|_{\mathcal{H}^{2p}}^{2p} \leq C\mathbb{E}\Big[|\xi_1 - \xi_2|^{2pq} + \Big(\int_0^T |g^1(s, Y_s^2, Z_s^2) - g^2(s, Y_s^2, Z_s^2)| \mathrm{d}s\Big)^{2pq}\Big]^{\frac{1}{q}}.$$

We also deduce the following stability result.

Corollary 3.10. *Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a sequence of bounded $\mathfrak{F}_T$-adapted random variables that converges $\mathbb{P}$-a.s. to ξ. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of drivers satisfying Assumption 3.2 with the same constant Λ and the same function (f), such that for $\mathrm{d}t \times \mathrm{d}\mathbb{P}$-a.s. $(t, \omega) \in [0,T] \times \Omega$ $(g_n)_{n \in \mathbb{N}}(t, \omega, y, z)$ converges to $(g)(t, \omega, y, z)$ locally uniformly in $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. Let $(Y^n, Z^n) \in \mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ be the solution to the BSDE (2.1) with parameters (ξ_n, g_n). Then the BSDE (2.1) has a unique solution $(Y, Z) \in \mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ such that $\mathbb{P}$-a.s. Y_t^n converges to Y_t uniformly in $t \in [0,T]$ and Z^n converges to Z in $\mathcal{H}^2(\mathbb{R}^d)$*

The proof of Corollary 3.10 is straightforward from Lemma 3.9, we will not provide it here.

4. Differentiability of parameterized quadratic BSDEs

In this section we discuss the both the Malliavin and the classical differentiablility of the solutions to the BSDEs (2.1). We provide below, sufficient conditions for these differentiabilities to hold.

4.1. **Malliavin Differentiability.** In this sebsection we show under some weak conditions on the generator that the solution (Y, Z) to the BSDE (2.1) is Malliavin differentiable and the Malliavin derivatives $(D_u Y_t, D_u Z_t)_{u,t \in [0,T]}$ are given as solution to a BSDE. In addition to Assumptions 3.1-3.2, we will suppose the following additional assumptions

Assumption 4.1.

(M1) (i) *The function $g : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ is adapted, measurable and continuously differentiable in (y, z). There exist constants $\Lambda_y, \Lambda_z > 0$, and a non-decreasing function f (the same function as in Assumption 3.2) such that for all $(t, y, z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$ and $\alpha \in (0,1)$*

$$|\nabla_y g(t, y, z)| \leq \Lambda_y (1 + |z|^\alpha) \text{ a.s.},$$
$$|\nabla_z g(t, y, z)| \leq \Lambda_z (1 + f(|y|)|z|) \text{ a.s.}$$

(ii) *The random variable ξ belongs to $\mathbb{D}^{1,\infty}$.*

(M2) *For each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, it holds that $(g(t, y, z))_{t \in [0,T]} \in \mathbb{L}_{1,2p}(\mathbb{R})$ for all $p \geq 1$. Its Malliavin derivative denoted by $(D_u g(t, y, z))_{u,t \in [0,T]}$ satisfies*

$$|D_u g(t, y, z)| \leq K_u(t)(1 + |y| + [f(|y|)|z|]^\alpha) + \tilde{K}_u(t)(1 + |z|^\alpha + f(|y|)|z|) \text{ a.s.}$$

for any $(u,t,y,z) \in [0,T]\times[0,T]\times\mathbb{R}\times\mathbb{R}^d$, $\alpha \in (0,1)$. *Here* $(K_u(t))_{u,t\in[0,T]}$ *and* $(\tilde{K}_u(t))_{u,t\in[0,T]}$ *are two positive adapted processes such that for all* $p \geq 1$

$$\sup_{0\leq t\leq T}\int_0^T \mathbb{E}|K_u(t)|^{2p}\mathrm{d}u + \|\tilde{K}_u(t)\|^{2p}_{\mathcal{S}^{2p}} < \infty.$$

Below, we state the main result in this subsection:

Theorem 4.2. *Suppose g and ξ satisfy Assumptions 3.1-3.2, and 4.1. Then the solution process (Y,Z) to the BSDE (2.1) is in $\mathbb{L}_{1,2}\times(\mathbb{L}_{1,2})^d$ and a version of $(D_uY_t, D_uZ_t)_{u,t\in[0,T]}$ is the unique solution to*

$$\begin{aligned}
&D_uY_t = 0 \text{ and } D_uZ_t = 0, \text{ if } t\in[0,u),\\
&D_uY_t = D_u\xi - \int_t^T D_uZ_s\mathrm{d}B_s\\
&\qquad + \int_t^T \left[(D_ug)(s,\Theta_s) + \langle(\nabla g)(s,\Theta_s), D_u\Theta_s\rangle\right]\mathrm{d}s, \text{ if } t\in[u,T].
\end{aligned} \tag{4.1}$$

Moreover, $\{D_tY_t : 0\leq t\leq T\}$ is a version of $\{Z_t : 0\leq t\leq T\}$.

The strategy to prove the above theorem is also well known and analogous to [1, 19, 20, 38]. It is performed in two main steps. We first build a family of truncated BSDEs that approximate the BSDE (2.1) and then, prove some uniform bounds for solutions to the truncated BSDEs. Second we apply a compactness result (Lemma 2.2) to derive the desired result.

4.1.1. *Family of truncated generators.* : Let us consider the family $(\tilde{\rho}_n)_{n\in\mathbb{N}}$ of smooth (continuously differentiable) real valued functions, that truncated the identity on the real line. We use this family of functions to truncate the variables y and z simultaneously in the driver $g(\cdot,y,z)$, for $(y,z)\in\mathbb{R}\times\mathbb{R}^d$. If the procedure is well known (see for example [1, 20, 38]), it is worth mentioning that truncating the two variables at the same time is not so common in the literature. This approach is motivated by the form of the driver which is not uniformly Lipschitz in y.

The family of functions $(\tilde{\rho}_n)_{n\in\mathbb{N}}$ are such that:

(a) $(\tilde{\rho}_n)_{n\in\mathbb{N}}$ converges uniformly to the identity. For all $n\in\mathbb{N}$ and $x\in\mathbb{R}$

$$\tilde{\rho}_n(x) = \begin{cases} n+1, & x > n+2,\\ x, & |x|\leq n,\\ -(n+1), & x < -(n+2). \end{cases} \tag{4.2}$$

In addition $|\tilde{\rho}_n(x)| \leq |x|$ and $|\tilde{\rho}_n(x)|\leq n+1$.

(b) The derivative $\nabla\tilde{\rho}_n$ is absolutely uniformly bounded by 1, and converges to 1 locally uniformly.

Let $(g_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$g_n(t,y,z) := g(t,\tilde{\rho}_n(y),\rho_n(z)) \text{ for } (\omega,t,y,z)\in\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d,\ n\in\mathbb{N}, \tag{4.3}$$

where $\rho_n : \mathbb{R}^d\to\mathbb{R}^d, z\mapsto\rho_n(z) = (\tilde{\rho}_n(z_1),\ldots,\tilde{\rho}_n(z_d)), n\in\mathbb{N}$.

Let us consider the following sequence $(Y^n,Z^n)_{n\geq1}$ satisfying the BSDE

$$Y_t^n = \xi + \int_t^T g_n(s,Y_s^n,Z_s^n)\mathrm{d}s - \int_t^T Z_s^n\mathrm{d}B_s, \qquad t\in[0,T], n\in\mathbb{N}. \tag{4.4}$$

Fix $n\in\mathbb{N}$. Using (M1), the family $(g_n)_{n\in\mathbb{N}}$ satisfies: for $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$

$$|\nabla_y g_n(t,y,z)| + |\nabla_z g_n(t,y,z)| \leq \Lambda_y(1+(n+1)^\alpha) + \Lambda_z(1+(n+1)f(n+1)).$$

In addition, using the mean value theorem, we obtain for all $t\in[0,T]$, $y,y'\in\mathbb{R}$, and $z,z'\in\mathbb{R}^d$ $|g_n(t,y,z)-g_n(t,y',z')| \leq \Lambda_n(|y-y'|+|z-z'|)$. Thus for each $n\in\mathbb{N}$, the family of functions $(g_n)_{n\in\mathbb{N}}$ is Lipschitz continuous in its spatial variables.

The next Lemma gives the uniform bounds of (Y^n,Z^n) solution to the BSDE (4.4) in the Banach space $\mathcal{S}^\infty(\mathbb{R})\times\mathcal{H}^2(\mathbb{R}^d)$.

Lemma 4.3. *For each $n \in \mathbb{N}$, the BSDE (4.4) admits a unique solution (Y^n, Z^n) which is uniformly bounded in $\mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$. In addition, the process $Z^n \in \mathcal{H}_{BMO}$, and $\sup_{n\in\mathbb{N}} \|\mathcal{E}(Z^n * B)\|_{BMO} \leq \Upsilon^{(2)}$, where $\Upsilon^{(2)}$ is given in Theorem 3.4. Furthermore, there exists $r > 1$ independent of n such that $\sup_{n\in\mathbb{N}} \|\mathcal{E}(Z^n * B)\|_{L^r} < \infty$.*

Proof. The existence and uniqueness of solution to BSDE (4.4) follows directly from standard results in the theory of BSDEs, since the generator g_n satisfies for each $n \in \mathbb{N}$ a Lipschitz condition in the space variables. Let us remark that the function g_n satisfies (thanks to condition (M1)) a similar condition to equation (3.1), that is for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned} |g_n(t, y, z)| &\leq \Lambda_0 + \Lambda_y |\tilde{\rho}_n(y)| + \Lambda_z(|\rho_n(z)| + f(|\tilde{\rho}_n(y)|)|\rho_n(z)|^2) \\ &\leq \Lambda_0 + \Lambda_y|y| + \Lambda_z(|z| + f(|y|)|z|^2). \end{aligned}$$

Hence conditions on the existence of a maximal solution to the BSDE (4.4) in Theorem 3.4 are satisfied. Therefore, $\sup_{n\in\mathbb{N}} \Big[\|\mathcal{E}(Z^n * B)\|_{BMO}\Big] < \infty$.

The existence of such a constant r for which the bound holds follows directly from (P2) in Lemma A.1. □

Remark 4.4. *As a consequence of Corollary 3.10, the sequence $(Y^n)_{n\in\mathbb{N}}$ converges to Y uniformly on $[0, T]$, the sequence $(Z^n)_{n\in\mathbb{N}}$ converges to Z in $\mathcal{H}^2(\mathbb{R}^d)$ and (Y, Z) solves the BSDE (2.1).*

Recall that $\mathbb{L}_{1,2}$ stands for the set of progressively measurable processes $(\eta_t)_{0\leq t\leq T}$ which are Malliavin differentiable, with $(D_s\eta_t)_{s\leq t\leq T}$ having a progressively measurable version and such that $\|\eta\|^2_{1,2} := \mathbb{E}\Big[\int_0^T |\eta_t|\mathrm{d}t + \int_0^T\int_0^T |D_s\eta_t|^2\mathrm{d}s\mathrm{d}t\Big] < +\infty$.

Lemma 4.5. *Suppose $\xi \in \mathbb{D}^{1,\infty}$ and for each $n \in \mathbb{N}$, let g_n be as in (4.3). Then the solution $\Theta^n = (Y^n, Z^n)_{n\in\mathbb{N}}$ to the BSDE (4.4) belongs to $\mathbb{L}_{1,2} \times (\mathbb{L}_{1,2})^d$. A version of $\{(D_uY^n_t, D_uZ^n_t), 0 \leq u, t \leq T\}$ is given by*

$$\begin{aligned} D_uY^n_t =&0 \text{ and } D_uZ^n_t = 0, \text{ if } t \in [0, u), \\ D_uY^n_t =&D_u\xi - \int_t^T D_uZ^n_s\mathrm{d}B_s \\ &+ \int_t^T [(D_ug_n)(s, \Theta^n_s) + \langle(\nabla g_n)(s, \Theta^n_s), D_u\Theta^n_s\rangle]\,\mathrm{d}s, \text{ if } t \in [u, T]. \end{aligned} \tag{4.5}$$

Moreover $\{D_tY^n_t, 0 \leq t \leq T\}$ defined by the above equation is a version of $\{Z^n_t, 0 \leq t \leq T\}$. Furthermore for any $p > 1$, the following holds:

$$\sup_{n\in\mathbb{N}} \int_0^T \mathbb{E}\Big[\|D_uY^n\|^{2p}_{\mathcal{S}^{2p}} + \|D_uZ^n\|^{2p}_{\mathcal{H}^{2p}}\Big]\mathrm{d}u < \infty. \tag{4.6}$$

Proof. The proof of the first statement concerning the Malliavin derivatives (DY^n, DZ^n) of $\Theta^n = (Y^n, Z^n)_{n\in\mathbb{N}}$ and the representation (4.5) follows from Theorem 2.1 under Assumptions 4.1.

