On the local time of a Weierstrass function

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14h18, 9^{th} May, 2018 (File `localtime-weierstrass5.tex`)

Abstract

We show that a Weierstrass function that is $\frac{1}{2}$-Hölder continuous possesses a square integrable local time.

2000 AMS subject classifications: primary ; secondary .
Key words and phrases:

1 Introduction/Notes

Figure 1: Graphic of W for $x \in [0,1]$; $\{(x, W(x)) : x \in [0,1]\} \subset \mathbb{R}^2$.

2 The curve as attractor of a dynamical system

Our aim is to investigate the local time of the one-dimensional Weierstrass curve given by

$$W(x) = \sum_{n=0}^{\infty} 2^{-\frac{n}{2}} \cos(2\pi 2^n x), \quad x \in [0,1]. \tag{2.1}$$

Our access to the analysis of this function is via the theory of dynamical systems. In fact, we shall describe a dynamical system on $[0,1]^2$, alternatively $\Omega = \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$, which produces the graph of the function as its attractor. For elements of Ω we write for convenience $\omega = ((\omega_{-n})_{n\geq 0}, (\omega_n)_{n\geq 1})$; one understands Ω as the space of 2-dimensional sequences of Bernoulli random variables. Denote by θ the canonical shift on Ω, given by

$$\theta : \Omega \to \Omega, \omega \mapsto (\omega_{n+1})_{n\in\mathbb{Z}}.$$

Ω is endowed with the product σ-algebra, and the infinite product $\nu = \otimes_{n\in\mathbb{Z}}(\frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\{1\}})$ of Bernoulli measures on $\{0,1\}$. We recall that θ is ν-invariant.

Now let

$$T = (T_1, T_2) : \Omega \to [0,1]^2, \quad \omega \mapsto (\sum_{n=0}^{\infty} \omega_{-n} 2^{-(n+1)}, \sum_{n=1}^{\infty} \omega_n 2^{-n}).$$

Let us denote by T_1 the first component of T, and by T_2 the second one. It is well known that ν is mapped by the transformation T to λ^2 (i.e. $\nu = \lambda^2 \circ T$), the 2-dimensional Lebesgue measure. It is also well known that the inverse of T, the dyadic representation of the two components from $[0,1]^2$, is uniquely defined apart from the dyadic pairs. For these we define the inverse to map to the sequences not finally containing only 0. Let

$$B = T \circ \theta \circ T^{-1}.$$

We call B the *Baker's transformation*. The θ-invariance of ν directly translates into the B-invariance of λ^2:

$$\lambda^2 \circ B^{-1} = (\lambda^2 \circ T) \circ \theta^{-1} \circ T^{-1} = (\nu \circ \theta^{-1}) \circ T^{-1} = \nu \circ T^{-1} = \lambda^2.$$

For $(\xi, x) \in [0,1]^2$ let us note

$$T^{-1}(\xi, x) = \big((\overline{\xi}_{-n})_{n\geq 0}, (\overline{x}_n)_{n\geq 1}\big).$$

Let us calculate the action of B and its entire iterates on $[0,1]^2$.

Lemma 2.1 *Let $(\xi, x) \in [0,1]^2$. Then for $k \geq 0$*

$$B^k(\xi, x) = \Big(2^k \xi (mod\ 1), \frac{\overline{\xi}_{-k+1}}{2} + \frac{\overline{\xi}_{-k+2}}{2^2} + \cdots + \frac{\overline{\xi}_0}{2^k} + \frac{x}{2^k}\Big),$$

for $k \geq 1$

$$B^{-k}(\xi, x) = \Big(\frac{\xi}{2^k} + \frac{\overline{x}_1}{2^k} + \frac{\overline{x}_2}{2^{k-1}} + \cdots + \frac{\overline{x}_k}{2}, 2^k x (mod\ 1)\Big).$$

Proof: By definition of θ^k for $k \geq 0$

$$B^k(\xi, x) = \Big(\sum_{n \geq 0} \overline{\xi}_{-n+k} 2^{-(n+1)}, \frac{\overline{\xi}_{-k+1}}{2} + \frac{\overline{\xi}_{-k+2}}{2^2} + \cdots + \frac{\overline{\xi}_0}{2^k} + \sum_{n \geq 1} \overline{x}_n 2^{-(k+n)} \Big).$$

Now we can write

$$\sum_{n \geq 0} \overline{\xi}_{-n+k} 2^{-(n+1)} = 2^k \xi (\text{mod } 1) \quad \text{and} \quad \sum_{n \geq 1} \overline{x}_n 2^{-(k+n)} = \frac{x}{2^k}.$$

