TAKAGI TYPE FUNCTIONS AND DYNAMICAL SYSTEMS: THE SMOOTHNESS OF THE SBR MEASURE AND THE EXISTENCE OF LOCAL TIME

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Abstract. We investigate Takagi-type functions with roughness parameter γ that are Hölder continuous with coefficient $H = \frac{\log \gamma}{\log \frac{1}{2}}$. Analytical access is provided by an embedding into a dynamical system related to the baker transform where the graphs of the functions are identified as their global attractors. They possess stable manifolds hosting Sinai-Bowen-Ruelle (SBR) measures. We show that the SBR measure is absolutely continuous for large enough γ. Dually, where duality is related to time reversal, we prove that for large enough γ a version of the Takagi-type curve centered around fibers of the associated stable manifold possesses a square integrable local time.

1. Introduction

The interest in the subject of this paper, rough Takagi-type curves, arose from a two dimensional example of such functions studied in the context of the Fourier analytic approach of rough path analysis or rough integration theory laid out in [9] and [10]. In [10], the construction of a Stratonovich type integral of a rough function f with respect to another rough function g is based on the notion of paracontrol of f by g. This Fourier analytic concept generalizes the original notion of control introduced by Gubinelli [8]. In search of a good example of two-dimensional functions for which no component is controlled by the other one, in [15] we come up with a pair of Weierstrass functions $W = (W_1, W_2)$. One of them fluctuates on all dyadic scales in a sinusoidal manner, the other one in a cosinusoidal one. Hence while the first one has minimal increments, the second one has maximal ones, and vice versa. This is seen to mathematically underpin in a rigorous way the fact that they are mutually not controlled. It is also seen that the Lévy areas of the approximating finite sums of the representing series do not converge. This geometric pathology motivated us to look for further geometric properties of the pair, or of its single components. Here we look at a relative of the Weierstrass curves, Takagi-type curves with similar regularity parameters. In contrast to the former, they are more easily accessible to the analysis we employ for the investigation of their geometric properties. They are given by

$$\mathcal{T}(x) = \sum_{n=0}^{\infty} \gamma^n \Phi(2^n x), \quad x \in [0, 1],$$

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with $\Phi(x) = d(x, \mathbb{Z})$, the distance of x to the closest point in $\mathbb{Z}$, and a roughness parameter $\gamma \in]\frac{1}{2}, 1[$ (see Figure 1). They are Hölder continuous with Hurst parameter $H = \frac{\log \gamma}{\log \frac{1}{2}}$.

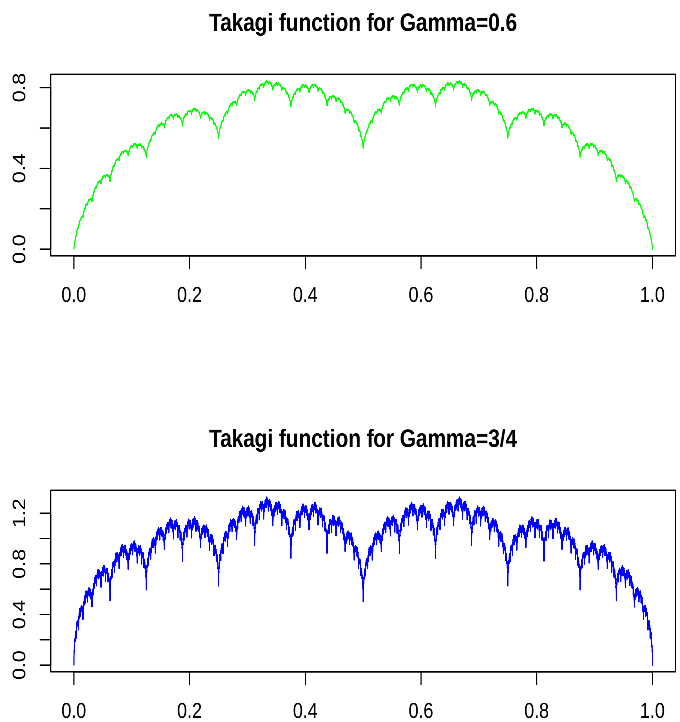

FIGURE 1. Takagi curve for differents values of γ

We continue the study of geometric properties of such functions by asking the question: under which condition on γ does $\mathcal{T}$ possess a local time? In fact, we shall answer this question for a modification H of $\mathcal{T}$ defined in (5.1), obtained by perturbation of $\mathcal{T}$ with a very smooth path, naturally given by geometric properties of the associated dynamical system our analysis is based on. The answer to this question resonates back to rough path analysis, as is impressively shown in [6]. There it is proved that curves possessing smooth local times have a regularizing effect, if added to an ill-posed ODE. More precisely, in [6] the notion of (ρ, γ)-irregularity is introduced. It is proved that adding a (ρ, γ)-irregular function to an ill-posed ODE typically gives rise to a well-posed equation. This notion of irregularity is based on a Sobolev smoothness of the occupation measure given in terms of the decay of its Fourier modes. From the stochastic analysis point of view, at least for one-dimensional problems, it is more natural to study regularity of the occupation measure in terms of local times.

It had been noticed in a series of papers (see [12], [3], [4], [5], [2], [16], [18]) on one-dimensional Weierstrass type curves that the number of iterations of the expansion by a real factor can be taken as a starting point in interpreting their graphs as pullback attractors of dynamical systems in which a baker transformation defines the dynamics. This observation marks, in many of the papers quoted, the point of departure for determining the Hausdorff dimension of graphs of one dimensional Weierstrass type functions. For a historical survey of this work the reader may consult [5]. For our curve we use the same metric dynamical system based on a suitable baker transformation as a starting point. This is done by introducing, besides a variable x that encodes expansion by the factor 2 forward in time, an auxiliary variable ξ describing contraction by the factor $\frac{1}{2}$ in turn, forward in time as well. The operation of expansion-contraction in both variables is described by the baker transformation $B = (B_1, B_2)$. Backward in time, the sense of expansion and contraction is interchanged. The action of applying forward expansion in one step just corresponds to stepping from one term in the series expansion of $\mathcal{T}$ to the following one. This indicates that $\mathcal{T}$ is an attractor of a three dimensional hyperbolic dynamical system F that, besides contracting a leading variable by the factor γ, adds the first term of the series to the result. So by definition of F, $\mathcal{T}$ is its attractor. Since $\frac{1}{2}$, the factor x in the forward fiber motion, is the smallest Lyapunov exponent of the linearization of F, there is a stable manifold related to this Lyapunov exponent. It is spanned by the vector which is given as another Weierstrass type series

$$S(\xi, x) = -\sum_{n=1}^{\infty} \kappa^n \Phi'\big(B_2^n(\xi, x)\big),$$

where $\kappa = \frac{1}{2\gamma} \in]\frac{1}{2}, 1[$ is a roughness parameter dual to γ. This will be explained below. The pushforward of the Lebesgue measure by $S(\cdot, x)$ for $x \in [0, 1]$ fixed, is the x-marginal of the Sinai-Bowen-Ruelle measure of F. The definition of F as a linear transformation added to a very smooth function may be understood as conveying the concept of *self-affinity* for the Takagi curve. Self-affinity can be seen as a concept providing the magnifying lens to zoom out microscopic properties of the underlying geometric object to a macroscopic scale. Our main tool of *telescoping relations* translates this rough idea into mathematical formulas, quite in the sense of Keller's paper [16]. Our telescoping is done in both time directions, forward and backward, and in doing this, we can, roughly, relate the Sinai-Bowen-Ruelle measure and the occupation measure underlying local time by duality through the operation of time reversal. More formally, we investigate the doubly infinite series

$$H(\xi, x) = \sum_{n\in\mathbb{Z}} \gamma^{-n} \left[\Phi\big(B_2^{-n}(\xi, x)\big) - \Phi\big(B_2^{-n}(\xi, 0)\big)\right], \quad \xi, x \in [0, 1].$$