Let us now focus on the proof of the bound (4.6). Note that, condition (M1) and Lemma 4.3 imply that: $\nabla_z g_n * B := \int_0^\cdot \nabla_z g_n(s, \Theta_s)\mathrm{d}B_s$ is a BMO martingale and the measure $\mathbb{Q}^n$ with density $\mathrm{d}\mathbb{Q}^n := \mathcal{E}(\nabla_z g_n * B)\mathrm{d}\mathbb{P}$ defines an equivalent measure to the probability $\mathbb{P}$. Then, Girsanov's theorem ensures that the process $B^n_t = B_t - \int_0^t \nabla_z g_n(s, \Theta^n_s)\mathrm{d}s$ is a Brownian motion under the new probability measure $\mathbb{Q}^n$. Thus (4.5) can be written is terms of B^n as follows

$$D_uY^n_t = D_u\xi - \int_t^T D_uZ^n_s\mathrm{d}B^n_s + \int_t^T F^n(s, D_uY^n_s, D_uZ^n_s)\mathrm{d}s,$$

where $F^n(s, D_uY^n_s, D_uZ^n_s) := (D_ug_n)(s, \Theta^n_s) + (\nabla_y g_n)(s, \Theta^n_s)D_u\Theta^n_s$. By using once more a standard linearisation technique and applying Itô's formula to the continuous semimartingale $(e^n_t D_uY^n_t)_{0\leq t\leq T}$, one obtains

$$e^n_t D_uY^n_t = e^n_T D_u\xi - \int_t^T e^n_s D_uZ^n_s\mathrm{d}B^n_s + \int_t^T e^n_s D_ug_n(s, \Theta^n_s)\mathrm{d}s,$$

where $e_t^n = \exp(\int_0^t a_s^n \mathrm{d}s)$ and $a_t^n = \frac{F^n(t,D_uY_t^n,D_uZ_t^n)-F^n(t,0,D_uZ_t^n)}{D_uY_t^n}1_{\{D_uY_t^n\neq 0\}}$. Therefore, the subsequent bound follows as in Lemma 3.9 i.e. for $p>1$, there exists $q\in(1,\infty)$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|D_uY_t^n|^{2p}+\Big(\int_0^T|D_uZ_s^n|^2\mathrm{d}s\Big)^p\Big]\leq C\mathbb{E}\Big(|D_u\xi|^{2pq}+\Big\{\int_0^T|(D_ug_n)(t,Y_t^n,Z_t^n)|\mathrm{d}t\Big\}^{2pq}\Big)^{\frac{1}{q}},$$

and q only depends on T,p and $\|Z^n*B\|_{BMO}$. Hence, the Jensen's inequality for concave functions leads to:

$$\begin{aligned}&\int_0^T\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|D_uY_t^n|^{2p}+\Big(\int_0^T|D_uZ_s^n|^2\mathrm{d}s\Big)^p\Big]\Big\}\mathrm{d}u\\ \leq& C\Big(\int_0^T\mathbb{E}\Big[|D_u\xi|^{2pq}+\Big(\int_0^T|(D_ug_n)(t,Y_t^n,Z_t^n)|\mathrm{d}t\Big)^{2pq}\Big]\mathrm{d}u\Big)^{\frac{1}{q}}.\end{aligned}$$

Since $\xi\in\mathbb{D}^{1,\infty}$, it follows that the first term on the right side of the above inequality is finite. As for the second term is also finite, let us et $\gamma=2pq$ and recall that the sequence of functions $(g_n)_{n\in\mathbb{N}}$ is given by equation (4.3). From assumption (M2), we deduce

$$\begin{aligned}\mathbb{E}\Big(\int_0^T|(D_ug_n)(t,Y_t^n,Z_t^n)|\mathrm{d}t\Big)^\gamma\leq& C\mathbb{E}\Big(\int_0^T K_u(t)(1+|\tilde\varphi(Y_t^n)|+[f(|\tilde\varphi(Y_t^n)|)|\varphi(Z_t^n)|]^\alpha)\mathrm{d}t\\ &+\sup_{t\in[0,T]}|\tilde K_u(t)|^2+\Big\{\int_0^T(1+|\varphi(Z_t^n)|^\alpha+f(|\tilde\varphi(Y_t^n)|)|\varphi(Z_t^n)|)\mathrm{d}t\Big\}^2\Big)^\gamma\end{aligned}$$

Using the properties of $\tilde\varphi$ and φ, the fact that f is increasing, Hölder's inequality and the inequality $|ab|\leq a^2+b^2$, we deduce the following:

$$\begin{aligned}\mathbb{E}\Big(\int_0^T|(D_ug_n)(t,Y_t^n,Z_t^n)|\mathrm{d}t\Big)^\gamma\leq& C\mathbb{E}\Big(\int_0^T K_u(t)(1+|Y_t^n|+[f(|Y_t^n|)|Z_t^n|]^\alpha)\mathrm{d}t+\sup_{t\in[0,T]}|\tilde K_u(t)|^2\\ &+\Big\{\int_0^T(1+|Z_t^n|^\alpha+f(|Y_t^n|)|Z_t^n|)\mathrm{d}t\Big\}^2\Big)^\gamma\\ \leq& C\mathbb{E}\Big(\int_0^T|K_u(t)|^2\mathrm{d}t+\int_0^T(1+|Y_t^n|+[f(|Y_t^n|)|Z_t^n|]^\alpha)^2\mathrm{d}t\\ &+\sup_{t\in[0,T]}|\tilde K_u(t)|^2+\Big\{\int_0^T(1+|Z_t^n|^\alpha+f(|Y_t^n|)|Z_t^n|)\mathrm{d}t\Big\}^2\Big)^\gamma<\infty.\end{aligned}$$

The proof is completed. □

We are now in position to prove the main theorem of this section.

Proof of Theorem 4.2 . Let us define by μ the measure $\mathrm{d}\mu=\mathrm{d}\mathbb{P}\otimes\mathrm{d}u\otimes\mathrm{d}t$. Using Lemmas 4.3 and 4.5, there exists a subsequence $(D_uY_t^n,D_uZ_t^n)$ (still indexed by n) that converges weakly to a limit process denoted by $(U_{u,t},V_{u,t}),0\leq u,t\leq T$ in the space of random variables with values in $L^2(\Omega\times[0,T])$. Thus, it is readily seen that for almost $t\in[0,T]$, the solution (Y_t,Z_t) to the BSDE (2.1) is Malliavin differentiable and $(D_uY_t,D_uZ_t)=(U_{u,t},V_{u,t})$ $\mathrm{d}\mu$-a.e. in $\Omega\times[0,T]\times[0,T]$. To conclude, we only need to prove that each term in equation (4.5) converges to its corresponding counterpart in equation (4.1) when n goes to infinity. The convergence off the stochastic intergal is well known and we do not reproduce its proof here (see [1] or [38, Theorem 3.2.3]).

Using assumption (M1) and the dominated convergence theorem, one can show that

$$\int_0^T\langle(\nabla g_n)(s,\Theta_s^n),D_u\Theta_s^n\rangle\mathrm{d}s \text{ converges to } \int_0^T\langle(\nabla g)(s,\Theta_s),D_u\Theta_s\rangle\mathrm{d}s$$

in the weak topology of $L^1(\Omega\times[0,T])$. Indeed, let ζ be any bounded $\mathfrak{F}_T$-adapted random variable. For $n\in\mathbb{N}$ and for almost all $u\in[0,T]$, using Hölder inequality, we have

$$\mathbb{E}\Big[\zeta\int_0^T D_uY_s^n\big(2+|Z_s|^\alpha+|Z_s^n|^\alpha\big)\mathrm{d}s\Big]\leq C\operatorname{essup}|\zeta|\mathbb{E}\Big[\sup_{s\in[0,T]}|D_uY_s^n|^2+\Big(\int_0^T\big(4+|Z_s|^{2\alpha}+|Z_s^n|^{2\alpha}\big)\mathrm{d}s\Big)\Big].$$

Using Lemma 4.3 and 4.5 and Theorem 3.4, it follows that the right side of the above inequality is uniformly bounded in n. In addition, the continuity of $\nabla_y g$, the uniform convergence of $(Y^n)_{n\in\mathbb{N}}$ to Y and the $\mathcal{H}^2$ convergence of $(Z^n)_{n\in\mathbb{N}}$ to Z yield the convergence of $(\nabla_y g_n)(s,\Theta_s^n)$ to $(\nabla_y g)(s,\Theta_s)$ in $\mathcal{H}^2(\Omega\times[0,T])$ as n goes to infinity. Since ζ is chosen arbitrarily, the uniform convergence of the process $(DY^n)_{n\in\mathbb{N}}$ to DY and the dominated convergence imply that

$$\lim_{n\to\infty}\int_0^T \nabla_y g_n(s,\Theta_s^n)D_uY_s^n\mathrm{d}s = \int_0^T \nabla_y g(s,\Theta_s)D_uY_s\mathrm{d}s.$$

Applying the same reasoning as above, we deduce that

$$\lim_{n\to\infty}\int_0^T \nabla_z g_n(s,\Theta_s^n)D_uZ_s^n\mathrm{d}s = \int_0^T \nabla_z g(s,\Theta_s)D_uZ_s\mathrm{d}s.$$

Let us now show that

$D_ug_n(s,Y_s^n,Z_s^n)$ converges to $D_ug(s,Y_s,Z_s)$ in the weak topology of $L^2(\Omega\times[0,T])$.

Using (4.3), assumption (M2) and the assumptions on $\varphi,\tilde\varphi$ and the function f, we have

$$\begin{aligned}\mathbb{E}\int_0^T\int_0^T |D_ug_n(s,Y_s^n,Z_s^n)|^2\mathrm{d}s\mathrm{d}u \le&\mathbb{E}\int_0^T\int_0^T |K_u(s)|^2(1+|\tilde\varphi(Y_s^n)|+[f(|\tilde\varphi(Y_s^n)|)|\varphi(Z_s^n)|]^\alpha)^2\mathrm{d}s\mathrm{d}u\\ &+\mathbb{E}\int_0^T\int_0^T |\tilde K_u(s)|^2(1+|\varphi(Z_s^n)|^\alpha+f(|\tilde\varphi(Y_s^n)|)|\varphi(Z_s^n)|)^2\mathrm{d}s\mathrm{d}u\\ \le&\mathbb{E}\int_0^T\int_0^T |K_u(s)|^2(1+|Y_s^n|+[f(|Y_s^n|)|Z_s^n|]^\alpha)^2\mathrm{d}s\mathrm{d}u\\ &+\mathbb{E}\int_0^T\int_0^T |\tilde K_u(s)|^2(1+|Z_s^n|^\alpha+f(|Y_s^n|)|Z_s^n|)^2\mathrm{d}s\mathrm{d}u\le C\end{aligned}$$

where the last inequality follows from Lemma 4.3. Therefore, Lemma 2.2 yields that $D_ug_n(s,Y_s^n,Z_s^n)$ converges to $D_ug(s,Y_s,Z_s)$ in the weak topology of $L^2(\Omega\times[0,T])$. The proof is completed. □

4.2. **Classical differentiability.** Throughout this section, we consider the following parameterised BSDE

$$Y_t^x=\xi(x)+\int_t^T g(s,\omega,x,Y_s^x,Z_s^x)\mathrm{d}s-\int_t^T Z_s^x\mathrm{d}B_s,\quad t\in[0,T],x\in\mathbb{R}^m. \tag{4.7}$$

We suppose the following assumption:

Assumption 4.6.

(C1) *Let $m,d\in\mathbb{N}$. Let $g:[0,T]\times\Omega\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ be an adapted measurable function, differentiable in the spatial variables with continuous partial derivatives in y and z. There exist a positive process $(K_t(x))_{t\in[0,T]}$ depending on $x\in\mathbb{R}^m$ and a locally bounded and non-decreasing function $f\in L^1_{loc}(\mathbb{R},\mathbb{R}_+)$ such that for all $(t,x,y,z)\in\times[0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d$, $\alpha\in(0,1)$*

$$\begin{aligned}|g(t,x,y,z)|&\le\Lambda_0+\Lambda_y|y|+\Lambda_z(|z|+f(|y|)|z|^2)\ a.s.,\\ |\nabla_x g(t,x,y,z)|&\le K_t(x)(1+|y|+[f(|y|)|z|]^\alpha)\ a.s.,\\ |\nabla_y g(t,x,y,z)|&\le\Lambda_y(1+|z|^\alpha)\ a.s.,\\ |\nabla_z g(t,x,y,z)|&\le\Lambda_z(1+f(|y|)|z|)\ a.s.\end{aligned}$$

Furthermore, the process $(K_t(x))_{t\in[0,T]}$ satisfies $\sup_{x\in\mathbb{R}^m}\int_0^T\mathbb{E}|K_s(x)|^{2p}\mathrm{d}s<\infty$ for any $p\ge1$.

(C2) (i) *For any $x\in\mathbb{R}^m$, the random variable $\xi(x)$ is $\mathfrak{F}_T$-adapted and $\sup_{x\in\mathbb{R}^m}\|\xi(x)\|_{L^\infty(\Omega)}<\infty$ a.s.*

(ii) *For all $p\ge1$ the mapping $x\mapsto\xi(x)$ from $\mathbb{R}^m$ to $L^{2p}(\Omega)$ is differentiable and $\sup_{x\in\mathbb{R}^m}\|\nabla_x\xi(x)\|_{L^{2p}(\Omega)}<\infty$.*

(C3) *The function $x\mapsto\nabla_x\xi(x)$ is continuous.*

Note that even if the process $(K_t(x))_{t\in[0,T]}$ does not satisfy one of the standard requirements in the literature (that is $\mathbb{E}\sup_{0\le t\le T}|K_t(x)|^p<\infty$), we can still use the Hölder and the Minkowski's integral type inequalities to prove the desire result.

Lemma 4.7. *Suppose Assumptions (C1) and (C2) are valid. For all $p>1$ and $i\in\{1,\dots,m\}$ there exists $C>0$ such that for all $x,x'\in\mathbb{R}^m$ and $h,h'\in\mathbb{R}$ for which $(x+he_i)$ and $(x'+h'e_i)$ belongs to $\mathbb{R}^m$ we have*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Y_t^{x+he_i}-Y_t^{x'+h'e_i}|^{2p}+\Big(\int_0^T|Z_s^{x+he_i}-Z_s^{x'+h'e_i}|^2\mathrm{d}s\Big)^p\Big]\le C(|x-x'|^2+|h-h'|^2)^p, \tag{4.8}$$

where (Y^r,Z^r) is the solution to the BSDE (4.7) with parameter $r\in\{x+he_i,x'+h'e_i\}$.