This gives the first formula. For the second, note that by definition of θ^{-k} for $k \geq 1$

$$B^{-k}(\xi, x) = \Big(\sum_{n \geq 0} \overline{\xi}_{-n} 2^{-(n+1+k)} + \frac{\overline{x}_1}{2^k} + \frac{\overline{x}_2}{2^{k-1}} + \cdots + \frac{\overline{x}_k}{2}, \sum_{n \geq 1} \overline{x}_{n+k} 2^{-n} \Big).$$

Again, we identify

$$\sum_{n \geq 1} \overline{x}_{n+k} 2^{-n} = 2^k x (\text{mod } 1) \qquad \text{and} \qquad \sum_{n \geq 0} \overline{\xi}_{-n} 2^{-(n+1+k)} = \frac{\xi}{2^k}.$$

□

For $k \in \mathbb{Z}, (\xi, x) \in [0,1]^2$ we abbreviate the k-th Baker transform of (ξ, x) as

$$B^k(\xi, x) = (\xi_k, x_k),$$

where for $k \geq 0$

$$\xi_k = 2^k \xi (\text{mod } 1), \quad \text{and} \quad x_k = \frac{\overline{\xi}_{-k+1}}{2} + \frac{\overline{\xi}_{-k+2}}{2^2} + \cdots + \frac{\overline{\xi}_0}{2^k} + \frac{x}{2^k},$$

and for $k \geq 1$

$$\xi_{-k} = \frac{\xi}{2^k} + \frac{\overline{x}_1}{2^k} + \frac{\overline{x}_2}{2^{k-1}} + \cdots + \frac{\overline{x}_k}{2}, \quad \text{and} \quad x_{-k} = 2^k x (\text{mod } 1).$$

Following Baranski [1, 2, 3], Shen [12], Hunt [7], we will next interpret the Weierstrass curve W by a transformation on our base space $[0,1]^2$. Let

$$\begin{aligned} F : [0,1]^2 \times \mathbb{R} &\to [0,1]^2 \times \mathbb{R}, \\ (\xi, x, y) &\mapsto \Big(B(\xi, x), 2^{-\frac{1}{2}} y + \cos\big(2\pi B_2(\xi, x)\big) \Big). \end{aligned}$$

Here we note $B = (B_1, B_2)$ for the two components of the Baker transform B.

For convenience, we extend W from $[0,1]$ to $[0,1]^2$ by setting

$$W(\xi, x) = W(x), \quad \xi, x \in [0,1].$$

To see that the graph of W is an attractor for F, the skew-product structure of F with respect to B plays a crucial role.

Lemma 2.2 *For any $\xi, x \in [0,1]$ we have*

$$F\big(\xi, x, W(\xi,x)\big) = \Big(B(\xi,x), W\big(B(\xi,x)\big)\Big).$$

Proof: We have by the 2π-periodicity of the trigonometric functions

$$\begin{aligned} W\big(B_2(\xi,x)\big) = W\Big(\frac{\overline{\xi}_0 + x}{2}\Big) &= \sum_{n=0}^{\infty} 2^{-\frac{n}{2}} \cos\Big(2\pi 2^n \frac{\overline{\xi}_0 + x}{2}\Big) \\ &= \cos\Big(2\pi \frac{\overline{\xi}_0 + x}{2}\Big) + \sum_{n=1}^{\infty} 2^{-\frac{n}{2}} \cos(2\pi 2^{n-1} x) \\ &= \cos\Big(2\pi \frac{\overline{\xi}_0 + x}{2}\Big) + 2^{-\frac{1}{2}} \sum_{n=0}^{\infty} 2^{-\frac{n}{2}} \cos(2\pi 2^n x) \\ &= \cos\big(2\pi B_2(\xi,x)\big) + 2^{-\frac{1}{2}} W(x). \end{aligned}$$

Hence by definition of F

$$\Big(B(\xi,x), W(B(\xi,x))\Big) = \Big(B(\xi,x), W(B_2(\xi,x))\Big) = F\Big(\xi, x, W(\xi,x)\Big).$$

□

To assess stability properties of the dynamical system generated by F, let us calculate its Jacobian. We obtain for $\xi, x \in [0,1], y \in \mathbb{R}$

$$DF(\xi,x,y) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\pi \sin\big(2\pi B_2(\xi,x)\big) & 2^{-\frac{1}{2}} \end{bmatrix}.$$

Hence the Lyapunov exponents of the dynamical system associated with F are given by 2, $\frac{1}{2}$, and $\gamma := 2^{-\frac{1}{2}}$. The corresponding invariant vector fields are given by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X(\xi,x) = \begin{pmatrix} 0 \\ 1 \\ 2\pi \sum_{n=1}^{\infty} \gamma^n \sin\big(2\pi B_2^n(\xi,x)\big) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

as is straightforwardly verified. Hence we have in particular for $\xi, x \in [0,1], y \in \mathbb{R}$

$$DF(\xi,x,y) X(\xi,x) = \frac{1}{2}\, X\big(B(\xi,x)\big).$$

Note that the vector X spans an invariant stable manifold, and does not depend on y.