A key equation relates $H, \mathcal{T}$ and the stable process S by the formula

$$H(\xi, y) - H(\xi, x) = \mathcal{T}(y) - \mathcal{T}(x) - \int_x^y S(\xi, z)\mathrm{d}z.$$

For a geometric interpretation of the increments of H, define the stable fiber through a point $(x, \mathcal{T}(x))$ of the graph of $\mathcal{T}$ by solutions of the initial value problem of the ODE

$$\frac{d}{dv} l_{(\xi,x,w)}(v) = S(\xi, v), \quad l_{(\xi,x,w)}(x) = w,$$

where we set $w = \mathcal{T}(x)$. Then vertical distances on different stable fibers are just given by the increments of H:

$$l_{(\xi,y,\mathcal{T}(y))}(y) - l_{(\xi,x,\mathcal{T}(x))}(y) = H(\xi, y) - H(\xi, x), \quad \xi, x, y \in [0, 1].$$

To study the Sinai-Bowen-Ruelle measure, we will start by looking at the measure ρ given by the pushforward of three-dimensional Lebesgue measure with the transformation

$$(\xi, \eta, x) \mapsto S(\xi, x) - S(\eta, x).$$

For the investigation of the occupation measure, in a dual step, we shall start with considering χ, the pushforward of three-dimensional Lebesgue measure with the transformation

$$(x, y, \xi) \mapsto H(\xi, y) - H(\xi, x).$$

For both measures we shall derive *telescoping equations* relating them with macroscopic versions $\hat{\rho}$ resp. $\hat{\chi}$ which live on the macroscopic sets $\{\overline{\xi}_0 \neq \overline{\eta}_0\}$ where $\overline{\xi}_0$ resp. $\overline{\eta}_0$ denote the first components in the dyadic expansion of ξ resp. η. The key element of our approach is the observation that the behaviour of S on these macroscopic sets is easy to describe, at least for specific ranges of the roughness parameters κ resp. γ. The related macroscopic properties are close to *transversality* properties used in many of the papers cited earlier. The first appearance of this notion describing a quality of the flow related to the map $x \mapsto S(\xi, x) - S(\eta, x)$ for (ξ, η) in the macroscopic set $\{\overline{\xi}_0 \neq \overline{\eta}_0\}$ is in Tsujii [19]. The particular role played by Takagi-type curves among more general Weierstrass curves is the relative simplicity of the stable manifold map S stating that $S(\xi, \cdot)$ is constant on its domain $[0, 1]$, for $\xi \in [0, 1]$. In this situation, transversality is just expressed by positivity of $S(\xi, 0)$ resp. $S(\xi, 0) - S(\eta, 0)$ uniformly on macroscopic sets $\{\overline{\xi}_0 = 1\}$ resp. $\{\frac{1}{2} < |\xi - \eta|\}$. This makes the usually tedious investigation of transversality relatively simple for Takagi-type curves. This property of positivity is established for ranges of κ resp. γ obtained from a representation of $S(\xi, \cdot) - S(\eta, \cdot)$ by series depending only on the *jump times* τ_k at which the dyadic components $\overline{\xi}_{-n}$ and $\overline{\eta}_{-n}$ representing ξ resp. η differ for the kth time (for details see Subsection 4.1 below). The telescoping relations used in Fourier analytic criteria for the smoothness of SBR resp. occupation measures reduce to simpler Fourier analytic criteria for the macroscopic restrictions of the measures that are deduced using transversality in the main theorems of the paper.

The paper is organized along these lines of reasoning in the following way. In Section 2, repeating [3], [12] or [16], we explain the interpretation of our Takagi-type curve in terms of dynamical systems based on the baker transform. In Section 3, we describe the measures related to the SBR measure, deduce telescoping relationships between them, and representation formulas for $S(\xi, \cdot) - S(\eta, \cdot)$ using the jump times $\tau_n, n \in \mathbb{N}$ outlined above. In Section 4, we treat the absolute continuity of the SBR measure. In the short Subsection 4.1 we establish transversality on suitable ranges of κ resp. γ. This provides the basis for the proof of absolute continuity of the SBR measure in Subsection 4.3, after the relationship between measures related to the SBR measure and their macroscopic versions have been clarified in Section 4.2. Dually, in Section 5, we deal with measures related to the occupation measure of H, and use a Fourier analytic criterion to show absolute continuity of the occupation measure of $H = \mathcal{T} - \int_0^{\cdot} S(\xi, z)\mathrm{d}z$.

2. The curve as the attractor of a dynamical system

Let $\gamma \in]\frac{1}{2}, 1[$. Our aim is to investigate the fine structure geometry of the one-dimensional Takagi type curves given by

$$\mathcal{T}(x) = \sum_{n=0}^{\infty} \gamma^n \Phi(2^n x), \quad x \in [0, 1], \tag{2.1}$$

where $\Phi(y) = d(y, \mathbb{Z}), y \in \mathbb{R}$. Let us first determine the Hölder exponent of $x \mapsto \mathcal{T}(x)$ (see [2] for an overview).

Proposition 2.1. *$\mathcal{T}$ is Hölder continuous with exponent $-\frac{\log \gamma}{\log 2}$.*

Proof. Let $x, y \in [0, 1]$ and choose an integer $k \geq 0$ such that

$$2^{-(k+1)} \leq |x - y| \leq 2^{-k}.$$

Then we have, using the Lipschitz continuity of the distance function

$$\begin{aligned}|\mathcal{T}(x)-\mathcal{T}(y)| &\leq \sum_{n=1}^{k}\gamma^n|d(2^n x,\mathbb{Z})-d(2^n y,\mathbb{Z})|+2\sum_{n=k+1}^{\infty}\gamma^n\\ &\lesssim \sum_{n=1}^{k}(2\gamma)^n|x-y|+\gamma^k\lesssim(2\gamma)^k\ 2^{-k}+\gamma^k\simeq\gamma^k=2^{-k\frac{\log\gamma}{\log\frac{1}{2}}}\\ &\lesssim |x-y|^{-\frac{\log\gamma}{\log 2}}.\end{aligned}$$

This shows that $\frac{\log\gamma}{\log\frac{1}{2}}$ is an upper bound for the Hölder exponent of $\mathcal{T}$. To see that it is also a lower bound, for $n\in\mathbb{N}$ choose $x_n=0, y_n=2^{-n}$. Then we may write

$$\begin{aligned}|\mathcal{T}(x_n)-\mathcal{T}(y_n)| &= \Big|\sum_{k=1}^{\infty}\gamma^k d(2^{k-n},\mathbb{Z})\Big|\\ &= \sum_{k=1}^{n-1}\gamma^k 2^{k-n}\simeq 2^{-n\frac{\log\gamma}{\log\frac{1}{2}}}=|x_n-y_n|^{-\frac{\log\gamma}{\log 2}}.\end{aligned}$$

Since $|x_n-y_n|\to 0$ as $n\to\infty$, this shows that $-\frac{\log\gamma}{\log 2}$ is also a lower bound for the Hölder exponent of $\mathcal{T}$. The argument can be extended to the other points in the interval. □

Our access to the analysis and geometry of $\mathcal{T}$ is via the theory of dynamical systems. In fact, we shall describe a dynamical system on $[0,1]^2$, alternatively $\Omega=\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$ the attractor of which is given by the graph of the function. For elements of Ω we write for convenience $\omega=((\omega_{-n})_{n\geq 0},(\omega_n)_{n\geq 1})$; one understands Ω as the space of 2-dimensional sequences of Bernoulli random variables. Denote by θ the canonical shift on Ω, given by

$$\theta:\Omega\to\Omega,\quad \omega\mapsto(\omega_{n+1})_{n\in\mathbb{Z}}.$$

Ω is endowed with the product σ-algebra, and the infinite product $\iota=\otimes_{n\in\mathbb{Z}}(\frac{1}{2}\delta_{\{0\}}+\frac{1}{2}\delta_{\{1\}})$ of Bernoulli measures on $\{0,1\}$. We recall that θ is ι-invariant.