Remark 4.8. *It follows rom the Kolmogorov's continuity criterion,for $0\le t\le T$ the mapping $x\mapsto Y_t^x$ has a continuous version for which almost all sample paths are β-Hölder continuous in $\mathbb{R}^m$ for any $\beta\in(0,1)$. For $(t,x)\in[0,T]\times\mathbb{R}^m$ the mapping $t\mapsto Y_t^x(\omega)$ is continuous $\mathbb{P}$-a.s. $\omega\in\Omega$. This is a necessary condition to obtain a classical differentiability result for the solution process (Y^x,Z^x) to the BSDE (4.7) under Assumption 4.6.*

Proof. Set $\bar{x}=x-x',\bar{h}=h-h'$ and define the following processes: $\delta Y_t:=(Y_t^{x+he_i}-Y_t^{x'+h'e_i})$, $\delta Z_t:=(Z_t^{x+he_i}-Z_t^{x'+h'e_i})$ and $\delta\xi:=(\xi(x+he_i)-\xi(x'+h'e_i))$. From assumption (C1) $(\delta Y,\delta Z)$ satisfies the equation

$$\delta Y_t=\delta\xi-\int_t^T\left[I_s^y(\bar{x}+\bar{h}e_i)+I_s^x\delta Y_s+I_s^z\delta Z_s\right]\mathrm{d}s-\int_t^T\delta Z_s\mathrm{d}B_s,$$

where the processes I^x,I^y and I^z are given by

$$\begin{aligned}
I_s^x&=\int_0^1(\nabla_x g)(s,x+he_i+\theta(\bar{x}+\bar{h}e_i),Y_s^{x+he_i},Z_s^{x+he_i})\mathrm{d}\theta,\\
I_s^y&=\int_0^1(\nabla_y g)(s,x'+h'e_i,Y_s^{x+he_i}-\theta\delta Y_s,Z_s^{x+he_i})\mathrm{d}\theta,\\
I_s^z&=\int_0^1(\nabla_z g)(s,x'+h'e_i,Y_s^{x+he_i},Z_s^{x+he_i}-\theta\delta Z_s)\mathrm{d}\theta.
\end{aligned}$$

Using (C1), the following bounds can be obtained:

$$\begin{aligned}
|I_t^x|&\le\int_0^1\Big[(1+|Y_t^{x+he_i}|+[f(|Y_t^{x+he_i}|)|Z_t^{x+he_i}|]^\alpha)K_t(x+he_i+\theta(\bar{x}+\bar{h}e_i))\Big]\mathrm{d}\theta,\\
|I_t^y|&\le\Lambda_y(1+|Z_t^{x+he_i}|^\alpha),\\
|I_t^z|&\le\Lambda_z\big(1+f(|Y_t^{x+he_i}|)(|Z_t^{x+he_i}|+|Z_t^{x'+h'e_i}|)\big).
\end{aligned}$$

Observe that (I^z*B) is a BMO martingale and

$$\sup_{x\in\mathbb{R}^m,h\in\mathbb{R}}(\|I^z*B\|_{BMO})\le\sup_{x\in\mathbb{R}^m,h\in\mathbb{R}}(\|Z^{x+he_i}*B\|_{BMO}+\|Z^x*B\|_{BMO})<\infty.$$

Moreover, from (C2), $\delta\xi$ is bounded. Thus, Lemma 3.9 (see also [22, Lemma A 1]) yields that for any $p>1$:

$$\|\delta Y\|_{\mathcal{S}^{2p}}^{2p}+\|\delta Z\|_{\mathcal{H}^{2p}}^{2p}\le C\mathbb{E}\Big[|\delta\xi|^{2pq}+\Big(\int_0^T|I_s^x|(|x-x'|+|h-h'|)\mathrm{d}s\Big)^{2pq}\Big]^{\frac{1}{q}}, \tag{4.9}$$

where $q\in(1,\infty)$. From (C2) and the mean value theorem, we deduce the existence of a constant $C>0$ such that

$$\mathbb{E}|\delta\xi|^{2pq}\le C(|x-x'|^2+|h-h'|^2)^{pq},$$

We now focus on the second term of (4.9). For simplicity, we will write $K_t^{\bar{x},\bar{h}}(\theta):=K_t(x+he_i+\theta(\bar{x}+\bar{h}e_i))$. For any $\gamma\ge2$, using the Hölder inequality and Minkowski's integral type inequality we have

$$\begin{aligned}
&\Big(\int_0^T |I_t^x|\mathrm{d}t\Big)^\gamma\\
\leq&\Big(\int_0^T (1+|Y_t^{x+he_i}|+\big[f(|Y_t^{x+he_i}|)|Z_t^{x+he_i}|\big]^\alpha)\int_0^1 K_t^{\bar{x},\bar{h}}(\theta)\mathrm{d}\theta\mathrm{d}t\Big)^\gamma\\
\leq&\Big(\int_0^T \big(1+|Y_t^{x+he_i}|+\big[f(|Y_t^{x+he_i}|)|Z_t^{x+he_i}|\big]^\alpha\big)^{\frac{1}{\alpha}}\mathrm{d}t\Big)^{\alpha\gamma}\Big(\int_0^T \Big(\int_0^1 K_t^{\bar{x},\bar{h}}(\theta)\mathrm{d}\theta\Big)^{\frac{1}{1-\alpha}}\mathrm{d}t\Big)^{\gamma(1-\alpha)}\\
\leq&C\Big(\int_0^T \big(1+|Y_t^{x+he_i}|^{\frac{1}{\alpha}}+\sup_{0\leq y\leq 2\Upsilon^{(1)}}\varphi(y)+f(|Y_t^{x+he_i}|)|Z_t^{x+he_i}|^2\big)\mathrm{d}t\Big)^{\alpha\gamma}\Big(\int_0^1 \Big(\int_0^T |K_t^{\bar{x},\bar{h}}(\theta)|^{\frac{1}{1-\alpha}}\mathrm{d}t\Big)^{(1-\alpha)}\mathrm{d}\theta\Big)^{\gamma}.
\end{aligned}$$

Taking the expectation on both sides of the above inequality and applying the Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^T |I_t^x|\mathrm{d}t\Big)^\gamma\\
\leq&C\Big[\mathbb{E}\Big(\int_0^T \big(1+|Y_t^{x+he_i}|^{\frac{1}{\alpha}}+f(|Y_t^{x+he_i}|)|Z_t^{x+he_i}|^2\big)\mathrm{d}t\Big)^{\gamma}\Big]^{\alpha}\Big[\mathbb{E}\Big(\int_0^1 \Big(\int_0^T |K_t^{\bar{x},\bar{h}}(\theta)|^{\frac{1}{1-\alpha}}\mathrm{d}t\Big)^{(1-\alpha)}\mathrm{d}\theta\Big)^{\frac{\gamma}{1-\alpha}}\Big]^{(1-\alpha)}\\
\leq&C\Big[\mathbb{E}\Big(\int_0^T \big(1+|Y_t^{x+he_i}|^{\frac{1}{\alpha}}+f(|Y_t^{x+he_i}|)|Z_t^{x+he_i}|^2\big)\mathrm{d}t\Big)^{\gamma}\Big]^{\alpha}\Big[\mathbb{E}\int_0^1 \Big(\int_0^T |K_t^{\bar{x},\bar{h}}(\theta)|^{\frac{1}{1-\alpha}}\mathrm{d}t\Big)^{(1-\alpha)\cdot\frac{\gamma}{1-\alpha}}\mathrm{d}\theta\Big]^{(1-\alpha)}\\
\leq&CT^{\frac{1}{(1-\gamma)}}\Big[\mathbb{E}\Big(\int_0^T \big(1+|Y_t^{x+he_i}|^{\frac{1}{\alpha}}+f(|Y_t^{x+he_i}|)|Z_t^{x+he_i}|^2\big)\mathrm{d}t\Big)^{\gamma}\Big]^{\alpha}\Big[\int_0^1\int_0^T \mathbb{E}|K_t^{\bar{x},\bar{h}}(\theta)|^{\frac{\gamma}{1-\alpha}}\mathrm{d}t\mathrm{d}\theta\Big]^{(1-\alpha)}.
\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^m,h\in\mathbb{R}}\mathbb{E}\Big(\int_0^T |I_t^x|\mathrm{d}t\Big)^\gamma\\
\leq&C\sup_{x\in\mathbb{R}^m,h\in\mathbb{R}}\Big(1+\|Y^{x+he_i}\|_{\mathcal{S}^\infty}^\gamma+\|\sqrt{f(|Y^{x+he_i}|)}Z^{x+he_i}\|_{\mathcal{H}^\gamma}^{2\gamma\alpha}\Big)\sup_{r\in\mathbb{R}^m}\Big(\int_0^T \mathbb{E}|K_t(r)|^{\frac{\gamma}{1-\alpha}}\mathrm{d}t\Big)^{(1-\alpha)}<\infty.
\end{aligned}$$

The proof is completed. □

It is then possible to identify the couple process $(\nabla Y^x, \nabla Z^x)$ as the derivatives of (Y^x, Z^x) under Assumption 4.6 such that the following BSDE

$$\nabla Y_t^x=\nabla_x\xi(x)-\int_t^T\nabla_x Z_s^x\mathrm{d}B_s+\int_t^T\Big(\nabla_x g(s,\Theta_s^x)+\nabla_y g(s,\Theta_s^x)\nabla Y_s^x+\nabla_z g(s,\Theta_s^x)\nabla Z_s^x\Big)\mathrm{d}s, \tag{4.10}$$

makes sense for all $x\in\mathbb{R}^m$, $t\in[0,T]$, where $\Theta_\cdot^x=(x,Y_\cdot^x,Z_\cdot^x)$.

We only prove the differentiability of (Y^x,Z^x) with respect to the natural topological structure on the Banach space $\mathcal{S}^{2p}\times\mathcal{H}^{2p}$, $p>1$. Under some additional assumptions (for example smoothness of the parameters) the pathwise differentiablity of the maps $x\mapsto(Y_t^x(\omega),Z_t^x(\omega))$ for almost all $(\omega,t)\in\Omega\times[0,T]$ can be obtained without major difficulties.

The main result of this subsection is the following

Theorem 4.9 (Differentiability)**.** *Suppose the coefficients of the BSDE (4.7) satisfy Assumption 4.6. Then for any parameter $x\in\mathbb{R}^n$ and all $p>1$ the solution function: $\mathbb{R}^m\to\mathcal{S}^{2p}\times\mathcal{H}^{2p}$, $x\mapsto(Y^x,Z^x)$ is differentiable in the norm topology and the couple (derivatives) $x\mapsto(\nabla Y^x,\nabla Z^x)$ is solution to the BSDE (4.10). In particular for $x,x'\in\mathbb{R}^m$ we have*

$$\lim_{x\to x'}\Big\{\|\nabla_x Y_t^x-\nabla_x Y_t^{x'}\|_{\mathcal{S}^{2p}}^{2p}+\|\nabla_x Z_t^x-\nabla_x Z_t^{x'}\|_{\mathcal{H}^{2p}}^{2p}\Big\}=0.$$

The proof of Theorem 4.9 follows from well known techniques related to differentiability of BSDEs with drivers that grow quadratically in the control variable. We refer the reader for example to [1, 17, 20, 38]. For the sake of better understanding, we sketch the proof below

Proof of Theorem 4.9. We start by observing the following: Assumption (H1) guarantees the existence of a maximal solution (Y^x, Z^x) to the BSDE (4.7) in $\mathcal{S}^\infty \times \mathcal{H}_{BMO}$ such that the norms of Y^x (resp. Z^x) in $\mathcal{S}^\infty$ (resp. $\mathcal{H}_{BMO}$) are uniformly bounded in x. Using Lemma 4.7 we also deduce that for all $p > 1$, $x \in \mathbb{R}^m$, $h, h' \in \mathbb{R}$ and $i \in \{1, \cdots, m\}$

$$\lim_{h\to 0} \left\{ \left\| Y_t^{x+he_i} - Y_t^x \right\|_{\mathcal{S}^{2p}} + \left\| Z_t^{x+he_i} - Z_t^x \right\|_{\mathcal{H}^{2p}} \right\} = 0,$$

Then, for any candidate $(\nabla_{x_i} Y_t^x, \nabla_{x_i} Z_t^x)$ for the partial derivatives satisfying BSDE (4.10), the following can be achieved as in [**?** , P33]DosReis:

$$\lim_{h\to 0} \left\{ \left\| \frac{Y_t^{x+he_i} - Y_t^x}{h} - \nabla_{x_i} Y_t^x \right\|_{\mathcal{S}^{2p}} + \left\| \frac{Z_t^{x+he_i} - Z_t^x}{h} - \nabla_{x_i} Z_t^x \right\|_{\mathcal{H}^{2p}} \right\} = 0.$$

The above guarantees the existence of partial derivatives of (Y^x, Z^x). Using Lemma 4.7 once more, and the continuity of the derivatives of g, it can be shown that $(\nabla_{x_i} Y^x, \nabla_{x_i} Z^x)$ are continuous with respect to any $x \in \mathbb{R}^m$. This concludes the proof. □

5. Differentiability of quadratic FBSDEs with rough drift

In this section, we study the smoothness properties of the solution (X^x, Y^x, Z^x) of the following FBSDE

$$X_t^x = x + \int_0^t b(s, X_s^x) \mathrm{d}s + B_t, \tag{5.1}$$

$$Y_t^x = \phi(X_T^x) + \int_t^T g(s, X_s^x, Y_s^x, Z_s^x) \mathrm{d}s - \int_t^T Z_s^x \mathrm{d}B_s, \tag{5.2}$$

where the functions b, ϕ and g are both deterministic explicit functional that are Borel measurables and satisfying some conditions that will be made precise below.