3 The regularity of the SBR measure

In Tsujii [9] it has been proved that the Sinai-Bowen-Ruelle (SBR) measure of

$$S(\xi,x) = 2\pi \sum_{n=1}^{\infty} \gamma^n \sin\big(2\pi B_2^n(\xi,x)\big), \quad \xi, x \in [0,1],\ \gamma = 2^{-\frac{1}{2}},$$

is absolutely continuous with respect to Lebesgue measure. We shall now tackle a proof of this statement which rest upon the scaling properties of S alone, and argues via a Fourier analytic characterization of regularity of measures. It not only simplifies the proof by Tsujii [9], but is also *dual* to a derivation of the smoothness of the occupation measure of W in the subsequent section.

To recall the SBR measure of F, let us first calculate the action of S on the λ^2-measure preserving map B. For $\xi, x \in [0,1]$ we have

$$\begin{aligned} S(B(\xi,x)) &= 2\pi \sum_{n=1}^{\infty} \gamma^n \sin\Big(2\pi B_2^n(B_2(\xi,x))\Big) \\ &= 2\pi \sum_{n=1}^{\infty} \gamma^n \sin\big(2\pi B_2^{n+1}(\xi,x)\big) \\ &= 2\pi 2^{\frac{1}{2}} \sum_{k=1}^{\infty} \gamma^k \sin\big(2\pi B_2^k(\xi,x)\big) - 2\pi \sin\big(2\pi B_2(\xi,x)\big) \\ &= 2^{\frac{1}{2}} S(\xi,x) - 2\pi \sin\big(2\pi B_2(\xi,x)\big). \end{aligned}$$

So we may define the Anosov skew product

$$\begin{aligned} G : [0,1]^2 \times \mathbb{R} &\to [0,1]^2 \times \mathbb{R}, \\ (\xi,x,v) &\mapsto \Big(B(\xi,x), 2^{\frac{1}{2}} v - 2\pi \sin\big(2\pi B_2(\xi,x)\big)\Big). \end{aligned}$$

Then the equation proved before yields the next result (compare with Lemma 2.2).

Lemma 3.1 *For* $\xi, x \in [0,1]$ *we have*

$$\Gamma\big(\xi,x,S(\xi,x)\big) = \big(B(\xi,x), S(B(\xi,x))\big).$$

The measure

$$\psi = \lambda^2 \circ (id, S)^{-1}$$

on $\mathcal{B}([0,1]^2) \otimes \mathcal{B}(\mathbb{R})$ *is* Γ*-invariant.*

Proof: The first equation has been verified above. The Γ-invariance of ψ is a direct consequence of the B-invariance of λ^2. □

Define $\pi_2 : [0,1]^2 \to [0,1], (\xi,x) \mapsto x$ and let

$$\mu = \lambda^2 \circ (S, \pi_2)^{-1}.$$

The measure μ is called *Sinai-Bowen-Ruelle measure* of Γ.

We now define a map on our probability space that exhibits certain increments of S in a self similar way. Let

$$G(\xi,x) = 2\pi \sum_{n\in\mathbb{Z}} 2^{\frac{n}{2}} \big[\sin\big(2\pi B_2^{-n}(\xi,x)\big) - \sin\big(2\pi B_2^{-n}(0,x)\big)\big], \quad \xi, x \in [0,1].$$

Then we have the following simple relationship between G and S.