Now let

$$T=(T_1,T_2):\Omega\to[0,1]^2,\quad \omega\mapsto(\sum_{n=0}^{\infty}\omega_{-n}2^{-(n+1)},\sum_{n=1}^{\infty}\omega_n 2^{-n}).$$

Let us denote by T_1 the first component of T, and by T_2 the second one. It is well known that ι is mapped by the transformation T to λ^2 (i.e. $\iota=\lambda^2\circ T$), the 2-dimensional Lebesgue measure. It is also well known that the inverse of T, the dyadic representation of the two components from $[0,1]^2$, is uniquely defined apart from the dyadic pairs. For these we define the inverse to map to the sequences not converging to 0. Let

$$B=(B_1,B_2)=T\circ\theta\circ T^{-1}.$$

We call $B=(B_1,B_2)$ the *baker's transformation*. The θ-invariance of ι directly translates into the B-invariance of λ^2:

$$\lambda^2\circ B^{-1}=(\lambda^2\circ T)\circ\theta^{-1}\circ T^{-1}=(\iota\circ\theta^{-1})\circ T^{-1}=\iota\circ T^{-1}=\lambda^2. \tag{2.2}$$

For $(\xi,x)\in[0,1]^2$ let us note

$$T^{-1}(\xi,x)=\big((\overline{\xi}_{-n})_{n\geq 0},(\overline{x}_n)_{n\geq 1}\big).$$

Let us calculate the action of B and its entire iterates on $[0,1]^2$.

Lemma 2.2. *Let $(\xi, x) \in [0,1]^2$. Then for $k \geq 0$*

$$B^k(\xi, x) = \Big(2^k \xi \,(mod\ 1), \frac{\overline{\xi}_{-k+1}}{2} + \frac{\overline{\xi}_{-k+2}}{2^2} + \cdots + \frac{\overline{\xi}_0}{2^k} + \frac{x}{2^k}\Big),$$

for $k \geq 1$

$$B^{-k}(\xi, x) = \Big(\frac{\xi}{2^k} + \frac{\overline{x}_1}{2^k} + \frac{\overline{x}_2}{2^{k-1}} + \cdots + \frac{\overline{x}_k}{2}, 2^k x(mod\ 1)\Big).$$

Proof: By definition of θ^k for $k \geq 0$

$$B^k(\xi, x) = \Big(\sum_{n\geq 0} \overline{\xi}_{-n+k} 2^{-(n+1)}, \frac{\overline{\xi}_{-k+1}}{2} + \frac{\overline{\xi}_{-k+2}}{2^2} + \cdots + \frac{\overline{\xi}_0}{2^k} + \sum_{n\geq 1} \overline{x}_n 2^{-(k+n)}\Big).$$

Now we can write

$$\sum_{n\geq 0} \overline{\xi}_{-n+k} 2^{-(n+1)} = 2^k \xi (\text{mod } 1) \quad \text{and} \quad \sum_{n\geq 1} \overline{x}_n 2^{-(k+n)} = \frac{x}{2^k}.$$

This gives the first formula. For the second, note that by definition of θ^{-k} for $k \geq 1$

$$B^{-k}(\xi, x) = \Big(\sum_{n\geq 0} \overline{\xi}_{-n} 2^{-(n+1+k)} + \frac{\overline{x}_1}{2^k} + \frac{\overline{x}_2}{2^{k-1}} + \cdots + \frac{\overline{x}_k}{2}, \sum_{n\geq 1} \overline{x}_{n+k} 2^{-n}\Big).$$

Again, we identify

$$\sum_{n\geq 1} \overline{x}_{n+k} 2^{-n} = 2^k x(\text{mod } 1) \qquad \text{and} \qquad \sum_{n\geq 0} \overline{\xi}_{-n} 2^{-(n+1+k)} = \frac{\xi}{2^k}.$$

□

For $k \in \mathbb{Z}, (\xi, x) \in [0,1]^2$ we abbreviate the k-fold iterate of the baker transform of (ξ, x) as

$$B^k(\xi, x) = \big(B_1^k(\xi, x), B_2^k(\xi, x)\big) = (\xi_k, x_k),$$

where for $k \geq 0$

$$\xi_k = 2^k \xi (\text{mod } 1), \quad \text{and} \quad x_k = \frac{\overline{\xi}_{-k+1}}{2} + \frac{\overline{\xi}_{-k+2}}{2^2} + \cdots + \frac{\overline{\xi}_0}{2^k} + \frac{x}{2^k},$$

and for $k \geq 1$

$$\xi_{-k} = \frac{\xi}{2^k} + \frac{\overline{x}_1}{2^k} + \frac{\overline{x}_2}{2^{k-1}} + \cdots + \frac{\overline{x}_k}{2}, \quad \text{and} \quad x_{-k} = 2^k x(\text{mod } 1).$$

Following Baranski [3, 4, 5], Shen [18], Hunt [12] and [14], we will next interpret the Takagi curve $\mathcal{T}$ by a transformation on our base space $[0,1]^2$. Let

$$\begin{aligned} F : [0,1]^2 \times \mathbb{R} \quad &\to \quad [0,1]^2 \times \mathbb{R}, \\ (\xi, x, y) \quad &\mapsto \quad \Big(B(\xi, x), \gamma y + \Phi(B_2(\xi, x))\Big). \end{aligned}$$

Here we note $B = (B_1, B_2)$ for the two components of the baker transform B.

For convenience, we extend $\mathcal{T}$ from $[0,1]$ to $[0,1]^2$ by setting

$$\mathcal{T}(\xi, x) = \mathcal{T}(x), \quad \xi, x \in [0,1].$$

To see that the graph of $\mathcal{T}$ is an attractor for F, the skew-product structure of F with respect to B plays a crucial role.

Lemma 2.3. *For any $\xi, x \in [0,1]$ we have*

$$F\big(\xi, x, \mathcal{T}(\xi, x)\big) = \Big(B(\xi, x), \mathcal{T}\big(B(\xi, x)\big)\Big).$$

Proof: By the definition of the baker's transform we may write

$$\mathcal{T}(\xi, x) = \sum_{n=0}^{\infty} \gamma^n \Phi\Big(B_2^{-n}(\xi, x)\Big), \quad \xi, x \in [0,1].$$

Hence, setting $k = n-1$, for $\xi, x \in [0,1]$

$$\begin{aligned} \mathcal{T}\big(B_2(\xi, x)\big) &= \sum_{n=0}^{\infty} \gamma^n \Phi\Big(B_2^{-n+1}(\xi, x)\Big) \\ &= \Phi\big(B_2(\xi, x)\big) + \gamma \sum_{k=0}^{\infty} \gamma^k \Phi\Big(B_2^{-k}(\xi, x)\Big) \\ &= \Phi\big(B_2(\xi, x)\big) + \gamma \mathcal{T}(x). \end{aligned}$$

Hence by definition of F

$$\Big(B(\xi, x), \mathcal{T}(B(\xi, x))\Big) = \Big(B(\xi, x), \mathcal{T}(B_2(\xi, x))\Big) = F\Big(\xi, x, \mathcal{T}(\xi, x)\Big).$$

□

To assess the stability properties of the dynamical system generated by F, we calculate its Jacobian. We obtain for $\xi, x \in [0,1], y \in \mathbb{R}$

$$DF(\xi, x, y) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2}\Phi'\big(B_2(\xi, x)\big) & \gamma \end{bmatrix}.$$

Hence the Lyapunov exponents of the dynamical system associated with F are given by 2, $\frac{1}{2}$, and γ. The corresponding invariant vector fields are given by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X(\xi, x) = \begin{pmatrix} 0 \\ 1 \\ \sum_{n=1}^{\infty} \big(\frac{1}{2\gamma}\big)^n \Phi'\big(B_2^n(\xi, x)\big) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

as is straightforwardly verified. Note that X is well defined, since by our choice of γ we have $2\gamma > 1$. Hence we have in particular for $\xi, x \in [0,1], y \in \mathbb{R}$

$$DF(\xi, x, y) X(\xi, x) = \frac{1}{2}\, X\big(B(\xi, x)\big).$$

Note that the vector X spans an invariant stable manifold and does not depend on y.