In the sequel, the driver g and the terminal value ϕ satisfy the following assumptions:

(AY): The function $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, measurable and uniformly bounded; $g : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a measurable function satisfying: $\|g(t,0,0,0)\|_\infty \leq \Lambda_0$ and there exist positive constants Λ_x, Λ_y and Λ_z such that for all $(t,x,y,z) \in \times[0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, $(t, x', y', z') \in \times[0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, $\alpha \in (0,1)$

$$|g(t,x,y,z) - g(t,x',y,z)| \leq \Lambda_x(1 + |y| + [f(|y|)|z|]^\alpha)|x - x'|,$$
$$|g(t,x,y,z) - g(t,x,y',z')| \leq \Lambda_y(1 + (|z| + |z'|))|y - y'| + \Lambda_z(1 + (f(|y|) + f(|y'|))(|z| + |z'|))|z - z'|),$$

where $f \in L^1_{loc}(\mathbb{R}, \mathbb{R}_+)$ is locally bounded and non-decreasing.

(AY1): The functions ϕ and g are differentiable in x and g is continuously differentiable in (x,y,z). There exist non negative constants $\Lambda_x, \Lambda_y, \Lambda_z$ and Λ_ϕ such that for all $(t,x,y,z) \in \times[0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, $\alpha \in (0,1)$

$$|\nabla_x g(t,x,y,z)| \leq \Lambda_x(1 + |y| + [f(|y|)|z|]^\alpha),$$
$$|\nabla_y g(t,x,y,z)| \leq \Lambda_y(1 + |z|^\alpha),$$
$$|\nabla_z g(t,x,y,z)| \leq \Lambda_z(1 + f(|y|)|z|),$$
$$|\nabla_x \phi| \leq \Lambda_\phi.$$

5.1. The case of SDEs with bounded drift.

In this section, we assume that the drfit b in the forward SDE (5.1) satisfies

(AX): $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is uniformly bounded and Borel measurable.

It is well known that under (AX), the SDE (5.1) has a unique strong Malliavin and Sobolev differentiable solution (see for example [30, 32]). Let us recall the following results whose proofs can be found in [30, 32].

Theorem 5.1. *Suppose (AX) is valid.*

(i) *The SDE* (5.1) *has a unique strong solution* $X_t^x \in L^2(\Omega; W^{1,2}_{loc}(\mathbb{R}^d))$. *Moreover, for all* $s,t\in[0,T]$, $x,y\in\mathbb{R}$ *it holds*

$$\mathbb{E}\left[|X_t^x - X_s^y|^2\right] \leq C(\|b\|_\infty)\left(|t-s| + |x-y|^2\right). \tag{5.3}$$

(ii) *The strong solution* X^x *is Malliavin diffferentiable and for* $0\leq s\leq t\leq T$, *the Malliavin derivative* $D_sX_t^x$ *satisfies*

$$\sup_{x\in\mathbb{R}}\sup_{s\in[0,t]}\mathbb{E}\left[|D_sX_t^x|^p\right] \leq C \tag{5.4}$$

for all $p\geq 1$, *where* C *is an increasing function off* $\|b\|_\infty$. *Moreover, the following relation holds*

$$D_sX_t^x\left(\partial_x X_s^x\right) = \partial_x X_t^x \quad \mathbb{P}\text{-a.s.}, \tag{5.5}$$

where $\partial_x X_t^x$ *denotes the first variation of the process* $x\mapsto X^x$.

The next result pertains with the existence and representation of the Malliavin derivative of the solution to the FBSDE (5.1)-(5.2).

Theorem 5.2. *Suppose Assumptions (AX), (AY) and (AY1) are in force. Then the solution process* (Y^x, Z^x) *belongs to* $\mathbb{L}_{1,2}\times(\mathbb{L}_{1,2})^d$ *and a version of* $(D_uY_t^x, D_uZ_t^x)_{u,t\in[0,T]}$ *is the unique solution to the linear BSDE*

$$\begin{aligned}
&D_uY_t^x = 0 \text{ and } D_uZ_t^x = 0, \text{ if } t\in[0,u),\\
&D_uY_t^x = \nabla_x\phi(X_T^x)D_uX_T^x + \int_t^T \nabla_x g(s,X_s^x,Y_s^x,Z_s^x)D_uX_s^x\mathrm{d}s + \int_t^T \nabla_y g(s,X_s^x,Y_s^x,Z_s^x)D_uY_s^x\mathrm{d}s,\\
&\qquad + \int_t^T \nabla_z g(s,X_s^x,Y_s^x,Z_s^x)D_uZ_s^x\mathrm{d}s - \int_t^T D_uZ_s^x\mathrm{d}B_s, \quad t\in[u,T],
\end{aligned} \tag{5.6}$$

where D_uX^x *is the Malliavin derivative of the process* X^x.

Proof. We just need to prove as in [1, 20, 19, 38] that the conditions of the theorem imply that assumptions (M1) and (M2) in Section 4.1 are satisfied.

Here $\xi(x) = \phi(X_T^x)$ and for $u\in[0,T]$ we have

$$|D_u\xi(x)| = |D_u\phi(X_T^x)| = |(D_uX_T^x)\nabla_x\phi(X_T^x)| \leq C|D_uX_T^x|.$$

Using (5.4), it holds that for any $p\geq 1$

$$\sup_{x\in\mathbb{R}}\sup_{u\in[0,T]}\mathbb{E}\left[|D_u\xi(x)|^p\right] \leq C\sup_{x\in\mathbb{R}}\sup_{u\in[0,T]}\mathbb{E}\left[|D_uX_T^x|^p\right] < \infty.$$

On the other hand, consider

$$\begin{aligned}
\bar{g} : \Omega\times[0,T]\times\mathbb{R} &\to \mathbb{R}\\
(\omega,t,x,y,z) &\mapsto g(t,X_t^x(\omega),y,z).
\end{aligned}$$

From (AY1), it is readily seen that $\bar{g}$ satisfies (M1) almost surely. Furthermore,

$$|D_u\bar{g}(t,x,y,z)| \leq \Lambda_x|D_uX_t^x|(1+|y|+[f(|y|)|z|]^\alpha).$$

Using once more (5.4) one has $\sup_{0\leq t\leq T}\int_0^T\mathbb{E}[|D_uX_t^x|^{2p}]\mathrm{d}u < \infty$ for any $p\geq 1$. Thus (M2) is satisfied.

□

Remark 5.3. *The above result remains true when* ϕ *satisfies the polynomial growth condition, i.e.,* $|\nabla_x\phi(x)|\leq C(1+|x|^n)$, *for* $n\in\mathbb{N}$. *Suppose for example that* ϕ *is of linear growth then for all* $p\geq 1$

$$\sup_{u\in[0,T]}\mathbb{E}\left[(1+|X_T^x|)^{2p}|D_uX_T^x|^p\right] \leq C\left[\mathbb{E}(1+|X_T^x|)^{2p}\right]^{\frac{1}{2}}\sup_{u\in[0,T]}\left[\mathbb{E}(|D_uX_T^x|^{2p})\right]^{\frac{1}{2}} < \infty.$$

Remark 5.4. *In the one dimensional case, under assumption (AX), the first varaition process and the Malliavin derivative of the solution to the SDE (5.1) can be represented explicitely (see for example [4, 31]) with respect to the local tome space-time integral as follows*

$$\partial_x X_t^x = \exp\Big(- \int_0^t \int_{\mathbb{R}} b(u,y) L^{X^x}(\mathrm{d}u,\mathrm{d}y)\Big) \quad \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{-}a.s.,$$

$$D_s X_t^x = \exp\Big(- \int_s^t \int_{\mathbb{R}} b(u,y) L^{X^x}(\mathrm{d}u,\mathrm{d}y)\Big).$$

See also Appendix C.

Remark 5.5. *Theorem 5.2 is still valid when the process $(X_t)_{t\in[0,T]}$ is the solution of the following SDE with random coefficients:*

$$X_t^\theta(\omega) = \theta + \int_0^t b\left(s,\omega,X_s(\omega)\right) \mathrm{d}s + \int_0^t \sigma\left(s,\omega,X_s(\omega)\right) \mathrm{d}B_s, \tag{5.7}$$

where $\theta \in L^0(\mathfrak{F}_0;\mathbb{P},\mathbb{R}^d)$ and b,σ satisfies the conditions in [21].

Indeed, from [21, Theorem 3.2], the SDE (5.7) has a unique Malliavin differentiable solution X in $\mathcal{S}^p$. In addition, there exist two adapted processes U and V such that for $0 \le s \le t \le T$:

$$\begin{aligned} D_sX_t(\omega) &= \sigma(s,\omega,X_s(\omega)) + \int_s^t \nabla_x b\left(r,\omega,X_r(\omega)\right) D_sX_r(\omega)\mathrm{d}r + \int_s^t \nabla_x \sigma\left(r,\omega,X_r(\omega)\right) D_sX_r(\omega)\mathrm{d}B_r \\ &\quad + \int_s^t U(s,r,\omega)\mathrm{d}r + \int_s^t V(s,r,\omega)\mathrm{d}B_r, \end{aligned} \tag{5.8}$$

$D_sX_t(\omega) = 0$ for $s > t$.

Here, $s \mapsto D_sX_.$ is not continuous but only square integrable

5.2. **The case of SDEs with C_b^β drift.** In this section, we assume that the drift b of fthe forward SDE satisfies:

Assumption 5.6. *There exists $\beta \in (0,1)$ such that $b \in L^\infty([0,T]; C_b^\beta(\mathbb{R}^d;\mathbb{R}^d))$.*

The following result is from [23]

Theorem 5.7. *Suppose Assumption 5.6 and fix any $\beta' \in (0,\beta)$. Then*

(1) *(Pathwise uniqueness) For every $s \ge 0, x \in \mathbb{R}^d$ the SDE (5.1) has a unique continuous adapted solution $X^{s,x} = (X_t^{s,x}(\omega), t \ge s, \omega \in \Omega)$.*
(2) *(Differentiable flow) There exists a stochastic flow $X_t^{s,x} = \phi_{s,t}(x)$ of diffeomorphisms for equation (5.1). The flow is also of class $C^{1+\beta'}$ and for any $p \ge 1$*

$$\sup_{x\in\mathbb{R}} \sup_{0\le s\le T} \mathbb{E}\Big[\sup_{s\le t\le T} \|D\phi_{s,t}(x)\|^p\Big] < \infty.$$

(3) *(Stability) Let $(b^n)_{n\in\mathbb{N}} \subset L^\infty(0,T;C_b^\beta(\mathbb{R}^d;\mathbb{R}^d))$ be a sequence of vector fields and ϕ^n be the associate stochastic flows. If $b^n \to b$ in $L^\infty(0,T;C_b^{\beta'}(\mathbb{R}^d;\mathbb{R}^d))$ for some $\beta' > 0$ then for any $p \ge 1$*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{0\le s\le T} \mathbb{E}\Big[\sup_{r\in[s,T]} |\phi_{s,r}^n(x) - \phi_{s,r}(x)|^p\Big] = 0, \tag{5.9}$$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{0\le s\le T} \mathbb{E}\Big[\sup_{r\in[s,T]} \|D\phi_{s,r}^n(x) - D\phi_{s,r}(x)\|^p\Big] = 0. \tag{5.10}$$

Similar to [16], we have

Proposition 5.8. *Under Assumption 5.6, the solution $(X_t^x, 0 \le t \le T)$ to equation (5.1) is Malliavin differentiable and for any $p \ge 2$*

$$\sup_{0\le s\le t} \mathbb{E}\big[\sup_{s\le t\le T} |D_sX_t^x|^p\big] < \infty. \tag{5.11}$$

Let $(b_n)_{n\geq 1}$ be a sequence of compactly supported smooth functions approximating b and let $(X^{n,x})_{n\geq 1}$ be a sequence of corresponding solution to the SDE (5.1). Then we have the following stability result: For all $p \geq 2$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \mathbb{E}\Big[\sup_{0\leq s\leq t\leq T} |D_s X_t^{n,x} - D_s X_t^x|^p\Big] = 0. \tag{5.12}$$

Proof. Fix $\lambda > 0$ and consider the following backward Kolmogorov PDE

$$\partial_t u_\lambda + \mathcal{L}u_\lambda - \lambda u_\lambda = -b, \quad (t,x) \in [0,+\infty) \times \mathbb{R}^d,$$

where $\mathcal{L}u_\lambda = 1/2\Delta u_\lambda + b \cdot Du_\lambda$. It can be shown (see [16, Lemma 6]) that $u_\lambda \in L^\infty([0,\infty); C_b^{2+\beta}(\mathbb{R}^d))$. In addition for λ large enough, the map $\Psi_\lambda(t,x) = x + u_\lambda(t,x)$ satisfies (see [16, Lemma 6]) :

(i) uniformly in t, Ψ_λ has bounded first and second derivatives and the second (Fréchet) derivative $D^2\Psi_\lambda$ is globally β-Hölder continuous.
(ii) For any $t \geq 0$, $\Psi_\lambda : \mathbb{R}^d \mapsto \mathbb{R}^d$ is non-singular diffeomorphism of class C^2.
(iii) Ψ_λ^{-1} has bounded first and second derivatives uniformly in $t \in [0,\infty)$.