Lemma 3.2 *For $x, \xi, \eta \in [0,1]$ we have*

$$G(\xi, x) - G(\eta, x) = S(\xi, x) - S(\eta, x).$$

Proof: For $x, \xi, \eta \in [0,1]$ we have indeed

$$\begin{aligned} G(\xi, x) - G(\eta, x) &= \sum_{n \in \mathbb{Z}} 2^{\frac{n}{2}} \left[\sin\left(2\pi B_2^{-n}(\xi, x)\right) - \sin\left(2\pi B_2^{-n}(\eta, x)\right) \right] \\ &= \sum_{k=1}^{\infty} 2^{-\frac{k}{2}} \left[\sin\left(2\pi B_2^{k}(\xi, x)\right) - \sin\left(2\pi B_2^{k}(\eta, x)\right) \right] \\ &= S(\xi, x) - S(\eta, x). \end{aligned}$$

This completes the proof. □

Let us next assess the scaling properties of G. This will be crucial for the proof of the absolute continuity of the SBR measure. For this purpose, denote by ρ the image measure of λ^3 on $[0,1]^3$ by the mapping $(x, \xi, \eta) \mapsto G(\xi, x) - G(\eta, x) = S(\xi, x) - S(\eta, x)$. The next result concerns scaling and self-similarity properties of G and ρ.

Proposition 3.3 (Scaling of G) *For $\xi, x \in [0,1]$ we have*

$$G(B^{-1}(\xi, x)) = \gamma G(\xi, x).$$

Let C be a Borel set in $\mathbb{R}$. Then

$$\rho(\gamma C) = \gamma^2 \rho(C).$$

Proof: First note that by definition, defining $n + 1 = k$, for $\xi, x \in [0,1]$

$$\begin{aligned} G(B^{-1}(\xi, x)) &= 2\pi \sum_{n \in \mathbb{Z}} 2^{\frac{n}{2}} \left[\sin\left(2\pi B^{-n-1}(\xi, x)\right) - \sin\left(2\pi B^{-n-1}(0, x)\right) \right] \\ &= \gamma \sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}} \left[\sin\left(2\pi B^{-k}(\xi, x)\right) - \sin\left(2\pi B^{-k}(0, x)\right) \right] \\ &= \gamma G(\xi, x). \end{aligned}$$

For the second claim, note that the first one gives

$$\begin{aligned} &\int_{[0,1]^3} \mathbb{1}_C\left(|G(B^{-1}(\xi, x)) - G(B^{-1}(\eta, x))|\right) \mathrm{d}x \mathrm{d}\xi \mathrm{d}\eta \\ &\quad = \int_{[0,1]^3} \mathbb{1}_C\left(\gamma |G(\xi, x) - G(\eta, x)|\right) \mathrm{d}x \mathrm{d}\xi \mathrm{d}\eta = \rho(\gamma^{-1} C). \end{aligned}$$

On the other hand, using the definition of B^{-1}, we may calculate

$$\int_{[0,1]^3} \mathbb{1}_C\Big(\big|G\big(B^{-1}(\xi,x)\big) - G\big(B^{-1}(\eta,x)\big)\big|\Big)\mathrm{d}x\mathrm{d}\xi\mathrm{d}\eta$$
$$= \int_{[0,1]^3} \mathbb{1}_C\Big(\Big|G\Big(\frac{\xi+\overline{x}_1}{2}, 2x(\mathrm{mod}\ 1)\Big) - G\Big(\frac{\eta+\overline{x}_1}{2}, 2x(\mathrm{mod}\ 1)\Big)\Big|\Big)\mathrm{d}x\mathrm{d}\xi\mathrm{d}\eta$$
$$= \frac{1}{2}\int_{[0,1]^3} \mathbb{1}_C\Big(\Big|G\Big(\frac{\xi}{2}, 2x(\mathrm{mod}\ 1)\Big) - G\Big(\frac{\eta}{2}, 2x(\mathrm{mod}\ 1)\Big)\Big|\Big)\mathrm{d}x\mathrm{d}\xi\mathrm{d}\eta$$
$$+\frac{1}{2}\int_{[0,1]^3} \mathbb{1}_C\Big(\Big|G\Big(\frac{\xi+1}{2}, 2x(\mathrm{mod}\ 1)\Big) - G\Big(\frac{\eta+1}{2}, 2x(\mathrm{mod}\ 1)\Big)\Big|\Big)\mathrm{d}x\mathrm{d}\xi\mathrm{d}\eta$$
$$= (\frac{1}{2}+\frac{1}{2})\,2\int_{[0,1]^3} \mathbb{1}_C\Big(\big|G(\xi',x') - G(\eta',x')\big|\Big)\mathrm{d}x'\mathrm{d}\xi'\mathrm{d}\eta' = 2\rho(C).$$

For obtaining the first equality in the last line, we set $x' = 2x(\mathrm{mod}\,1), \xi' = \frac{\xi}{2}, \eta' = \frac{\eta}{2}$ resp. $x' = 2x(\mathrm{mod}\,1), \xi' = \frac{\xi+1}{2}, \eta' = \frac{\eta+1}{2}$. Combining the two preceding equations, we obtain altogether

$$\gamma^{-2}\rho(C) = \rho(\gamma^{-1}C).$$

Replacing C with γC and multiplying the equation by γ^2, we obtain the desired equation. □

Something seems not to fit here: if I take $C = \mathbb{R}$, then $\gamma C = C$, so the measure should not change when scaling.