3. The Sinai-Bowen-Ruelle measure

Abbreviate $\kappa = \frac{1}{2\gamma} \in]0,1[$. In Tsujii [19] the problem of the absolute continuity of the Sinai-Bowen-Ruelle (SBR) measure on the stable manifold described by

$$S(\xi, x) = \sum_{n=1}^{\infty} \kappa^n \Phi'\big(B_2^n(\xi, x)\big), \quad \xi, x \in [0,1],$$

with respect to Lebesgue measure has been treated. It has been related to the *transversality* of the map $x \mapsto S(\xi, x) - S(\eta, x)$ for $\xi, \eta \in [0,1]$ such that $\overline{\xi}_0 \neq \overline{\eta}_0$. We shall now tackle a proof of this statement for a reasonably big range of κ by giving the problem of transversality of S a closer look. Our proof rests upon a comparison of the measures ρ, image measure of three dimensional Lebesgue measure under the map

$$(x, \xi, \eta) \mapsto S(\xi, x) - S(\eta, x),$$

and its restriction to the set $\{\frac{1}{2} < |\xi - \eta|\}$ which contains $\{\overline{\xi}_0 \neq \overline{\eta}_0\}$, namely $\hat{\rho} = \rho(\cdot \cap \{\frac{1}{2} < |\xi - \eta|\})$. This comparison will simplify the derivation of smoothness of the SBR measure from transversality in the spirit of Tsuji [19].

To recall the SBR measure of F, let us first calculate the action of S on the λ^2-measure preserving map B. For $\xi, x \in [0,1]$ we have

$$\begin{aligned} S(B(\xi,x)) &= \sum_{n=1}^{\infty} \kappa^n \Phi'\Big(B_2^n\big(B_2(\xi,x)\big)\Big) \\ &= \sum_{n=1}^{\infty} \kappa^n \Phi'\big(B_2^{n+1}(\xi,x)\big) \\ &= \kappa^{-1} \sum_{k=1}^{\infty} \kappa^k \Phi'\big(B_2^k(\xi,x)\big) - \Phi'\big(B_2(\xi,x)\big) \\ &= 2\gamma S(\xi,x) - \Phi'\big(B_2(\xi,x)\big). \end{aligned}$$

So we may define the Anosov skew product

$$\begin{aligned} \Gamma : [0,1]^2 \times \mathbb{R} &\to [0,1]^2 \times \mathbb{R}, \\ (\xi, x, v) &\mapsto \Big(B(\xi,x), 2\gamma v - \Phi'\big(B_2(\xi,x)\big)\Big). \end{aligned}$$

Then the equation just obtained yields the following result (compare with Lemma 2.3).

Lemma 3.1. *For $\xi, x \in [0,1]$ we have*

$$\Gamma\big(\xi, x, S(\xi,x)\big) = \big(B(\xi,x), S(B(\xi,x))\big).$$

The push-forward of the Lebesgue measure in $\mathbb{R}^2$ to the graph of S given by

$$\psi = \lambda^2 \circ (id, S)^{-1}$$

on $\mathcal{B}([0,1]^2) \otimes \mathcal{B}(\mathbb{R})$ is Γ-invariant.

Proof: The first equation has been verified above. The Γ-invariance of ψ is a direct consequence of the B-invariance of λ^2. □

Define $\pi_2 : [0,1]^2 \to [0,1], (\xi, x) \mapsto x$ and define the measure

$$\mu = \lambda^2 \circ (\pi_2, S)^{-1} \tag{3.1}$$

on $\mathcal{B}([0,1]^2)$. The measure μ is called the *Sinai-Bowen-Ruelle measure* of Γ. Its marginals in $x \in [0,1]$ are denoted $\mu_x = \lambda \circ S(\cdot, x)^{-1}$.

We now define a map on our probability space that exhibits certain increments of S in a self similar way. Let

$$G(\xi,x) = \sum_{n\in\mathbb{Z}} \kappa^{-n} \big[\Phi'\big(B_2^{-n}(\xi,x)\big) - \Phi'\big(B_2^{-n}(0,x)\big)\big], \quad \xi, x \in [0,1].$$

Then we have the following simple relationship between G and S.

Lemma 3.2. *For $x, \xi, \eta \in [0,1]$ we have*

$$G(\xi,x) - G(\eta,x) = S(\xi,x) - S(\eta,x).$$

Proof: For $x, \xi, \eta \in [0,1]$, the equation

$$\begin{aligned} G(\xi, x) - G(\eta, x) &= \sum_{n\in\mathbb{Z}} \kappa^{-n} \big[\Phi'\big(B_2^{-n}(\xi, x)\big) - \Phi'\big(B_2^{-n}(\eta, x)\big)\big] \\ &= \sum_{k=1}^{\infty} \kappa^{k} \big[\Phi'\big(B_2^{k}(\xi, x)\big) - \Phi'\big(B_2^{k}(\eta, x)\big)\big] \\ &= S(\xi, x) - S(\eta, x), \end{aligned}$$

holds. Here we used that the first sum for non-negative integers n is zero.

This completes the proof. □

The following result describes the scaling properties of G.

Lemma 3.3. *For $\xi, x \in [0,1]$ we have*

$$G\big(B^{-1}(\xi, x)\big) = \kappa G(\xi, x).$$

Proof: Note that by definition, setting $n+1 = k$, for $\xi, x \in [0,1]$

$$\begin{aligned} G\big(B^{-1}(\xi, x)\big) &= \sum_{n\in\mathbb{Z}} \kappa^{-n} \big[\Phi'\big(B^{-n-1}(\xi, x)\big) - \Phi'\big(B^{-n}(B_1^{-1}(\xi, x), 0)\big)\big] \\ &+ \sum_{n\in\mathbb{Z}} \kappa^{-n} \big[\Phi'\big(B^{-n-1}(\xi, x)\big) - \Phi'\big(B^{-n-1}(\xi, 0)\big)\big] \\ &+ \sum_{n\in\mathbb{Z}} \kappa^{-n} [\Phi'\big(B_2^{-n-1}(\xi, 0)\big) - \Phi'\big(B_2^{-n}(B_1^{-1}(\xi, x), 0)\big)\big] \\ &= \kappa G(\xi, x) + \sum_{k=1}^{\infty} \kappa^{k} [\Phi'(B_2^{k-1}(\xi, 0)) - \Phi'(B_2^{k}(B_1^{-1}(\xi, x), 0))] \\ &= \kappa G(\xi, x). \end{aligned}$$

For the last equality, note that Φ' is constant on the two halves of the unit interval, and that

$$B_2^{k-1}(\xi, 0) = \frac{\bar{\xi}_0}{2^{k-1}} + \cdots + \frac{\bar{\xi}_{-k+2}}{2}$$

and

$$B_2^{k}(B_1^{-1}(\xi, x), 0) = \frac{\bar{x}_1}{2^k} + \frac{\bar{\xi}_0}{2^{k-1}} + \cdots + \frac{\bar{\xi}_{-k+2}}{2}$$

belong to the same half. This provides the claimed identity.□

We finish this section by giving a representation which will be the starting point for our subsequent approach of transversality of S. To this end, fix $\xi, \eta \in [0,1]$. We recursively define the following sequence of times of disagreement of dyadic components of ξ and η. For $n \in \mathbb{N}$ let

(3.2) $$\tau_1 = \inf\{\ell \geq 0 : \bar{\xi}_{-\ell} \neq \bar{\eta}_{-\ell}\}, \text{ and } \tau_{n+1} = \inf\{\ell > \tau_n : \bar{\xi}_{-\ell} \neq \bar{\eta}_{-\ell}\},$$

and for $x \in [0,1]$

$$\begin{aligned} g(x) &:= \sum_{m=0}^{\infty} \kappa^m \Big[\Phi'\Big(B_2^m(0, \frac{1+x}{2})\Big) - \Phi'\Big(B_2^m(0, \frac{x}{2})\Big)\Big] \\ &= \sum_{m=0}^{\infty} \kappa^m \Big[\Phi'\Big(\frac{1+x}{2^{m+1}}\Big) - \Phi'\Big(\frac{x}{2^{m+1}}\Big)\Big]. \end{aligned}$$

We have the following result.