Let us consider the following SDE

$$\tilde{X}_t = y + \int_s^t \tilde{b}(v,\tilde{X}_v)\mathrm{d}v + \int_s^t \tilde{\sigma}(v,\tilde{X}_v)\mathrm{d}B_v, \quad t \in [s,T], \tag{5.13}$$

where $\tilde{b}(t,y) = -\lambda u_\lambda(t,\Psi_\lambda^{-1}(t,y))$ and $\tilde{\sigma}(t,y) = D\Psi_\lambda(t,\Psi_\lambda^{-1}(t,y))$. It is then clear that: $\tilde{b} \in L^\infty([0,\infty); C_b^{2+\beta}(\mathbb{R}^d))$ and $\tilde{\sigma} \in L^\infty([0,\infty); C_b^{1+\beta}(\mathbb{R}^d))$. Thus, equation (5.13) has a unique strong Malliavin differentiable solution (see [35, Theorem 2.2.1]) such that for any $p \geq 2$

$$\sup_{0\leq s\leq t} \mathbb{E}[\sup_{s\leq t\leq T} |D_s\tilde{X}_t|^p] < \infty.$$

Let $(\tilde{X}_t, 0 \leq t \leq T)$ be the solution to the SDE (5.13). Then we deduce that $X_t = \Psi_\lambda^{-1}(t,\tilde{X}_t)$ is solution to SDE (5.1). Using the chain rule for Malliavin calculus and the fact that Ψ_λ^{-1} has bounded first derivative, we have

$$\mathbb{E}[\sup_{s\leq t\leq T} |D_s X_t|^p] \leq C\mathbb{E}[\sup_{s\leq t\leq T} |D_s\tilde{X}_t|^p] < \infty.$$

Thus, (5.11) follows. The proof of (5.12) follows in an anologous way as in [16, Theorem ???]) by observing that in the smooth case both the equations of $D_sX_t^x$ and $\nabla_x X_t^x$ are similar. □

Remark 5.9. *By using the same argumen as above, one can show that :*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |(\nabla_x X_t)|^p\Big] < \infty, \ \forall p \in]-\infty, 0[\cup]0,+\infty[^1.$$

Indeed, the solution $\tilde{X}$ to the equation (5.13) (with Lipschitz continuous coefficients $\tilde{b}$ and $\tilde{\sigma}$) is differentiable with respect to x and $(\nabla_x\tilde{X}_t)^{-1}$ satisfies a linear SDE such that $\mathbb{E}\left[\sup_{0\leq t\leq T} |(\nabla_x\tilde{X}_t)^{-1}|^p\right] < \infty$, for any $p > 1$ (see [28]). Then, by using the fact that Ψ^{-1} is differentiable with bounded derivative, we obtain the desired result.

Since b is not differentiable, the representation of the Malliavin derivative $D_sX_t^x$ in the classical sense of Lebesgue integrals does not hold. Recently, the representation for the first variation process $\nabla_x X_t^x$ was given in terms of a system of Young type equations in [29]. The main ingredient in the proof of this result is to rigorously establish the well posedness of the process $V_t^{k,j}(b,X) = \mathcal{A}_t^X[\partial_k b^j]$ for every $t \geq 0$ and $j,k \in \{1,\ldots,d\}$ via the so called stochastic sewing lemma.

Theorem 5.10 (Theorem 4.1 in [29]). *Suppose Assumption 5.6. For any $\beta' \in (0,\beta)$, the first variation process ∇X_t^x satisfies the following system of Young-type equations*

$$\partial_{x_i} X_t^{j,x} = \delta_{i,j} + \sum_{k=1}^d \int_0^t \partial_{x_i} X_s^{j,x} \mathrm{d}V_s^{k,j}(b,X), \quad \forall i,j \in \{1,\cdots,d\} \tag{5.14}$$

[1] The case $p > 1$ was treated in [16]

where $\delta_{i,j}$ stands for the Kronecker delta symbol, $V_t^{k,j}(b,X) = \mathcal{A}_t^X[\partial_k b^j]$ and $(\mathcal{A}_t)_{s\leq t\leq T}$ is the unique (up to modifications) stochastic process with values in $\mathbb{R}^d$ satisfying conditions of Lemma D.1 (see [29, Theorem 2.3]). Moreover, the map $t \to \nabla X_t^x$ is a.s. $\frac{1+\beta'}{2}$-Hölder continuous for every $\beta' \in (0,\beta)$.

Combining the arguments in Theorem 5.10 and Corollary 5.8, we have

Corollary 5.11. *Suppose Assumption 5.6. For any $\beta' \in (0,\beta)$, the Malliavin derivative of the solution X_t^x to SDE (5.1) is given by:*

$$(D_s^i X_t^x)^j = \delta_{i,j} + \sum_{k=1}^d \int_s^t (D_s^i X_u^x)^j \mathrm{d}V_s^{k,j}(b,X), \quad \forall i,j \in \{1,\cdots,d\}. \tag{5.15}$$

Moreover, the map $s \mapsto D_s X^x(\cdot)$ is a.s. $\frac{1+\beta'}{2}$–Hölder continuous for every $\beta' \in (0,\beta)$.

We have the subsequent results as a combination of Theorem 5.7 and Theorem 4.9.

Theorem 5.12. *Suppose Assumption 5.6, (AY) and (AY1) hold. Let $x \in \mathbb{R}^d$ and $p > 1$. Then the map $x \mapsto (Y^x, Z^x)$ solution to (5.2) is differentiable in the norm topology and the derivative process $(\nabla_x Y^x, \nabla_x Z^x)$ solves the BSDE*

$$\begin{aligned}\nabla Y_t^x =& \nabla_x\phi(X_T^x)\nabla X_T^x + \int_t^T \nabla_x g(s,X_s^x,Y_s^x,Z_s^x)\nabla_x X_s^x \mathrm{d}s + \int_t^T \nabla_y g(s,X_s^x,Y_s^x,Z_s^x)\nabla_x Y_s^x \mathrm{d}s \\ &+ \int_t^T \nabla_z g(s,X_s^x,Y_s^x,Z_s^x)\nabla_x Z_s^x \mathrm{d}s - \int_t^T \nabla_x Z_s^x \mathrm{d}B_s,\end{aligned} \tag{5.16}$$

where $\nabla_x X^x$ satisfies equation (5.14).

Theorem 5.13. *Suppose Assumptions 5.6, (AY) and (AY1) are in force. Then the solution process (Y^x, Z^x) belongs to $\mathbb{L}_{1,2} \times (\mathbb{L}_{1,2})^d$ and a version of $(D_u Y_t^x, D_u Z_t^x)_{u,t\in[0,T]}$ is the unique solution to the BSDE*

$$\begin{aligned}&D_u Y_t^x = 0 \text{ and } D_u Z_t^x = 0, \text{ if } t \in [0,u), \\ &D_u Y_t^x = \nabla_x\phi(X_T^x)D_u X_T^x + \int_t^T \nabla_x g(s,X_s^x,Y_s^x,Z_s^x)D_u X_s^x \mathrm{d}s + \int_t^T \nabla_y g(s,X_s^x,Y_s^x,Z_s^x)D_u Y_s^x \mathrm{d}s, \\ &\quad + \int_t^T \nabla_z g(s,X_s^x,Y_s^x,Z_s^x)D_u Z_s^x \mathrm{d}s - \int_t^T D_u Z_s^x \mathrm{d}B_s, \quad t \in [u,T],\end{aligned} \tag{5.17}$$

where $D_u X^x$ satisfies equation (5.15).

Moreover, $\{D_t Y_t^x : 0 \leq t \leq T\}$ is continuous a version of $\{Z_t^x : 0 \leq t \leq T\}$ and

$$D_u Y_t^x(\nabla_x X_u^x) = \nabla_x Y_t^x, \tag{5.18}$$

$$Z_t^x(\nabla_x X_t^x) = \nabla_x Y_t^x, \tag{5.19}$$

$$D_u Z_t^x(\nabla_x X_t^x) = \nabla_x Z_t^x \tag{5.20}$$

Proof. The first part follows from Theorem 5.2

For the second part, it is enough to establish (5.18) under the additional assumption on b. The relation (5.19) follows from (5.18) for $u = t$. We also have

$$D_u Y_t^x \nabla_x X_u^x = \nabla_x\phi(X_T^x)D_u X_T^x \nabla_x X_u^x + \int_t^T \langle \nabla g(s,\Theta_s^x), D_u\Theta_s^x \nabla_x X_u^x\rangle \mathrm{d}s - \int_t^T D_u Z_s^x \nabla_x X_u^x \mathrm{d}B_s,$$

with terminal value and generator satisfying conditions (C1),(C2) and (C3). Then, from the unique solvability of BSDE (4.10) we obtain (5.18) and (5.20). Moreover, $\{D_t Y_t^x : 0 \leq t \leq T\}$ is a version of $\{Z_t^x : 0 \leq t \leq T\}$. We end the proof by observing that the process DY admits a continuous version, since it is represented in terms of continuous processes $t \to \nabla Y_t$ and $t \to (\nabla_x X_t)^{-1}$ (see Theorem 5.12 and Theorem 5.10). Thus, the existence of a continuous version to $t \mapsto Z_t$ follows. □

6. Path regularity and explicit convergence rate

In this section, we study the path regularity and the rate of convergence of a numerical scheme to the FBSDE (5.1)-(5.2).

Lemma 6.1. *Under Assumptions 5.6, (AY) and (AY1), we obtain for all $p > 1$*

$$\mathbb{E}\Big[\sup_{0\le s\le t\le T}|D_sY_t|^{2p}\Big] < \infty, \tag{6.1}$$

$$\mathbb{E}\Big[\sup_{0\le t\le T}|Z_t|^{2p}\Big] < \infty. \tag{6.2}$$

In addition for all $p \geq 2$, there exists a positive constant $C_p > 0$, such that for $0 \le s \le t \le T$

$$\mathbb{E}\Big[\sup_{s\le r\le t}|Y_r - Y_s|^p\Big] \le C_p|t-s|^{p/2}.$$

Proof. Under the assumptions of the Lemma, it follows from Lemma 5.13 that the process $\{\nabla_x Y_t(\nabla_x X_s)^{-1} : 0 \le s \le t \le T\}$ is a version of $\{D_sY_t : 0 \le s \le t \le T\}$. Using Hölder inequality and Remark 5.9, for any $p > 1$ we deduce that:

$$\begin{aligned}\mathbb{E}[\sup_{0\le s\le t\le T}|D_sY_t|^{2p}] &\le \mathbb{E}\Big[\sup_{0\le t\le T}|\nabla_xY_t|^{4p}\Big]^{\frac12}\mathbb{E}\Big[\sup_{0\le s\le T}|(\nabla_xX_s)^{-1}|^{4p}\Big]^{\frac12}\\ &\le \|\nabla_xY\|_{\mathcal{S}^{2p}}\mathbb{E}\Big[\sup_{0\le s\le T}|(\nabla_xX_s)^{-1}|^{4p}\Big]^{\frac12} < \infty,\end{aligned}$$

the bound (6.1) follows. In particular, for $s = t$ we obtain the bound (6.2). On the other hand, we recall that for all $s \le v \le t$,

$$Y_s = \phi(X_t) + \int_s^t g(v, X_v, Y_v, Z_v)\mathrm{d}v - \int_s^t Z_v\mathrm{d}B_v.$$

Then from BDG inequality we deduce that

$$\mathbb{E}\Big[\sup_{s\le r\le t}|Y_r - Y_s|^p\Big] \le C(p)\Big\{\mathbb{E}\Big(\int_s^t|g(v,X_v,Y_v,Z_v)|\mathrm{d}v\Big)^p + \mathbb{E}\Big(\int_s^t|Z_v|^2\mathrm{d}v\Big)^{p/2}\Big\}.$$

By using the bound: $|g(v, X_v.Y_v.Z_v)| \le K(1 + |Y_v| + (1 + f(|Y_v|))|Z_v|^2)$ and the fact that Y_v is bounded we obtain that:

$$\mathbb{E}\Big[\sup_{s\le r\le t}|Y_r - Y_s|^p\Big] \le C(p)\Big\{|t-s|^p + \mathbb{E}\Big[\Big(\int_s^t(1+f(|Y_v|))|Z_v|^2\mathrm{d}v\Big)^p + \Big(\int_s^t|Z_v|^2\mathrm{d}v\Big)^{p/2}\Big]\Big\}.$$

From to the local boundedness of f and the bound (6.2), we deduce the following

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\le r\le t}|Y_r - Y_s|^p\Big]\\ \le& C(p)\Big\{|t-s|^p + |t-s|^p\mathbb{E}[\sup_{s\le v\le t}|Z_v|^{2p}] + |t-s|^{p/2}\mathbb{E}[\sup_{s\le v\le t}|Z_v|^{2p}]\Big\}\\ \le& C(p)\Big\{|t-s|^p + |t-s|^{p/2}\Big\}.\end{aligned}$$

The proof is completed. □

In the sequel, Δ_N stands for the collection of all partitions of the interval $[0, T]$ by finite families of real numbers. Particular partitions will be denoted by $\delta_N = \{t_i : 0 = t_0 < \cdots < t_N = T\}$ with $N \in \mathbb{N}$. We define the mesh size of partitions as $|\delta_N| = \max_{0\le i\le N}|t_{i+1} - t_i|$

Theorem 6.2 (Path regularity)**.** *Suppose Assumption 5.6, (AY) and (AY1). Then for all $p \geq 2$, there exists a positive C_p such that for any partition δ_N of $[0, T]$ with $(N+1) \in \mathbb{N}$ points and mesh size $|\delta_N|$, we have*

$$\sum_{i=0}^{N-1}\mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}}|Z_t - Z_{t_i}|^2\mathrm{d}t\Big)^{p/2}\Big] \le C(p)|\delta_N|^{p/2}.$$

Proof. For $t \in [t_i, t_{i+1}]$, using the representation (5.19), we have

$$\begin{aligned} Z_t - Z_{t_i} &= \nabla Y_t (\nabla_x X_t)^{-1} - \nabla Y_{t_i} (\nabla_x X_{t_i})^{-1} \\ &= \Big(\nabla Y_t - \nabla Y_{t_i}\Big)(\nabla_x X_{t_i})^{-1} + \nabla Y_t \Big((\nabla_x X_t)^{-1} - (\nabla_x X_{t_i})^{-1}\Big) \\ &= I_1 + I_2. \end{aligned}$$

Thus from Hölder's inequality, we have

$$\begin{aligned} \mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} |Z_t - Z_{t_i}|^2 \mathrm{d}t\Big)^{p/2}\Big] &\leq |\delta_N|^{p/2-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\Big[|Z_t - Z_{t_i}|^p\Big] \mathrm{d}t \\ &\leq |\delta_N|^{p/2-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|I_1|^p + |I_2|^p\right] \mathrm{d}t. \end{aligned}$$

Let us first focus on the term I_2. We first assume that b is a bounded and compactly supported smooth function. Indeed, if b is not differentiable, then, by denseness of the set of compactly supported and differentiable functions in the set of bounded functions, there exists a sequence $(b_n)_{n\geq 1}$ of compactly supported and smooth functions converging to b a.e. on $[0,T] \times \mathbb{R}^d$ and the desired result is obtained from the Vitali's convergence theorem. Then, the map $t \mapsto (\nabla_x X_t)^{-1}$ satisfies the following linear equation :

$$\begin{cases} \dfrac{\mathrm{d}(\nabla_x X_t)^{-1}}{\mathrm{d}t} &= -(\nabla_x X_t)^{-1} b'(t, X_t), \\ (\nabla_x X_0)^{-1} &= I_d. \end{cases}$$

Iterating the above equation gives for all $0 \leq s \leq t$

$$|(\nabla_x X_t)^{-1} - (\nabla_x X_s)^{-1}| = \Big|\sum_{k=1}^{\infty} \int_{s<s_1<\ldots<s_n<t} b'(s_1, X^x_{s_1}) : \cdots : b'(s_k, X^x_{s_k}) \mathrm{d}s_1 \cdots \mathrm{d}s_k\Big|,$$

where the symbol " : " stands for the matrix multiplication.