▷gon: the sets should be taken from the support of the measure which is contained in a compact?

From the preceding scaling statement we can easily deduce the following practical corollary.

Corollary 3.4 *Let $L > 0$ be such that $[-L, L]$ is the support of ρ, which is symmetric. Then for $n \in \mathbb{N}$ we have*

$$\rho\big(]2^{-\frac{n+1}{2}}L, 2^{-\frac{n}{2}}L]\big) = \gamma^{2n}\,\rho\big(]2^{-\frac{1}{2}}L, L]\big).$$

Proof: Choose $C =]2^{-\frac{1}{2}}L, L]$ in Proposition 3.3, and iterate. □

Equipped with the scaling properties of G deduced above, we are finally able to address the main result of this section. We aim at studying the absolute continuity of μ with respect to Lebesgue measure. For this purpose we consider the Fourier transforms of the marginals $\mu_x, x \in [0,1]$. Let

$$\phi_x(u) = \int_{\mathbb{R}} \exp(iuy)\mu_x(\mathrm{d}y), \quad u \in \mathbb{R}.$$

By definition of μ and the integral transform theorem, we have

$$\phi_x(u) = \int_0^1 \exp(iuS(\xi, x))\mathrm{d}\xi, \quad u \in \mathbb{R}, x \in [0,1].$$

To prove absolute continuity of μ_x, we have to prove that ϕ_x is square integrable on $\mathbb{R}$. Therefore, to prove that μ is absolutely continuous, it will be sufficient to prove

$$\int_0^1 \int_{\mathbb{R}} |\phi_x(u)|^2 \mathrm{d}u\mathrm{d}x = \int_{\mathbb{R}} \int_{[0,1]^3} \exp\Big(iu\big(S(\xi,x) - S(\xi',x)\big)\Big)\mathrm{d}\xi\mathrm{d}\xi'\mathrm{d}x\mathrm{d}u < \infty.$$

Theorem 3.5 *For almost every $x \in [0,1]$ the function*

$$\xi \mapsto S(\xi, x)$$

has an absolutely continuous law with respect to Lebesgue measure with a square integrable density. In particular, the SBR measure of Γ is absolutely continuous with respect to Lebesgue measure and possesses a square integrable density.

Proof: We have to show that

$$\int_{\mathbb{R}} \int_{[0,1]^3} \exp\Big(iu\big(S(\xi,x) - S(\xi',x)\big)\Big)\mathrm{d}\xi\mathrm{d}\xi'\mathrm{d}x\mathrm{d}u < \infty.$$

Let $K > 0$ be fixed. We shall show that

$$\int_{-K}^{K} \int_{[0,1]^3} \exp\Big(iu\big(S(\xi,x) - S(\eta,x)\big)\Big)\mathrm{d}x\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}u,$$

is bounded by a constant independent of $K > 0$. Recall that ρ is symmetric with respect to reflection at the origin, and its compact support $[-L, L]$. We have

$$\begin{aligned}\int_{-K}^{K} \int_{[0,1]^3} \exp\Big(iu\big(S(\xi,x) - S(\eta,x)\big)\Big)\mathrm{d}x\mathrm{d}\xi\mathrm{d}\eta &= \int_{-K}^{K}\int_{-L}^{L} \exp(iuy)\rho(\mathrm{d}y)\mathrm{d}u \\ &= 2\int_{-K}^{K}\int_{0}^{L} \exp(iuy)\rho(\mathrm{d}y)\mathrm{d}u \\ &= 4\int_{0}^{L}\int_{0}^{K} \cos(uy)\mathrm{d}u\rho(\mathrm{d}y) \\ &= 4\int_0^L [\frac{1}{y}\sin(Ky)]\rho(\mathrm{d}y).\end{aligned}$$

Since sin is bounded, it remains to show that

$$\int_0^L \frac{1}{y}\rho(\mathrm{d}y) < \infty.$$

But this clearly follows from the scaling properties of ρ, Corollary 3.4, in writing

$$\int_0^L \frac{1}{y}\rho(\mathrm{d}y) \le L\sum_{n=0}^\infty 2^{\frac{n+1}{2}}\rho\big(]2^{-\frac{n+1}{2}}L, 2^{-\frac{n}{2}}L]\big) = L\sum_{n=0}^\infty 2^{\frac{1-n}{2}}\,\rho\big(]2^{-\frac{1}{2}}L, L]\big)$$
$$\le L\rho\big(]2^{-\frac{1}{2}}L, L]\big)\frac{2^{1/2}}{1-2^{-1/2}} < \infty.$$

□

The last estimate in the proof of the preceding Theorem indicates that the density of the SBR measure has more regularity than just square integrability.