Proposition 3.4. *Let $\xi, \eta, x \in [0,1]$. Then*

$$S(\xi,x) - S(\eta,x) = \sum_{\ell=1}^{\infty} \kappa^{\tau_\ell+1} (-1)^{(1-\bar{\xi}_{-\tau_\ell})} g\big(B_2^{\tau_\ell}(\xi,x)\big). \tag{3.3}$$

Proof: It follows from the definition of $\tau_n, n \in \mathbb{N}$, that ξ can be written

$$\xi = (\overline{\eta}_0, \dots, \overline{\eta}_{\tau_1+1}, \overline{\xi}_{-\tau_1}, \overline{\eta}_{-\tau_1-1}, \cdots, \overline{\eta}_{-\tau_2+1}, \overline{\xi}_{-\tau_2}, \overline{\eta}_{-\tau_2-1}, \cdots\cdots, \overline{\eta}_{-\tau_n+1}, \overline{\xi}_{-\tau_n}, \overline{\eta}_{-\tau_n-1}, \cdots).$$

For $n \in \mathbb{N}$ let ξ^n be the sequence which up to τ_n represents the dyadic expansion of ξ, and then switches to the representing sequence of η. Then for $n \in \mathbb{N}$ we have

$$\xi^n = (\overline{\eta}_0, \dots, \overline{\eta}_{\tau_1+1}, \overline{\xi}_{-\tau_1}, \overline{\eta}_{-\tau_1-1}, \cdots, \overline{\eta}_{-\tau_2+1}, \overline{\xi}_{-\tau_2}, \overline{\eta}_{-\tau_2-1}, \cdots\cdots, \overline{\eta}_{-\tau_n+1}, \overline{\xi}_{-\tau_n}, \overline{\eta}_{-\tau_n-1}, \cdots, \overline{\eta}_{-m}, \cdots).$$

Note that $\lim_{n\to\infty} \xi^n = \xi$. We can therefore rewrite the left hand side of (3.3) in the telescoping form

$$S(\xi,x) - S(\eta,x) = \sum_{l=1}^{\infty} \Big(S(\xi^\ell, x) - S(\xi^{\ell-1}, x) \Big),$$

where $\xi^0 = \eta$.

For $\ell \in \mathbb{N}$ let us calculate $S(\xi^\ell, \cdot) - S(\xi^{\ell-1}, \cdot)$. Since $\overline{\xi}^\ell_{-k} = \overline{\xi}^{\ell-1}_{-k}$ for $k \le \tau_\ell - 1$, we have

$$\begin{aligned} S(\xi^\ell, x) - S(\xi^{\ell-1}, x) &= \sum_{n=1}^{\infty} \kappa^n \Big[\Phi'\big(B_2^n(\xi^\ell, x)\big) - \Phi'\big(B_2^n(\xi^{\ell-1}, x)\big) \Big] \\ &= \sum_{n=\tau_\ell+1}^{\infty} \kappa^n \Big[\Phi'\big(B_2^n(\xi^\ell, x)\big) - \Phi'\big(B_2^n(\xi^{\ell-1}, x)\big) \Big] \\ &= \kappa^{\tau_\ell} \sum_{m=1}^{\infty} \kappa^m \Big[\Phi'\Big(B_2^m\big(B_2^{\tau_\ell}(\xi^\ell, x)\big)\Big) - \Phi'\Big(B_2^m\big(B_2^{\tau_\ell}(\xi^{\ell-1}, x)\big)\Big) \Big]. \end{aligned}$$

Now

$$B_2^{\tau_\ell}(\xi^\ell, x) = B_2^{\tau_\ell}(\xi, x) = B_2^{\tau_\ell}(\xi^{\ell-1}, x).$$

And in case $\overline{\xi}_{-\tau_\ell} = 1$ we have

$$B_2^1\big(B_2^{\tau_\ell}(\xi^\ell, x)\big) = B_2^{\tau_\ell+1}(\xi^\ell, x) = \frac{1 + B_2^{\tau_\ell}(\xi^\ell, x)}{2} = \frac{1 + B_2^{\tau_\ell}(\xi, x)}{2},$$

while

$$B_2^1\big(B_2^{\tau_\ell}(\xi^{\ell-1}, x)\big) = B_2^{\tau_\ell+1}(\xi^{\ell-1}, x) = \frac{B_2^{\tau_\ell}(\xi, x)}{2}.$$

In case $\overline{\xi}_{-\tau_\ell} = 0$, we have in contrast

$$B_2^1\big(B_2^{\tau_\ell}(\xi^\ell, x)\big) = B_2^{\tau_\ell+1}(\xi^\ell, x) = \frac{B_2^{\tau_\ell}(\xi^\ell, x)}{2} = \frac{B_2^{\tau_\ell}(\xi, x)}{2},$$

while

$$B_2^1\big(B_2^{\tau_\ell}(\xi^{\ell-1}, x)\big) = B_2^{\tau_\ell+1}(\xi^{\ell-1}, x) = \frac{1 + B_2^{\tau_\ell}(\xi^\ell, x)}{2} = \frac{1 + B_2^{\tau_\ell}(\xi, x)}{2}.$$

So we may write by definition of g

$$\begin{aligned} &S(\xi^\ell, x) - S(\xi^{\ell-1}, x) \\ &\quad = (-1)^{(1-\bar{\xi}_{-\tau_\ell})} \kappa^{\tau_\ell} \sum_{m=1}^{\infty} \kappa^m \Big[\Phi'\Big(B_2^{m-1}\Big(\frac{1 + B_2^{\tau_\ell}(\xi, x)}{2}\Big)\Big) - \Phi'\Big(B_2^{m-1}\Big(\frac{B_2^{\tau_\ell}(\xi, x)}{2}\Big)\Big) \Big] \\ &\quad = \kappa^{\tau_\ell+1} (-1)^{(1-\bar{\xi}_{-\tau_\ell})} g\big(B_2^{\tau_\ell}(\xi, x)\big). \end{aligned}$$

Hence we obtain the claimed representation

$$S(\xi, x) - S(\eta, x) = \sum_{\ell=1}^{\infty} \kappa^{\tau_\ell+1} (-1)^{(1-\overline{\xi}_{-\tau_\ell})} g\big(B_2^{\tau_\ell}(\xi, x)\big), \;\; \xi, \eta, x \in [0,1].$$

□

Let us calculate g. Here, as opposed to the trigonometric case in the stable manifold of Weierstrass curves (see [13]), the simplicity of ϕ implies the following surprising identity.

Lemma 3.5. *We have* $g(x) = 2, x \in [0,1]$.

Proof: Let us inspect the first term in the series decomposition of g. Here we have $B_2^0(0, \frac{1+x}{2}) = \frac{1+x}{2} \in [\frac{1}{2}, 1]$, while $B_2^0(0, \frac{x}{2}) = \frac{x}{2} \in [0, \frac{1}{2}]$. Hence the contribution of the first term is $1 + 1 = 2$. For $m \geq 1$ we have in contrast that both $B_2^m(0, \frac{1+x}{2}) = \frac{1+x}{2^{m+1}}$ and $B_2^m(0, \frac{x}{2}) = \frac{x}{2^{m+1}}$ belong to $[0, \frac{1}{2}]$. Hence the contribution of terms of order $m \geq 1$ vanishes. This implies the claimed identity.□

Lemma 3.5 gives the following simplification of the representation formula of Proposition 3.4.