For any $p \geq 2$, the Girsanov's theorem and Hölder's inequality yield

$$\begin{aligned} &\mathbb{E}|(\nabla_x X_t)^{-1} - (\nabla_x X_s)^{-1}|^p \\ &= \mathbb{E}\Big[\Big|\sum_{k=1}^{\infty} \int_{s<s_1<\ldots<s_n<t} b'(s_1, B^x_{s_1}) : \cdots : b'(s_k, B_{s_k}) \mathrm{d}s_1 \cdots \mathrm{d}s_k\Big|^p \times \mathcal{E}\Big(\int_0^t b(v, B_v) \mathrm{d}v\Big)\Big] \\ &\leq C(\|b\|_\infty) \mathbb{E}\Big[\Big|\sum_{k=1}^{\infty} \int_{s<s_1<\ldots<s_n<t} b'(s_1, B^x_{s_1}) : \cdots : b'(s_k, B_{s_k}) \mathrm{d}s_1 \cdots \mathrm{d}s_k\Big|^{2p}\Big]^{1/2}. \end{aligned}$$

Using [30, Proposition 3.7],we have

$$\mathbb{E}|(\nabla_x X_t)^{-1} - (\nabla_x X_s)^{-1}|^p \leq C(\|b\|_\infty)|t-s|^{p/2}$$

for all $p \geq 2$, where $C : [0,\infty) \to [0,\infty)$ is an increasing,continuous function, $\|\cdot\|$ is a matrix-norm on $\mathbb{R}^{d\times d}$ and $\|\cdot\|_\infty$ the supremum norm. Thus, the bound remains valid for only bounded and measurable drift b. Therefore,

$$\begin{aligned} \mathbb{E}\left[|I_2|^p\right] &\leq C\Big(\mathbb{E}\Big[\sup_{0\leq t\leq T} |\nabla_x Y_t|^{2p}\Big]\Big)^{\frac{1}{2}} \times \Big(\mathbb{E}\Big[|(\nabla_x X_t)^{-1} - (\nabla_x X_{t_i})^{-1}|^{2p}\Big]\Big)^{\frac{1}{2}} \\ &\leq C|\delta_N|^{p/2}. \end{aligned}$$

Let us turn now on I_1. We also claim:

$$\delta_N^{p/2-1} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|I_1|^p \mathrm{d}t\right] \leq C|\delta_N|^{p/2}.$$

Indeed, noticing that $(\nabla_x X_{t_i})^{-1}$ is $\mathfrak{F}_{t_i}$-adapted and using the tower property

$$\mathbb{E}[|(\nabla_x Y_t - \nabla_x Y_{t_i})(\nabla_x X^x_{t_i})^{-1}|^p] = \mathbb{E}\left[\mathbb{E}[|\nabla_x Y_t - \nabla_x Y_{t_i}|^p/\mathfrak{F}_{t_i}]|(\nabla_x X_{t_i})^{-1}|^p\right]$$

By writing the equation satisfied by the difference $\nabla Y_t - \nabla Y_{t_i}$ for all $t_i \leq t \leq t_{i+1}$ and using the conditional BDG's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[|\nabla_x Y_t - \nabla_x Y_{t_i}|^{2p}/\mathfrak{F}_{t_i}\Big]\\
\leq& C\mathbb{E}\Big[\Big|\int_{t_i}^t \langle \nabla g(s,\Theta_s), \nabla\Theta_s\rangle \mathrm{d}s\Big|^{2p} + \Big|\int_{t_i}^t \nabla Z_s \mathrm{d}B_s\Big|^{2p}\Big/\mathfrak{F}_{t_i}\Big]\\
\leq& C\mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} |\nabla g(s,\Theta_s)||\nabla\Theta_s|\mathrm{d}s\Big)^{2p} + \Big(\int_{t_i}^{t_{i+1}} |\nabla Z_s|^2\mathrm{d}s\Big)^p\Big/\mathfrak{F}_{t_i}\Big] := C\mathcal{X}_{[t_i,t_{i+1}]}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\delta_N^{p/2-1}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \mathbb{E}\left[|I_1|^p\right]\mathrm{d}t &\leq C|\delta_N|^{p/2}\sum_{i=0}^{N-1}\mathbb{E}\left[\mathcal{X}_{[t_i,t_{i+1}]}|(\nabla_x X_{t_i})^{-1}|^p\right]\\
&\leq C|\delta_N|^{p/2}\mathbb{E}\Big[\sup_{0\leq t\leq T}|(\nabla_x X_t)^{-1}|^p\sum_{i=0}^{N-1}\mathcal{X}_{[t_i,t_{i+1}]}\Big]\\
&\leq C|\delta_N|^{p/2}\mathbb{E}\Big[\sup_{0\leq t\leq T}|(\nabla_x X_t)^{-1}|^p\mathcal{X}_{[0,T]}\Big],
\end{aligned}$$

which is finite, thanks to Remark 5.9. This conclude the proof. □

Corollary 6.3 (Zhang's path regularity theorem)**.** *Under the assumptions of Theorem 6.2 , we deduce the following:*

$$\sum_{i=0}^{N-1}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|Z_t - \tilde{Z}_{t_i}^{\delta_N}|^2\mathrm{d}t\Big] \leq C|\delta_N|,$$

where

$$\tilde{Z}_{t_i}^{\delta_N} = \frac{1}{t_{i+1}-t_i}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} Z_s\mathrm{d}s\Big/\mathfrak{F}_{t_i}\Big], \tag{6.3}$$

is a family of random variables defined for all partition points t_i of δ_N and Z is the control process in the solution of FBSDE (5.1)-(5.2).

Proof. It is well known that the random variable $\tilde{Z}_{t_i}^{\delta_N}$ is the best $\mathfrak{F}_{t_i}$-adapted $\mathcal{H}^2([t_i,t_{i+1}])$ approximation of Z, i.e.,

$$\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|Z_s - \tilde{Z}_{t_i}^{\delta_N}|^2\mathrm{d}s\Big] = \inf_{Z_i\in L^2(\Omega,\mathfrak{F}_{t_i})}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|Z_s - Z_i|^2\mathrm{d}s\Big].$$

In particular,

$$\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|Z_s - \tilde{Z}_{t_i}^{\delta_N}|^2\mathrm{d}s\Big] \leq \mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|Z_s - Z_{t_i}|^2\mathrm{d}s\Big].$$

The result then follows from the Theorem 6.2 by taking $p = 2$. □

Let us introduce the following family of truncated FBSDE

$$Y_t^n = \phi(X_T) + \int_t^T g_n(s,X_s,Y_s^n,Z_s^n)\mathrm{d}s - \int_t^T Z_s^n\mathrm{d}B_s, \tag{6.4}$$

where

$$g_n(t,x,y,z) = g(t,x,\tilde{\rho}_n(y),\rho_n(z)) \text{ for } (t,x,y,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d,\ n\in\mathbb{N},$$

with $\tilde{\rho}_n$ given by (4.2), $\rho_n : \mathbb{R}^d \to \mathbb{R}^d, z \mapsto \rho_n(z) = (\tilde{\rho}_n(z_1),\cdots,\tilde{\rho}_n(z_d))$ and X stands for the solution to the SDE (5.1). It is clear that (6.4) satisfies assumptions of Theorem 5.2 provided (5.1) and (5.2) do satisfy them. Using similar arguments as in the proof of Theorem 3.4 and Lemma 4.3, we have

$$\max\Big\{\sup_{n\in\mathbb{N}}\|Z^n * B\|_{BMO}, \|Z * B\|_{BMO}\Big\} \leq \Upsilon^{(2)}. \tag{6.5}$$

Thus for all $n \in \mathbb{N}$ the sequence $(Z^n * B)_{n\in\mathbb{N}}$ satisfies the properties (P2) of Lemma A.1 i.e. there is a universal constant $r > 1$ such that $\mathcal{E}(Z^n * B) \in L^r$. Then, we deduce that (Y^n, Z^n) is differentiable in the sense of Theorem 5.12 and the following uniform bounds hold: $\sup_{n\in\mathbb{N}}\Big(\|\nabla Y^n\|_{\mathcal{S}^{2p}} + \|\nabla Z^n\|_{\mathcal{H}^{2p}}\Big) < \infty$. Moreover, we can show as in (6.2) that

$$\sup_{n\in\mathbb{N}}\mathbb{E}\big[\sup_{t\in[0,T]}|Z^n_t|^{2p}\big] < \infty. \tag{6.6}$$

Furthermore, the properties of $\tilde\rho$ yield

$$\Big(|\tilde\rho(Y^n_s) - Y^n_s| + |\rho(Z^n_s) - Z^n_s|\Big)^2 \le 8((\Upsilon^{(1)})^2\mathbb{I}_{\{|Y^n_s|>n\}} + |Z^n_s|^2\mathbb{I}_{\{|Z^n_s|>n\}}), \tag{6.7}$$

where $\sup_{n\in\mathbb{N}}|Y^n_s| \le \Upsilon^{(1)}$(see Theorem 3.4). Indeed, from definition of the function $\tilde\rho$ we deduce the following:

$$\begin{aligned}|\tilde\rho(Y^n_s) - Y^n_s| &= 1_{\{|Y^n_s|>n\}}|\tilde\rho(Y^n_s) - Y^n_s| + 1_{\{|Y^n_s|\le n\}}|\tilde\rho(Y^n_s) - Y^n_s|\\ &= 1_{\{|Y^n_s|>n\}}|\tilde\rho(Y^n_s) - Y^n_s|\\ &\le 2|Y^n_s|1_{\{|Y^n_s|>n\}}\end{aligned}$$

Using the same reasoning as above we deduce that $|\rho(Z^n_s) - Z^n_s| \le 2|Z^n_s|1_{\{|Z^n_s|>n\}}$.

Below we provide the convergence error of the truncation

Theorem 6.4. *Let assumptions of Theorem 5.2 be in force. Let (X, Y, Z) be the solution to equation (5.1)-(5.2) and (X, Y^n, Z^n) be the solution of (5.1)-(6.4), $n \ge 1$. Then for any $p > 1$ and $\kappa \ge 1$ there exist a positive finite constant $C(p, k)$ depending on $p, \kappa,$, (6.6), the bound in Lemma 4.3 and $\Upsilon^{(1)}$ such that for $n \in \mathbb{N}$*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Y^n_t - Y_t|^{2p} + \Big(\int_0^T |Z^n_s - Z_s|^2 \mathrm{d}s\Big)^p\Big] \le C(p,\kappa)(n)^{\frac{-\kappa}{4q}}$$

Proof. Let ζ^n be given by

$$\zeta^n_t := \frac{g(t, X_t, Y^n_t, Z^n_t) - g(t, X_t, Y^n_t, Z_t)}{|Z^n_t - Z_t|^2}(Z^n_t - Z_t)1_{\{Z^n_t - Z_t\neq 0\}}.$$

Then thanks to the assumptions of the theorem, $\zeta^n_t * B \in BMO(\mathbb{P})$. In addition for some constant $r > 1$, independent of n we have $\mathcal{E}(\zeta^n * B)^{-1} \in L^r(\mathbb{Q}^n)$. Let Π be define by $\Pi := \max\{\|\mathcal{E}(\zeta^n * B)\|_{L^r(\mathbb{P})}, \|\mathcal{E}(\zeta^n * B)^{-1}\|_{L^r(\mathbb{Q}^n)}\}$. Then, from the Bayes' rule and Hölder's inequality, we obtain for all $p > 1$ that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}|Y^n_t - Y_t|^{2p} + \Big(\int_0^T |Z^n_t - Z_t|^2\mathrm{d}t\Big)^p\Big]\\ \le&\Pi\Big[\mathbb{E}^{\mathbb{Q}^n}\Big(\sup_{t\in[0,T]}|Y^n_t - Y_t|^{2pq} + \Big(\int_0^T |Z^n_t - Z_t|^2\mathrm{d}t\Big)^{pq}\Big)\Big]^{\frac1q},\end{aligned} \tag{6.8}$$

where $\mathrm{d}\mathbb{Q}^n := \mathcal{E}(\zeta^n * B)\mathrm{d}\mathbb{P}$. Using Lemma 3.9, we have

$$\begin{aligned}&\Big(\mathbb{E}^{\mathbb{Q}^n}\Big[\sup_{t\in[0,T]}|Y^n_t - Y_t|^{2pq} + \Big(\int_0^T |Z^n_t - Z_t|^2\mathrm{d}t\Big)^{pq}\Big]\Big)^{\frac1q}\\ \le&C\Big(\mathbb{E}^{\mathbb{Q}^n}\Big[\int_0^T |g(s, X_s, Y^n_s, Z^n_s) - g(s, X_s, \tilde\rho(Y^n_s), \rho(Z^n_s))|\mathrm{d}s\Big]^{2pq}\Big)^{\frac1q}.\end{aligned}$$