4 The existence of a local time for W

In this section we use a similar Fourier analytic criterion as in the preceding one to show that the occupation measure associated with W possesses a square integrable density. This will be done in an indirect way. We shall first establish an intrinsic link between Weierstrass curve as the attractor of an underlying dynamical system and its stable manifold spanned by S. Then, we shall show, using a basic scaling equality and a Fourier analytic argument, that the local time of W shifted by smooth curves following the stable manifold exists. Finally, we shall get rid of the smooth curves to get a local time of W. In the following key lemma we establish the link between W and the stable manifold of F. For this purpose, we define

$$H(\xi,x) = \sum_{n\in\mathbb{Z}} 2^{-\frac{n}{2}}\big[\cos\big(2\pi B_2^{-n}(\xi,x)\big) - \cos\big(2\pi B_2^{-n}(\xi,0)\big)\big], \quad \xi, x\in[0,1].$$

Then we have the following relationship between W and S.

Lemma 4.1 *For $x, y, \xi \in [0,1]$ we have*

$$H(\xi,y) - H(\xi,x) = W(y) - W(x) - \int_x^y S(\xi,z)\mathrm{d}z.$$

Proof: For $x, y, \xi\in[0,1]$ we have indeed

$$\begin{aligned}
H(\xi,y) - H(\xi,x) &= \sum_{n\in\mathbb{Z}} 2^{-\frac{n}{2}}\big[\cos\big(2\pi B_2^{-n}(\xi,y)\big) - \cos\big(2\pi B_2^{-n}(\xi,x)\big)\big] \\
&= \sum_{n=0}^\infty 2^{-\frac{n}{2}}\big[\cos\big(2\pi B_2^{-n}(\xi,y)\big) - \cos\big(2\pi B_2^{-n}(\xi,x)\big)\big] \\
&\qquad + \sum_{k=1}^\infty 2^{\frac{k}{2}}\big[\cos\big(2\pi B_2^{k}(\xi,y)\big) - \cos\big(2\pi B_2^{k}(\xi,x)\big)\big] \\
&= W(y) - W(x) + \int_x^y (-2\pi)\sum_{k=1}^\infty 2^{-\frac{k}{2}}\sin\big(2\pi B_2^k(\xi,z)\big)\mathrm{d}z \\
&= W(y) - W(x) - \int_x^y S(\xi,z)\mathrm{d}z.
\end{aligned}$$

This completes the proof. □

We will next assess the scaling properties of H. This will be crucial for the proof of the existence of a local time.

Proposition 4.2 *For $\xi, x \in [0,1]$ we have*

$$H(B(\xi,x)) = \gamma H(\xi,x).$$

For $r > 0$ let

$$A_r = \{(x,y,\xi) \in [0,1]^3 : |H(\xi,y)) - H(\xi,x)| \leq r\}.$$

Then

$$\lambda^3(A_{\gamma r}) = \gamma^2 \lambda^3(A_r).$$

Proof: First note that by definition, setting $n-1=k$, for $\xi, x \in [0,1]$

$$\begin{aligned} H(B(\xi,x)) &= \sum_{n\in\mathbb{Z}} 2^{-\frac{n}{2}} \big[\cos\big(2\pi B^{-n+1}(\xi,x)\big) - \cos\big(2\pi B^{-n+1}(\xi,0)\big)\big] \\ &= \gamma \sum_{k\in\mathbb{Z}} 2^{-\frac{k}{2}} \big[\cos\big(2\pi B^{-k}(\xi,x)\big) - \cos\big(2\pi B^{-k}(\xi,0)\big)\big] \\ &= \gamma H(\xi,x). \end{aligned}$$

For the second claim, note that the first one gives

$$\int_{[0,1]^3} \mathbb{1}_{[0,r]}\Big(|H(B(\xi,y) - H(B(\xi,x))|\Big) \mathrm{d}x\mathrm{d}y\mathrm{d}\xi$$

$$\int_{[0,1]^3} \mathbb{1}_{[0,r]}\Big(\gamma|H(B(\xi,y)) - H(B(\xi,x)|\Big) \mathrm{d}x\mathrm{d}y\mathrm{d}\xi = \lambda^3(A_{\gamma^{-1}r}).$$