Corollary 3.6. *Let* $\xi, \eta, x \in [0,1]$. *Then*

$$S(\xi, x) - S(\eta, x) = 2 \sum_{\ell=1}^{\infty} \kappa^{\tau_\ell+1} (-1)^{(1-\bar{\xi}_{-\tau_\ell})}. \tag{3.4}$$

Proof: This follows by combining Lemma 3.5 and Proposition 3.4. □

Let us finally extend this property to $S(\xi, .)$ for $\xi \in [0,1]$. First observe that for any $x \in [0,1]$ we have

$$S(0, x) = \sum_{n=1}^{\infty} \kappa^n \Phi'(\frac{x}{2^n}) = \sum_{n=1}^{\infty} \kappa^n = \frac{\kappa}{1-\kappa}. \tag{3.5}$$

Now denote by $\tau_n^0, n \in \mathbb{N}$, the sequence of stopping times described above for the particular case $\eta = 0$. Then we obtain

Corollary 3.7. *Let* $\xi, x \in [0,1]$. *Then*

$$S(\xi, x) = \frac{\kappa}{1-\kappa} + 2 \sum_{\ell=1}^{\infty} \kappa^{\tau_\ell^0+1}. \tag{3.6}$$

Proof: Combine Corollary 3.6 with (3.5), and note that by definition $\overline{\xi}_{-\tau_k^0} = 1$ for all $k \geq 1$. □

4. Smoothness of the SBR measure

In this section we shall address the smoothness of the SBR measure defined in Section 3. More precisely, we shall prove that the SBR measure is absolutely continuous. To do this, in Subsection 4.2 we shall derive a telescoping relationship linking the image of three dimensional Lebesgue measure by the map

$$([0,1]^3 \ni (x, \xi, \eta) \mapsto S(\xi, x) - S(\eta, x)$$

to its *macroscopic* restriction to the set $\{\frac{1}{2} < |\xi - \eta\}$. Smoothness of the SBR measure will be seen to be crucially linked to transversality of S, i.e. positivity properties of $S(\xi, \cdot) - S(\eta, \cdot)$

on the macroscopic set $\{\frac{1}{2} < |\xi - \eta|\}$. This will be discussed in the following Subsection 4.1. Finally, in Subsection 4.3 we shall use a Fourier analytic approach to establish smoothness of the SBR measure.

4.1. **Transversality of S.** As a consequence of Corollary 3.6 we shall now tackle the transversality property for the stable manifold map S. The property will turn out to be crucial for the smoothness of the SBR measure deduced subsequently. Here, we simply say that S is *transversal* if $S(\xi,\cdot)-S(\eta,\cdot)$, a constant function according to Corollary 3.6, is bounded away from 0 on the set $\{\frac{1}{2} < |\xi - \eta|\}$. We will design an interval I in $]\frac{1}{2}, 1[$ such that for $\kappa \in I$ the map S is transversal.

Proposition 4.1. *For $\kappa \in I =]\frac{1}{2}, \frac{1}{\sqrt{2}}]$ the map S is transversal on the set $\{\frac{1}{2} < |\xi - \eta|\}$.*

Proof: Assume w.l.o.g. (modulo changing the roles of ξ and η) that $\xi - \eta > \frac{1}{2}$. This implies that $\overline{\xi}_0 = 1, \overline{\eta}_0 = 0$, hence $\tau_1 = 0$, and also $\overline{\xi}_{-\tau_1} = 1, \overline{\xi}_{-\tau_2} = 1$. As a consequence, we have

$$\begin{aligned} S(\xi,\cdot) - S(\eta,\cdot) &= 2(\kappa + \kappa^{\tau_2+1} + \sum_{\ell=3}^{\infty} \kappa^{\tau_\ell+1} (-1)^{(1-\bar{\xi}_{-\tau_\ell})}) \\ &\geq 2\kappa[1 + \kappa^{\tau_2} - \kappa^{\tau_3} \frac{1}{1-\kappa}]. \end{aligned} \tag{4.1}$$

Now by definition $\tau_3 \geq \tau_2 + 1, \tau_2 \geq 1$, and $\kappa > \frac{1}{2}$. Hence we may estimate

$$\begin{aligned} (1+\kappa^{\tau_2})(1-\kappa) - \kappa^{\tau_3} &\geq 1 - \kappa + \kappa^{\tau_2} - \kappa^{\tau_2+1} - \kappa^{\tau_2+1} \\ &= 1 - \kappa + \kappa^{\tau_2}(1-2\kappa) \geq 1 - \kappa + \kappa(1-2\kappa) = 1 - 2\kappa^2. \end{aligned}$$

And $1 - 2\kappa^2 > 0$ iff $\kappa < \frac{1}{\sqrt{2}}$. Using this in (4.1) yields the claimed uniform positivity of $S(\xi,\cdot) - S(\eta,\cdot)$. □

4.2. **The relationship between $\rho, \hat{\rho}$ and Lebesgue measure.** In this section we exploit the scaling properties of S, more precisely its self affinity in order to express ρ in terms of $\hat{\rho}$. Recall that ρ is the image measure of the three-dimensional Lebesgue measure by the map

$$[0,1]^3 \ni (x,\xi,\eta) \mapsto S(\xi,x) - S(\eta,x),$$

namely, for any Borel set $A \subset \mathbb{R}$, we define

$$\rho(A) = \lambda^3\Big(\big\{(x,\xi,\eta) \in [0,1]^3 : S(\xi,x) - S(\eta,x) \in A\big\}\Big).$$

and $\hat{\rho}(\cdot) = \rho(\cdot \cap \{\frac{1}{2} < |\xi-\eta|\})$. By its very definition, $\hat{\rho}$ lives on the set of pairs $(\xi,\eta) \in [0,1]^2$ for which $\frac{1}{2} < |\xi - \eta|$. On this set, transversality of S will allow a comparison of $\hat{\rho}$ with the Lebesgue measure. This will finally lead to conclusions about the regularity of the SBR measure. In the following formula, ρ is shown to be a weighted average of expansions measured by $\hat{\rho}$.

Proposition 4.2. *For Borel sets A on the real line, we have*

$$\rho(A) = \sum_{n=0}^{\infty} 2^{-n} \hat{\rho}(\kappa^{-n} A).$$

Proof: We have using Lemma 2.2

$$\begin{aligned}\rho(A) &= \sum_{n=0}^{\infty}\lambda^3\Big(\Big\{(\xi,\eta,x): S(\xi,x)-S(\eta,x)\in A, 2^{-(n+1)}<|\xi-\eta|\le 2^{-n}\Big\}\Big)\\ &= \sum_{n=0}^{\infty}\lambda^3\Big(\Big\{(\xi,\eta,x): S(\xi,x)-S(\eta,x)\in A, \frac{1}{2}<|B_1^n(\xi,x)-B_1^n(\eta,x)|\le 1\Big\}\Big)\\ &= \sum_{n=0}^{\infty}\lim_{\epsilon\to 0}\frac{1}{\lambda^4(D_\epsilon)}\lambda^4\Big(\Big\{(\xi,\eta,x,y): S(\xi,x)-S(\eta,y)\in A,\\ &\qquad\qquad \frac{1}{2}<|B_1^n(\xi,x)-B_1^n(\eta,y)|\le 1, |x-y|\le\epsilon\Big\}\Big),\end{aligned}$$

where for $\epsilon>0$ we let $D_\epsilon=\{(\xi,\eta,x,y):|x-y|\le\epsilon\}$. We next use the invariance of B (see (2.2)) to estimate term n of the preceding series. Then for $n\ge 0$