In addition for $K = \max(\Lambda_y, \Lambda_z)$

$$\begin{aligned}&|g(s, X_s, Y^n_s, Z^n_s) - g(s, X_s, \tilde\rho(Y^n_s), \rho(Z^n_s))|\\ \le&\Lambda_y(1 + |\rho(Z^n_s)|^\alpha + |Z^n_s|^\alpha)|\tilde\rho(Y^n_s) - Y^n_s| + \Lambda_z(1 + (f(|\tilde\rho(Y^n_s)|) + f(|Y^n_s|))|\rho(Z^n_s) - Z^n_s|\\ \le&K(1 + 2|Z^n_s|^\alpha + 4\sup_{0\le y\le 2\Upsilon^{(1)}}\varphi(y)|Z^n_s|)\big(|\tilde\rho(Y^n_s) - Y^n_s| + |\rho(Z^n_s) - Z^n_s|\big).\end{aligned}$$

By using the above bound and applying the Hölder's inequality, we deduce that

$$\Big(\mathbb{E}^{\mathbb{Q}^n}\Big[\sup_{t\in[0,T]}|Y_t^n - Y_t|^{2pq} + \Big(\int_0^T |Z_t^n - Z_t|^2\mathrm{d}t\Big)^{pq}\Big]\Big)^{\frac{1}{q}}$$
$$\leq C(q,p)\Big(\mathbb{E}^{\mathbb{Q}^n}\Big[\int_0^T \Big|K(1+2|Z_s^n|^\alpha + 4\sup_{0\leq y\leq 2\Upsilon^{(1)}}\varphi(y)|Z_s^n|)\Big|^2\mathrm{d}s\Big]^{2pq}\Big)^{\frac{1}{2q}}$$
$$\times\Big(\mathbb{E}^{\mathbb{Q}^n}\Big[\int_0^T \Big(|\tilde\rho(Y_s^n) - Y_s^n| + |\rho(Z_s^n) - Z_s^n|\Big)^2\mathrm{d}s\Big]^{2pq}\Big)^{\frac{1}{2q}}$$

Since Z^n belongs to $\mathcal{H}^{2p}$ and the function φ is continuous, we deduce that the first term in the above inequality is uniformly bounded in n. It then remains to prove that the second term is also uniformly bounded. Since, $\mathbb{P}$ and $\mathbb{Q}^n$ are equivalent, using (6.6) and (6.7), the Hölder and the Markov's inequalities give

$$\Big(\mathbb{E}^{\mathbb{Q}^n}\Big[\int_0^T \Big(|\tilde\rho(Y_s^n) - Y_s^n| + |\rho(Z_s^n) - Z_s^n|\Big)^2\mathrm{d}s\Big]^{2pq}\Big)^{\frac{1}{2q}}$$
$$\leq C\Big(\mathbb{E}^{\mathbb{Q}^n}\Big[\int_0^T ((\Upsilon^{(1)})^2\mathbb{I}_{\{|Y_s^n|>n\}} + |Z_s^n|^2\mathbb{I}_{\{|Z_s^n|>n\}})\mathrm{d}s\Big]^{2pq}\Big)^{\frac{1}{2q}}$$
$$\leq C\Big(\mathbb{E}^{\mathbb{Q}^n}\int_0^T \mathbb{I}_{\{|Y_s^n|>n\}}\mathrm{d}s\Big)^{\frac{1}{2q}} + C\Big(\mathbb{E}^{\mathbb{Q}^n}\int_0^T |Z_s^n|^{4pq}\mathbb{I}_{\{|Z_s^n|>n\}}\mathrm{d}s\Big)^{\frac{1}{2q}}$$
$$\leq C\Big(\int_0^T \mathbb{Q}^n\{|Y_s^n|>n\}\mathrm{d}s\Big)^{\frac{1}{2q}} + C\Big(\mathbb{E}^{\mathbb{Q}^n}\int_0^T |Z_s^n|^{8pq}\mathrm{d}s\Big)^{\frac{1}{4q}}\Big(\int_0^T \mathbb{Q}^n\{|Z_s^n|>n\}\mathrm{d}s\Big)^{\frac{1}{4q}}$$
$$\leq C(n)^{\frac{-\kappa}{2q}}\Big(\int_0^T \mathbb{E}^{\mathbb{Q}^n}\big[|Y_s^n|^{2\kappa}\big]\mathrm{d}s\Big)^{\frac{1}{2q}} + C(n)^{\frac{-\kappa}{4q}}\Big(\int_0^T \mathbb{E}^{\mathbb{Q}^n}|Z_s^n|^{8pq})\mathrm{d}s\Big)^{\frac{1}{4q}}\Big(\int_0^T \mathbb{E}^{\mathbb{Q}^n}\big[|Z_s^n|^{2\kappa}\big]\mathrm{d}s\Big)^{\frac{1}{4q}},$$

Using once more the uniform bounndedness of Y^n, the Bayes' rule and the Hölder's inequality, we derive the existence of a constant C that does not depend on κ and n such that:

$$\Big(\mathbb{E}^{\mathbb{Q}^n}\Big[\int_0^T \Big(|\tilde\rho(Y_s^n) - Y_s| + |\rho(Z_s^n) - Z_s^n|\Big)^2\mathrm{d}s\Big]^{2pq}\Big)^{\frac{1}{2q}}$$
$$\leq C(n)^{\frac{-\kappa}{2q}} + C(n)^{\frac{-\kappa}{4q}}\Big(\int_0^T \mathbb{E}\Big[\mathcal{E}(\Pi^n * B)\sup_{0\leq s\leq T}|Z_s^n|^{2\kappa}\Big]\mathrm{d}s\Big)^{\frac{1}{4q}}$$
$$\leq C(n)^{\frac{-\kappa}{2q}} + C\mathit{\Pi}(n)^{\frac{-\kappa}{4q}}\Big(\int_0^T \mathbb{E}\big[|Z_s^n|^{2\kappa q}\big]\mathrm{d}s\Big)^{\frac{1}{4q^2}}$$
$$\leq C(n)^{\frac{-\kappa}{2q}} + C\mathit{\Pi}(n)^{\frac{-\kappa}{4q}}\Big(\mathbb{E}\big[\sup_{0\leq s\leq T}|Z_s^n|^{2\kappa q}\big]\Big)^{\frac{1}{4q^2}}.$$

This completes the proof. □

Appendix A. Some Properties of BMO-Martingales

The following result give some properties of BMO-martingales which will be extensively used in this work. We refer the reader to [26] for more details on the subject.

Lemma A.1.

(P1) *Let M be a BMO martingale with quadratic variation $\langle M\rangle$. Let $(\mathcal{E}(M)_t)_{0\leq t\leq T}$ be the process defined by*

$$\mathcal{E}(M)_t := \exp\{M_T - 1/2\langle M\rangle_t\}.$$

Then $\mathbb{E}[\mathcal{E}(M)_T] = 1$ and the measure $\mathbb{Q}$ given by $\mathrm{d}\mathbb{Q} := \mathcal{E}(M)_T\mathrm{d}\mathbb{P}$ defines a probability measure.

(P2) *For a given BMO martingale M, there exists $r > 1$ such that $\mathcal{E}(M) \in L^r$. Moreover, for any stopping time $\tau \in [0,T]$ it holds that*

$$\mathbb{E}[\mathcal{E}(M)_T^r|\mathfrak{F}_\tau] \leq C(r)(\mathcal{E}(M)_\tau)^r,$$

(P3) *If* $\|M\|_{BMO} < 1$, *then for every stopping time* $\tau \in [0,T]$

$$\mathbb{E}\left[\exp\{\langle M\rangle_T - \langle M\rangle_\tau\}/\mathfrak{F}_\tau\right] < \frac{1}{1-\|M\|^2_{BMO}}.$$

(P4) *If* $\int_0 Z_s \mathrm{d}B_s \in BMO$, *then for every* $p \geq 1$ *it holds that* $Z \in \mathcal{H}^{2p}(\mathbb{R}^d)$ *i.e*

$$\mathbb{E}\Big[\Big(\int_0^T |Z_s|^2 \mathrm{d}s\Big)^p\Big] \leq p!\Big\|\int Z\mathrm{d}B\Big\|^{2p}_{BMO}.$$

Moreover for any $p \geq 1$ *and any* $\epsilon \in (0,2)$

$$\mathbb{E}\Big[\exp\Big(p\int_0^T |Z_s|^\epsilon \mathrm{d}s\Big)\Big] \leq C,$$

where C *depends on* p, ϵ *and* $\|\int Z\mathrm{d}B\|^2_{BMO}$.

Appendix B. Proofs of some Lemmas

proof of Lemma 3.8. Set

$$\begin{cases}\delta Y = Y^1 - Y^2, & \delta Z = Z^1 - Z^2\\ \delta\xi = \xi^1 - \xi^2, & \delta g = g^1(\cdot, Y^2, Z^2) - g^2(\cdot, Y^2, Z^2),\end{cases}$$

and define the processes

$$\Gamma_t := \frac{g^1(t, Y^1_t, Z^1_t) - g^1(t, Y^2_t, Z^1_t)}{Y^1_t - Y^2_t}1_{\{Y^1_t - Y^2_t \neq 0\}}, \quad e_t := \exp\Big(\int_0^t |\Gamma_s|\mathrm{d}s\Big), \tag{B.1}$$

$$\Pi_t := \frac{g^1(t, Y^2_t, Z^1_t) - g^1(t, Y^2_t, Z^2_t)}{|Z^1_t - Z^2_t|^2}(Z^1_t - Z^2_t)1_{\{|Z^1_t - Z^2_t| \neq 0\}}. \tag{B.2}$$

By following the same arguments developed in the proof of Theorem 3.4, we deduce the well posedness of the probability measure $\mathbb{Q}$ with density $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = \mathcal{E}(\int_0^\cdot \Pi.\mathrm{d}B.)$ and the following process $B^{\mathbb{Q}}_\cdot = B_\cdot - \int_0^\cdot \Pi_s \mathrm{d}s$ is a $\mathbb{Q}$-Brownian motion. We also have $\Gamma \in \mathcal{H}_{\mathrm{BMO}}$, thus $\Gamma \in \mathcal{H}^{2p}$, for every $p \geq 1$. Since $|\Gamma_t| \leq \Lambda_y(1 + 2|Z^1_t|^\alpha)$. The dynamics of (δY_t) is given by:

$$\delta Y_t = \delta\xi + \int_t^T \delta g_s \mathrm{d}s + \int_t^T \left(g^1(s, Y^1_s, Z^1_s) - g^1(s, Y^2_s, Z^2_s)\right)\mathrm{d}s - \int_t^T \delta Z_s \mathrm{d}B_s. \tag{B.3}$$

The Itô's formula yields

$$e_t\delta Y_t = e_t\delta\xi + \int_t^T e_s\delta g_s \mathrm{d}s - \int_t^T e_s\delta Z_s \mathrm{d}B^{\mathbb{Q}}_s, \tag{B.4}$$

where $\int_0^\cdot e_s\delta Z_s \mathrm{d}B^{\mathbb{Q}}_s$ is a true $\mathbb{Q}$-martingale. Taking the conditional expectation on both sides of (B) gives

$$e_t\delta Y_t = \mathbb{E}^{\mathbb{Q}}\Big[e_t\delta\xi + \int_t^T e_s\delta g_s \mathrm{d}s|\mathfrak{F}_t\Big]. \tag{B.5}$$

We conclude that $e_t\delta Y_t \leq 0$ and hence for all $t \in [0,T]$ we have $Y^1_t \leq Y^2_t$ $\mathbb{Q}$-a.s. and $\mathbb{P}$-a.s. In particular set $t = 0$ and suppose $\delta < 0$ or $\delta g < 0$ in a set of positive $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-measure. Then we obtain $Y^1_0 < Y^2_0$. □

Proof of Lemma 3.9. The proof follows as in [22, Lemma A.1]. For the sake of completeness, we briefly reproduce it here. We keep the same notations as in the proof of Lemma 3.8 at the exception of the process $(e_t)_{t\in[0,T]}$ defined as follows: $e_t := \exp(2\int_0^t |\Gamma_s|\mathrm{d}s)$. Using Itô's formula, and the Girsanov transform, the dynamics of the continuous semimartingale $(e_t\delta Y^2_t)_{t\geq 0}$ is given by

$$\begin{aligned}\mathrm{d}[e_t(\delta Y_t)^2] =& 2(\delta Y_t)^2|\Gamma_t|e_t\mathrm{d}t + 2e_t\delta Y_t\mathrm{d}\delta Y_t + e_t|\delta Z_t|^2\\ =& (\delta Y_t)^2|\Gamma_t|e_t\mathrm{d}t + 2e_t\delta Y_t\left\{-\delta g_t\mathrm{d}t - (g^1(t, Y^1_t, Z^1_t) - g^1(t, Y^2_t, Z^2_t))\mathrm{d}t + \delta Z_t\mathrm{d}B_t\right\} + e_t|\delta Z_t|^2\\ =& -2e_t\delta Y_t\delta g_t + e_t|\delta Z_t|^2 + \delta Z_t\mathrm{d}B^{\mathbb{Q}}_t.\end{aligned}$$