On the other hand, using the definition of B, we may calculate

$$\begin{aligned} &\int_{[0,1]^3} \mathbb{1}_{[0,r]}\Big(\big|H(B(\xi,y)) - H(B(\xi,x))\big|\Big) \mathrm{d}x\mathrm{d}y\mathrm{d}\xi \\ &= \int_{[0,1]^3} \mathbb{1}_{[0,r]}\Big(\Big|H\Big(2\xi (\mathrm{mod}\, 1), \frac{\overline{\xi}_0 + y}{2}\Big) - H\Big(2\xi (\mathrm{mod}\, 1), \frac{\overline{\xi}_0 + x}{2}\Big)\Big|\Big) \mathrm{d}x\mathrm{d}y\mathrm{d}\xi \\ &= \frac{1}{2}\int_{[0,1]^3} \mathbb{1}_{[0,r]}\Big(\Big|H\Big(2\xi (\mathrm{mod}\, 1), \frac{y}{2}\Big) - H\Big(2\xi (\mathrm{mod}\, 1), \frac{x}{2}\Big)\Big|\Big) \mathrm{d}x\mathrm{d}y\mathrm{d}\xi \\ &\quad + \frac{1}{2}\int_{[0,1]^3} \mathbb{1}_{[0,r]}\Big(\Big|H\Big(2\xi (\mathrm{mod}\, 1), \frac{1+y}{2}\Big) - H\Big(2\xi (\mathrm{mod}\, 1), \frac{1+x}{2}\Big)\Big|\Big) \mathrm{d}x\mathrm{d}y\mathrm{d}\xi \\ &= (\frac{1}{2} + \frac{1}{2})\, 2 \int_{[0,1]^3} \mathbb{1}_{[0,r]}\Big(\big|H(\xi',y') - H(\xi',x')\big|\Big) \mathrm{d}x'\mathrm{d}y'\mathrm{d}\xi' = 2\lambda^3(A_r). \end{aligned}$$

For obtaining the first equality in the last line, we set $\xi' = 2\xi (\mathrm{mod}\, 1), x' = \frac{x}{2}, y' = \frac{y}{2}$ resp. $\xi' = 2\xi (\mathrm{mod}\, 1), x' = \frac{x+1}{2}, y' = \frac{y+1}{2}$. Combining the two preceding equations, we obtain altogether

$$\gamma^{-2}\lambda^3(A_r) = \lambda^3(A_{\gamma^{-1}r}).$$

Replacing r with γr and multiplying the equation by γ^2, we obtain the desired equation. □

From the preceding scaling statement we can easily deduce the following practical corollary.

Corollary 4.3 *There are constants $c, C > 0$ such that for any $r > 0$ we have*

$$c\,r^2 \leq \lambda^3\Big(\big\{(\xi,x,y) \in [0,1]^3 : |H(\xi,y) - H(\xi,x)| < r\big\}\Big) \leq Cr^2.$$

Proof: Iterating the last statement of the preceding Proposition, we get for any $n \in \mathbb{N}$

$$\begin{aligned}&\lambda^3(\Big(\big\{(\xi,x,y) \in [0,1]^3 : |H(\xi,y) - H(\xi,x)| < \gamma^n\big\}\Big)\\&\quad = \gamma^{2n}\lambda^3(\Big(\big\{(\xi,x,y) \in [0,1]^3 : |H(\xi,y) - H(\xi,x)| < 1\big\}\Big).\end{aligned}$$

Choose $r > 0$. We may assume $r < 1$, since otherwise the claim is trivial. Next choose $l \in \mathbb{N}$ such that $\gamma^{l+1} \leq r \leq \gamma^l$. Then

$$\begin{aligned}&\lambda^3\Big(\big\{(\xi,x,y) \in [0,1]^3 : |H(\xi,y) - H(\xi,x)| < r\big\}\Big)\\&\quad \leq \lambda^3\Big(\big\{(\xi,x,y) \in [0,1]^3 : |H(\xi,y) - H(\xi,x)| < \gamma^l\big\}\Big)\\&\quad = \gamma^{2l}\lambda^3(\big\{(\xi,x,y) \in [0,1]^3 : |H(\xi,y) - H(\xi,x)| < 1\big\})\\&\quad \leq r^2\gamma^{-2}\lambda^3(\big\{(\xi,x,y) \in [0,1]^3 : |H(\xi,y) - H(\xi,x)| < 1\big\}).\end{aligned}$$