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{1}{\lambda^4(D_\epsilon)}\lambda^4\Big(\Big\{(\xi,\eta,x,y): S(\xi,x)-S(\eta,y)\in A,\\ &\qquad\qquad \frac{1}{2}<|B_1^n(\xi,x)-B_1^n(\eta,y)|\le 1, |x-y|\le\epsilon\Big\}\Big)\\ &=\lim_{\epsilon\to 0}\frac{1}{\lambda^4(D_\epsilon)}\lambda^4\Big(\Big\{(\xi,\eta,x,y): S(B^{-n}(\xi,x))-S(B^{-n}(\eta,y))\in A,\\ &\qquad\qquad \frac{1}{2}<|\xi-\eta|\le 1, |B_2^{-n}(\xi,x)-B_2^{-n}(\eta,y)|\le\epsilon\Big\}\Big)\\ &=\lim_{\epsilon\to 0}\frac{\lambda^4(D_{2^{-n}\epsilon})}{\lambda^4(D_\epsilon)}\frac{1}{\lambda^4(D_{2^{-n}\epsilon})}\lambda^4\Big(\Big\{(\xi,\eta,x,y): S(B^{-n}(\xi,x))-S(B^{-n}(\eta,y))\in A,\\ &\qquad\qquad \frac{1}{2}<|\xi-\eta|\le 1, |x-y|\le 2^{-n}\epsilon\Big\}\Big)\\ &=2^{-n}\lambda^3\Big(\Big\{(\xi,\eta,x): S(B^{-n}(\xi,x))-S(B^{-n}(\eta,x))\in A, \frac{1}{2}<|\xi-\eta|\le 1\Big\}\Big).\end{aligned}$$

Now we apply Lemma 3.3 to transform term n in the preceding chain of equations into

$$\lambda^3\Big(\Big\{(\xi,\eta,x):\kappa^n(S(\xi,x)-S(\eta,x))\in A, \frac{1}{2}<|\xi-\eta|\le 1\Big\}\Big)=\hat\rho\big(\kappa^{-n}A\big).$$

This implies the claimed equation. $\square$

4.3. **The smoothness of the SBR measure.** In this section we will finally draw our conclusions from the preceding two sections. In fact, we will derive a sufficient criterion for the absolute continuity of the SBR measure from Proposition 4.1. We note here that absolute continuity of the SBR measure, at least for the classical Weierstrass function, can be established for the entire range $[\frac{1}{2},1[$ for κ, as is shown in [18]. We stick to the simpler positivity criterion just for allowing an explicitly dual form for a criterion for absolute continuity of the occupation measure in section 5. For establishing absolute continuity, we consider the Fourier transforms of the marginals μ_x, $x\in[0,1]$, of the SBR measure μ defined in (3.1). Let

$$\phi_x(u)=\int_{\mathbb{R}}\exp(iuy)\mu_x(\mathrm{d}y),\quad u\in\mathbb{R}.$$

By definition of μ and the integral transform theorem we have

$$\phi_x(u)=\int_0^1\exp\big(iuS(\xi,x)\big)\mathrm{d}\xi,\quad u\in\mathbb{R}, x\in[0,1].$$

To prove the absolute continuity of μ_x we have to prove that ϕ_x is square integrable on $\mathbb{R}$. Therefore, to prove that μ is absolutely continuous, it will be sufficient to show

$$\begin{aligned}\int_0^1 \int_{\mathbb{R}} |\phi_x(u)|^2 \mathrm{d}u\mathrm{d}x &= \int_{\mathbb{R}} \int_{[0,1]^3} \exp\Big(iu\big(S(\xi,x) - S(\eta,x)\big)\Big)\mathrm{d}x\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}u \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \exp(iux)\rho(\mathrm{d}x)\mathrm{d}u < \infty.\end{aligned}$$

Theorem 4.3. *Let $\kappa \le \frac{1}{\sqrt{2}}$. Then for almost every $x \in [0,1]$ the function*

$$\xi \mapsto S(\xi, x)$$

has an absolutely continuous law with respect to the Lebesgue measure with square integrable density. In particular, the SBR measure (3.1) *is absolutely continuous with respect to the Lebesgue measure with square integrable density.*

Proof: By Proposition 4.2, the integral transformation formula and noting that $\frac{1}{2\kappa} = \gamma$, we may write

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp(iux)\rho(\mathrm{d}x)\mathrm{d}u &= \sum_{n=0}^{\infty} 2^{-n} \int_{\mathbb{R}}\int_{\mathbb{R}} \exp(iuy)\hat{\rho}(\kappa^{-n}\mathrm{d}y)\mathrm{d}u \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sum_{n=0}^{\infty} 2^{-n}\exp(iu\kappa^n y)\hat{\rho}(\mathrm{d}y)\mathrm{d}u \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sum_{n=0}^{\infty} \gamma^n \exp(iuy)\hat{\rho}(\mathrm{d}y)\mathrm{d}u \\ &= \frac{1}{1-\gamma}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp(iuy)\hat{\rho}(\mathrm{d}y)\mathrm{d}u.\end{aligned}$$

We have to show that

$$\limsup_{K\to\infty} \int_{-K}^{K}\int_{\mathbb{R}} \exp(iuy)\mathrm{d}u\hat{\rho}(\mathrm{d}y) < \infty.$$

Recall that $\hat{\rho}$ is antisymmetric with respect to reflection at the origin and has compact support $[-L, L]$. From Proposition 4.1 and Corollary 3.6 we further know that there is $k > 0$ such that $\hat{\rho}([-k, k]) = 0$. Hence we have

$$\begin{aligned}\int_{-K}^{K}\int_{\mathbb{R}} \exp(iuy)\mathrm{d}u\hat{\rho}(\mathrm{d}y) &= \int_{-K}^{K}\int_{-L}^{L} \exp(iuy)\hat{\rho}(\mathrm{d}y)\mathrm{d}u \\ &= 2\int_{-L}^{L}\int_{0}^{K} \cos(uy)\mathrm{d}u\hat{\rho}(\mathrm{d}y) \\ &= 2\int_{-L}^{L} \frac{\sin(Ky)}{y}\hat{\rho}(\mathrm{d}y) \\ &= 4\int_{k}^{L} \frac{\sin(Ky)}{y}\hat{\rho}(\mathrm{d}y). \qquad (4.2)\end{aligned}$$

But the integrand in the last line of (4.2) is uniformly bounded in K. This implies the claimed absolute continuity.

□

5. The existence of a local time for $\mathcal{T}$

In this section we use a similar criterion as in the preceding one to show that the occupation measure associated with $\mathcal{T}$ possesses a square integrable density. This will be done in an indirect way. We first establish an intrinsic link between the Takagi curve as the attractor of an underlying dynamical system and its stable manifold spanned by S. It will identify $H(\xi, x) = \mathcal{T}(x) - \int_0^x S(\xi, z)\mathrm{d}z = H(\xi, x) - xS(\xi, 0)$ for $x, \xi \in [0, 1]$ as a function having much in common with the function G of the preceding sections. To relate microscopic properties of H with macroscopic ones, we will deduce scaling relationships for H with respect to both its arguments. We will define measures χ and $\hat{\chi}$ related to increments of H in the second variable. This will be done in an analogous way as ρ and $\hat{\rho}$ in the preceding section. It will be crucial again to investigate the relationship between the macroscopic measures and the Lebesgue measure. This will be done by an argument as in subsection 4.2.

In the following key lemma we establish the link between $\mathcal{T}$ and the stable manifold of F. For this purpose, we define

$$H(\xi, x) = \sum_{n\in\mathbb{Z}} \gamma^n \left[\Phi\big(B_2^{-n}(\xi, x)\big) - \Phi\big(B_2^{-n}(\xi, 0)\big)\right], \quad \xi, x \in [0, 1]. \tag{5.1}$$

Then we have the following relationships between H and S.