Observe that $(e_t)_{t\in[0,T]}$ is strictly increasing and bounded from below by 1, and for any $0 \le t \le s \le T$

$$e_s(e_t)^{-1} = \exp\Big\{2\int_t^s |\Gamma_r|\mathrm{d}r\Big\} \le A_T := \exp\Big\{2\int_0^T K(1+|Z_s^1|^\alpha)\mathrm{d}s\Big\}.$$

Moreover since $Z^1 \in \mathcal{H}_{\mathrm{BMO}}$, $(e_t)_{t\in[0,T]} \in \mathcal{S}^p$ for all $p \ge 1$. Then, we deduce that

$$e_t(\delta Y_t)^2 + \int_t^T |\delta Z_s|^2\mathrm{d}s \le e_T\delta\xi^2 + 2\int_t^T e_s\delta Y_s\delta g_s\mathrm{d}s - 2\int_t^T e_s\delta Y_s\delta Z_s\mathrm{d}B_s^{\mathbb{Q}}. \tag{B.6}$$

Let us first provide an estimate of the norm of δY_t under the measure $\mathbb{Q}$. Dividing both sides of (B.6) by e_t, taking the conditional expectation on both sides and using the basic inequality $ab \le \frac{1}{\epsilon}a^2 + \epsilon b^2$ for $\epsilon > 0$, we obtain

$$\begin{aligned}\delta Y_t^2 \le& \mathbb{E}^{\mathbb{Q}}\Big[\frac{e_T}{e_t}\delta\xi^2 + 2\int_0^T \frac{e_s}{e_t}\delta Y_s|\delta g_s|\mathrm{d}s\Big|\mathfrak{F}_t\Big]\\ \le& \mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{\epsilon}\sup_{0\le t\le T}|\delta Y_t|^2 + A_T^2\mathcal{X}\Big|\mathfrak{F}_t\Big],\end{aligned} \tag{B.7}$$

where $\mathcal{X}$ is defined by $\mathcal{X} := |\delta\xi|^2 + \epsilon\Big(\int_0^T|\delta g_s|\mathrm{d}s\Big)^2$. Notice that $\mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{\epsilon}\sup_{0\le t\le T}|\delta Y_t|^2\Big|\mathfrak{F}_t\Big]$ (respectively $\mathbb{E}[A_T^2\mathcal{X}|\mathfrak{F}_t]$) is a right continuous martingale on $[0,T]$ with terminal random value given by $\frac{1}{\epsilon}\sup_{0\le t\le T}|\delta Y_t|^2$ (respectively $A_T^2\mathcal{X}$). Then by the Doob's martingale inequality we have that for all $p > 1$

$$\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}\mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{\epsilon}\sup_{0\le t\le T}|\delta Y_t|^2\Big|\mathfrak{F}_t\Big]\Big]^p \le \Big(\frac{p}{p-1}\Big)^p\mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{\epsilon^p}\sup_{0\le t\le T}|\delta Y_t|^{2p}\Big], \tag{B.8}$$

$$\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}\mathbb{E}^{\mathbb{Q}}[A_T^2\mathcal{X}|\mathfrak{F}_t]\Big]^p \le \Big(\frac{p}{p-1}\Big)^p\mathbb{E}^{\mathbb{Q}}[A_T^{2p}\mathcal{X}^p]. \tag{B.9}$$

Taking successively the absolute value, the p power, the supremum and the expectation on both sides of (B.7) and using (B.8) and (B.9), we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[\sup_{t\in[0,T]}|\delta Y_t|^{2p}] \le& \mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}\Big(\mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{\epsilon}\sup_{0\le t\le T}|\delta Y_t|^2 + A_T^2\mathcal{X}\Big|\mathfrak{F}_t\Big]\Big)^p\Big]\\ \le& C(p)\Big(\mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{\epsilon^p}\sup_{0\le t\le T}|\delta Y_t|^{2p}\Big] + \mathbb{E}^{\mathbb{Q}}[A_T^{2p}\mathcal{X}^p]\Big).\end{aligned}$$

Choosing ϵ such that $\frac{C(p)}{\epsilon^p} < 1$ and using the Hölder's inequality for $\nu \ge 1$, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[\sup_{t\in[0,T]}|\delta Y_t|^{2p}] \le& C(p)\mathbb{E}^{\mathbb{Q}}\Big[A_T^{2p}\mathcal{X}^p\Big]\\ \le& C(p)\mathbb{E}^{\mathbb{Q}}[A_T^{\frac{2p\nu}{\nu-1}}]^{\frac{\nu-1}{\nu}}\mathbb{E}^{\mathbb{Q}}\Big[|\delta\xi|^{2p\nu} + \Big(\int_0^T|\delta g_s|\mathrm{d}s\Big)^{2p\nu}\Big]^{\frac{1}{\nu}}.\end{aligned}$$

The first term on the right side of the above bound is finite (see Lemma A.1).

Since $\Pi * B$ is a $BMO(\mathbb{P})$ martingale it follows that $-\Pi * B^{\mathbb{Q}}$ and $\Pi * B^{\mathbb{Q}}$ are $BMO(\mathbb{Q})$ martingales. Then from Lemma A.1 there exists $r, r_1 > 1$ such that $\mathcal{E}(\Pi * B) \in L^r$ and $\mathcal{E}(-\Pi * B^{\mathbb{Q}}) \in L^{r'}$. In addition, since $\mathcal{E}(\Pi * B)^{-1} = \mathcal{E}(-\Pi * B^{\mathbb{Q}})$, we have $\mathrm{d}\mathbb{P} = \mathcal{E}(-\Pi * B^{\mathbb{Q}})\mathrm{d}\mathbb{Q}$. Let r' and r_1' be the Hölder conjugates of r and r_1, respectively. Then using Girsanov theorem, we have

$$
\begin{aligned}
\mathbb{E}[\sup_{t\in[0,T]}|\delta Y_t|^{2p}] &= \mathbb{E}^{\mathbb{Q}}[\mathcal{E}(-\Pi * B^{\mathbb{Q}})_T \sup_{t\in[0,T]}|\delta Y_t|^{2p}] \\
&\leq \mathbb{E}^{\mathbb{Q}}[\mathcal{E}(-\Pi * B^{\mathbb{Q}})_T^{r_1}]^{\frac{1}{r_1}}\mathbb{E}^{\mathbb{Q}}[\sup_{t\in[0,T]}|\delta Y_t|^{2pr_1'}]^{\frac{1}{r_1'}} \\
&\leq C\mathbb{E}^{\mathbb{Q}}\Big[|\delta\xi|^{2pr_1'\nu} + \Big(\int_0^T|\delta g_s|\mathrm{d}s\Big)^{2pr_1'\nu}\Big]^{\frac{1}{r_1'\nu}} \\
&\leq C\mathbb{E}[\mathcal{E}(\Pi * B)^r]^{\frac{1}{r}}\mathbb{E}\Big[|\delta\xi|^{2pr_1'\nu r'} + \Big(\int_0^T|\delta g_s|\mathrm{d}s\Big)^{2pr_1'\nu r'}\Big]^{\frac{1}{r_1'\nu r'}}
\end{aligned}
$$

We obtain the desired estimate for δY by taking $q = r_1'\nu r'$. Similar techniques can be used to provide the bound for δZ. This conclude the proof. □

Appendix C. Integration with respect to local time

Here we give some elements of calculus with respect to local time. For $a \in \mathbb{R}$ and $X = \{X_t, t \geq 0\}$ a continuous semimartingale, the local time $L^X(t,a)$ of X at level a can be defined by the following Tanaka-Meyer formula

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X(s)-a)\mathrm{d}X_s + L^X(t,a), \tag{C.1}$$

where $\operatorname{sgn}(x) = -1_{(-\infty,0]}(x) + 1_{(0,+\infty)}(x)$. The local time-space integral is defined for functions in the space $(\mathcal{H}_x, \|\cdot\|_x)$ (see e.g. [14]) where $(\mathcal{H}_x, \|\cdot\|_x)$ is the space of Borel measurable functions $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ with the norm

$$\|f\|_x^p := 2\mathbb{E}\Big[\Big(\int_0^t f^2(s,B_s)\mathrm{d}s\Big)^{p/2}\Big] + 2\mathbb{E}\Big[\Big|\int_0^t f(s,B_s)\frac{B_s}{s}\mathrm{d}s\Big|^p\Big],$$

for any $p \in [1,\infty)$. Let $f \in \mathcal{H}_x$. Then using [4, Lemma 2.11] we have the following representation

$$\int_0^t \partial_x f(s,X_s^x)\mathrm{d}s = -\int_0^t\int_{\mathbb{R}} f(s,z)L^{X^x}(\mathrm{d}s,\mathrm{d}z) \tag{C.2}$$

for all $t \in [0,T]$. In addition for $f \in \mathcal{H}_0$, we have (see for example [15, Theorem 2.1])

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}} f(s,z)L^{B^x}(\mathrm{d}s,\mathrm{d}z) &= \int_0^t f(s,B_s^x)\mathrm{d}B_s + \int_{T-t}^T f(T-s,\widehat{B}_s^x)\mathrm{d}W_s \\
&\quad - \int_{T-t}^T f(T-s,\widehat{B}^x(s))\frac{\widehat{B}_s}{T-s}\mathrm{d}s, \quad 0\leq t\leq T \text{ a.s.},
\end{aligned} \tag{C.3}
$$

where $L^{B^x}(\mathrm{d}s,\mathrm{d}z)$ denotes integration with respect to the local time of the Brownian motion B^x in both time and space, B^x is the Brownian motion started at x and $\widehat{B}$ is the time-reversed Brownian motion defined by

$$\widehat{B}_t := B_{T-t},\ 0 \leq t \leq T. \tag{C.4}$$

The process $W = \{W_t,\ 0 \leq t \leq T\}$ stands for an independent Brownian motion with respect to the filtration $\mathcal{F}_t^{\widehat{B}}$ generated by $\widehat{B}_t$, and satisfies:

$$W_t = \widehat{B}_t - B_T + \int_t^T \frac{\widehat{B}_s}{T-s}\mathrm{d}s. \tag{C.5}$$

Appendix D. Stochastic sewing lemma

In this section we recall the following lemma corresponding to [29, Theorem 2.1]

Lemma D.1 (Stochastic sewing lemma). *Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a complete probability space. Fix $p \geq 2$ and define*

$$\Delta_n^T := \{(t_1, \cdots, t_n) \in [0,T]^n,\ 0 \leq t_1 \leq \cdots, \leq t_n \leq T\}, \tag{D.1}$$

and let $A : \Delta_2^T \to \mathbb{R}$ be a stochastic process satisfying: $A_{s,s} = 0, A_{s,t}$ is $\mathfrak{F}_s$-measurable, and $(s,t) \mapsto A_{s,t}$ is right-continuous from Δ_2^T into $L^p(\Omega)$. Set $\delta_v A_{s,t} := A_{s,t} - A_{s,v} - A_{v,t}$ for $(s,v,t) \in \Delta_3^T$ and assume that there exists constants $\theta > 1$, $\kappa > \frac{1}{2}$, and $c_1, c_2 > 0$ such that

$$\|\mathbb{E}[\delta_v A_{s,t}|\mathfrak{F}_s]\|_{L^p(\Omega)} \leq c_1|t-s|^{\theta}, \tag{D.2}$$

$$\|\delta_v A_{s,t}\|_{L^p(\Omega)} \leq c_2|t-s|^{\kappa}. \tag{D.3}$$

Then, there exists a unique (up to modifications) $\{\mathfrak{F}_t\}_{t\geq 0}$-adapted stochastic process $\mathcal{A}$ satisfying the following properties:

1. *$\mathcal{A} : [0,T] \to L^p(\Omega)$ is right continuous, and $\mathcal{A}_0 = 0$.*
2. *There exists constants $C_1, C_2 > 0$ such that for $\mathcal{A}_{s,t} = \mathcal{A}_t - \mathcal{A}_s$*

$$\|\mathbb{E}[\mathcal{A}_{s,t} - A_{s,t}|\mathfrak{F}_s]\|_{L^p(\Omega)} \leq c_1 C_1|t-s|^{\theta}, \tag{D.4}$$

$$\|\mathcal{A}_{s,t} - A_{s,t}\|_{L^p(\Omega)} \leq c_1 C_1|t-s|^{\theta} + c_2 C_2|t-s|^{\kappa}. \tag{D.5}$$

Furthermore for all $(s,t) \in \Delta_2^T$ and for any partition $\mathcal{P}$ of the interval $[s,t]$, define

$$A_{s,t}^{\mathcal{P}} := \sum_{[u,v]\in\mathcal{P}} A_{u,v}. \tag{D.6}$$

Then $A_{s,t}^{\mathcal{P}}$ converge to $\mathcal{A}_{s,t}$ in $L^p(\Omega)$ as the mesh size $|\mathcal{P}| \to 0$.

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Humboldt-Universit'́at zu Berlin, Institut f'́ur Mathematik, Unter den Linden 6, D-10099 Berlin, Germany

Email address: imkeller@mathematik.hu-berlin.de

Department of Mathematics,University of Ghana, Accra, Ghana

African Institute for Mathematical Sciences, Ghana

Email address: rhoss@aims.edu.gh

African Institute for Mathematical Sciences, Ghana

Institute for Financial and Actuarial Mathematics, Department of Mathematical Sciences, University of Liverpool, L69 7ZL, United Kingdom

Email address: menoukeu@liverpool.ac.uk