Hence by setting $C = \gamma^{-2}\lambda^3(\big\{(\xi,x,y) \in [0,1]^3 : |H(\xi,y) - H(\xi,x)| < 1\big\})$, we get the right hand side of the claimed inequality. A similar argument for the left hand side reveals that setting $c = \gamma^2\lambda^3(\big\{(\xi,x,y) \in [0,1]^3 : |H(\xi,y) - H(\xi,x)| < 1\big\})$ finishes the proof. □

This corollary improves the one dimensional version in Keller [8] of our Lemma 4.8. in the manuscript on the Hausdorff dimension essentially. So the telescoping proof with the very complex and tedious arguments in Keller's paper is not necessary. One can improve on the one hand Keller's paper.

Equipped with the scaling properties of H deduced above, we are finally able to state and prove the main result of this section.

Theorem 4.4 *For almost every $\xi \in [0,1]$ the function*

$$H(\xi,x) = W(x) - W(0) - \int_0^x S(\xi,z)\mathrm{d}z, \quad x \in [0,1],$$

possesses a square integrable local time.

Proof: We just have to transfer the arguments of the proof of Theorem 3.5 from S to H and a corresponding measure. We have to show that

$$\int_{\mathbb{R}} \int_{[0,1]^3} \exp\Big(iu(H(\xi,y) - H(\xi,x))\Big) \mathrm{d}x\mathrm{d}y\mathrm{d}\xi\mathrm{d}u < \infty.$$

Let $K > 0$ be fixed. We shall show that

$$\int_{-K}^{K} \int_{[0,1]^3} \exp\Big(iu(H(\xi,y) - H(\xi,x))\Big) \mathrm{d}x\mathrm{d}y\mathrm{d}\xi\mathrm{d}u$$

is bounded by a constant independent of K. For this, denote by χ the image measure of λ^3 on $[0,1]^3$ by the mapping $(\xi,x,y) \mapsto H(\xi,y) - H(\xi,x)$. This measure is symmetric with respect to reflection at the origin, and has compact support, say $[-L,L]$. We have

$$\begin{aligned}
\int_{-K}^{K} \int_{[0,1]^3} \exp\Big(iu(H(\xi,y) - H(\xi,x))\Big) \mathrm{d}x\mathrm{d}y\mathrm{d}\xi du &= \int_{-K}^{K} \int_{-L}^{L} \exp(iuy)\chi(\mathrm{d}y)\mathrm{d}u \\
&= 2\int_{-K}^{K} \int_{0}^{L} \exp(iuy)\chi(\mathrm{d}y)\mathrm{d}u \\
&= 4\int_{0}^{L} \int_{0}^{K} \cos(uy)\mathrm{d}u\chi(\mathrm{d}y) \\
&= 4\int_{0}^{L} [\frac{1}{y}\sin(Ky)]\chi(\mathrm{d}y).
\end{aligned}$$

Since sin is bounded, it remains to show that

$$\int_0^L \frac{1}{y}\chi(dy) < \infty.$$

But this follows from Corollary 4.3. □

We finally have to translate the result of Theorem 4.4 to a statement of existence of a local time for W.

Theorem 4.5 *The function W possesses a square integrable local time.*

Proof: Let $\xi \in [0,1]$ and $L(\xi,\cdot)$ be the square integrable local time of $H(\xi,\cdot)$, according to Theorem 4.4. We have to show that for some $\xi \in [0,1]$ the function $W = W(0) + H + \int_0^{\cdot} S(\xi,z)\mathrm{d}z$ possesses a square integrable local time. For any $\xi \in [0,1]$, the function $f(\xi,t) = W(0) + \int_0^t S(\xi,z)\mathrm{d}z, t \in [0,1]$, is infinitely often continuously differentiable. The local time $L(\xi,\cdot)$ induces a family of measures on the Borel sets of $[0,1]$, the distribution functions of which are given by $L(\xi,x,t)$, where $L(\xi,\cdot,t)$ is the square integrable local time of $H(\xi,\cdot)$, restricted to the interval $[0,t], t \in [0,1]$. In these terms, it is easy to see that the square integrable local time M of W derives from L via the formula

$$M(x) = \int_0^1 L\big(\xi, x - f(\xi,t)\big)\mathrm{d}t.$$

This perturbation result is certainly known. I just did not find the right reference in the literature. Can you check? □

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