Lemma 5.1. *For $x, y, \xi \in [0, 1]$ we have*

$$H(\xi, y) - H(\xi, x) = \mathcal{T}(y) - \mathcal{T}(x) - \int_x^y S(\xi, z)\mathrm{d}z.$$

For $x, \xi, \eta \in [0, 1]$ we have

$$H(\eta, x) - H(\xi, x) = \int_0^x \big(S(\xi, z) - S(\eta, z)\big)\mathrm{d}z = x(S(\xi, 0) - S(\eta, 0)).$$

Proof: For $x, y, \xi \in [0, 1]$ we have indeed (recall $\kappa = \frac{1}{2\gamma}$)

$$\begin{aligned}
H(\xi, y) - H(\xi, x) &= \sum_{n\in\mathbb{Z}} \gamma^n \left[\Phi\big(B_2^{-n}(\xi, y)\big) - \Phi\big(B_2^{-n}(\xi, x)\big)\right] \\
&= \sum_{n=0}^{\infty} \gamma^n \left[\Phi\big(B_2^{-n}(\xi, y)\big) - \Phi\big(B_2^{-n}(\xi, x)\big)\right] \\
&\qquad + \sum_{k=1}^{\infty} \gamma^{-k} \left[\Phi\big(B_2^{k}(\xi, y)\big) - \Phi\big(B_2^{k}(\xi, x)\big)\right] \\
&= \mathcal{T}(y) - \mathcal{T}(x) - \int_x^y \sum_{k=1}^{\infty} (2\gamma)^{-k} \Phi'\big(B_2^k(\xi, z)\big)\mathrm{d}z \\
&= \mathcal{T}(y) - \mathcal{T}(x) - \int_x^y \sum_{k=1}^{\infty} \kappa^k \Phi'\big(B_2^k(\xi, z)\big)\mathrm{d}z \\
&= \mathcal{T}(y) - \mathcal{T}(x) - \int_x^y S(\xi, z)\mathrm{d}z.
\end{aligned}$$

To argue for the second equation, note that for $x, \xi, \eta \in [0,1]$ we have

$$\begin{aligned} H(\eta,x) - H(\xi,x) &= \sum_{n\in\mathbb{Z}} \gamma^n \Big\{ \big[\Phi\big(B_2^{-n}(\eta,x)\big) - \Phi\big(B_2^{-n}(\eta,0)\big)\big] \\ &\qquad\qquad - \big[\Phi\big(B_2^{-n}(\xi,x)\big) - \Phi\big(B_2^{-n}(\xi,0)\big)\big] \Big\} \\ &= \sum_{k=1}^{\infty} \gamma^{-k} \Big\{ \big[\Phi\big(B_2^{k}(\eta,x)\big) - \Phi\big(B_2^{k}(\eta,0)\big)\big] \\ &\qquad\qquad - \big[\Phi\big(B_2^{k}(\xi,x)\big) - \Phi\big(B_2^{k}(\xi,0)\big)\big] \Big\} \\ &= -\int_0^x \sum_{k=1}^{\infty} (2\gamma)^{-k} \big[\Phi'\big(B_2^k(\eta,z)\big) - \Phi'\big(B_2^k(\xi,z)\big)\big] \mathrm{d}z \\ &= -\int_0^x \sum_{k=1}^{\infty} \kappa^{k} \big[\Phi'\big(B_2^k(\eta,z)\big) - \Phi'\big(B_2^k(\xi,z)\big)\big] \mathrm{d}z \\ &= \int_0^x [S(\xi,z) - S(\eta,z)] \mathrm{d}z. \end{aligned}$$

This completes the proof of the second equation. □

We next address the scaling properties of H.

Lemma 5.2. *For $\xi, x \in [0,1]$ we have*

$$H\big(B(\xi,x)\big) = \gamma H(\xi,x) + \frac{\overline{\xi}_0}{2} S(2\xi,0).$$

Consequently, for $\xi, x, y \in [0,1]$

$$H(B(\xi,y)) - H(B(\xi,x)) = \gamma[H(\xi,y) - H(\xi,x)].$$

Proof: Let $\xi, x \in [0,1]$. By definition and setting $n-1 = k$ we obtain, defining $\hat{\xi}$ to be represented by the dyadic sequence $(0, \overline{\xi}_{-1}, \overline{\xi}_{-2}, \cdots)$,

$$\begin{aligned} H(B(\xi,x)) &= \sum_{n\in\mathbb{Z}} \gamma^n \big[\Phi\big(B^{-n+1}(\xi,x)\big) - \Phi\big(B^{-n}(B_1(\xi,x),0)\big)\big] \\ &= \sum_{n\in\mathbb{Z}} \gamma^n \big[\Phi\big(B^{-n+1}(\xi,x)\big) - \Phi\big(B^{-n+1}(\xi,0)\big)\big] \\ &+ \sum_{n\in\mathbb{Z}} \gamma^n \big[\Phi\big(B^{-n+1}(\xi,0)\big) - \Phi\big(B^{-n}(B_1(\xi,x),0)\big)\big] \\ &= \gamma \sum_{k\in\mathbb{Z}} \gamma^k \big[\Phi\big(B^{-k}(\xi,x)\big) - \Phi\big(B^{-k}(\xi,0)\big)\big] \\ &+ \sum_{k=1}^{\infty} \gamma^{-k} [\Phi\big(B_2^{k+1}(\xi,0)\big) - \Phi\big(B_2^{k+1}(\hat{\xi},0)\big)\big] \\ &= \gamma H(\xi,x) + \overline{\xi}_0 \sum_{k=1}^{\infty} \gamma^{-k} 2^{-k-1} \Phi'(B_2^{k+1}(\hat{\xi},0)) \\ &= \gamma H(\xi,x) + \frac{\overline{\xi}_0}{2} \sum_{k=1}^{\infty} \kappa^k \Phi'(B_2^k(2\xi,0)) \\ &= \gamma H(\xi,x) + \frac{\overline{\xi}_0}{2} S(2\xi,0). \end{aligned}$$

The last equation follows from the fact that the second term in the first formula only depends on ξ. $\square$

We finally give a representation of increments of H that can be considered dual to the representation of Proposition 3.4. It will serve as an entrance to investigating the smoothness of the occupation measure, i. e. the existence of local time. To this end, fix $x, y \in [0,1]$. We recursively define the following sequence of times of disagreement of dyadic components of x and y. For $n \in \mathbb{N}$ let

$$\sigma_1 = \inf\{\ell \geq 1 : \bar{x}_\ell \neq \bar{y}_\ell\}, \text{ and } \sigma_{n+1} = \inf\{\ell > \sigma_n : \bar{x}_\ell \neq \bar{y}_\ell\}. \tag{5.2}$$

For $x, y \in [0,1]$ denote by $x \wedge y$ the number in $[0,1]$ with dyadic representation sequence $\overline{x}_n \wedge \overline{y}_n, n \in \mathbb{N}$.

We have the following result.

Proposition 5.3. *Let $\xi, x, y \in [0,1]$. Then*

$$H(\xi, y) - H(\xi, x) = \sum_{\ell=1}^{\infty} \gamma^{\sigma_\ell} (-1)^{(1-\bar{y}_{\sigma_\ell})} S\big(B_1^{-\sigma_\ell}(\xi, x \wedge y), 0)\big). \tag{5.3}$$

Proof: It follows from the definition of $\sigma_n, n \in \mathbb{N}$, that y can be written

$$y = (\overline{x}_1, \ldots, \overline{x}_{\sigma_1 - 1}, \overline{y}_{\sigma_1}, \overline{x}_{\sigma_1+1}, \cdots, \overline{x}_{\sigma_2-1}, \overline{y}_{\sigma_2}, \overline{x}_{\sigma_2+1}, \cdots\cdots, \overline{x}_{\sigma_n-1}, \overline{y}_{\sigma_n}, \overline{x}_{\sigma_n+1}, \cdots).$$

For $n \in \mathbb{N}$ let y^n be the sequence which up to σ_n represents the dyadic expansion of y, and then switches to the representing sequence of x. Then for $n \in \mathbb{N}$ we have

$$y^n = (\overline{x}_1, \ldots, \overline{x}_{\sigma_1 - 1}, \overline{y}_{\sigma_1}, \overline{x}_{\sigma_1+1}, \cdots, \overline{x}_{\sigma_2-1}, \overline{y}_{\sigma_2}, \overline{x}_{\sigma_2+1}, \cdots\cdots, \overline{x}_{\sigma_n-1}, \overline{y}_{\sigma_n}, \overline{x}_{\sigma_n+1}, \cdots, \overline{x}_m, \cdots).$$

Note that $\lim_{n\to\infty} y^n = y$. Hence the left hand side of (5.3) has the telescoping representation

$$H(\xi, y) - H(\xi, x) = \sum_{l=1}^{\infty} \Big(H(\xi, y^\ell) - H(\xi, y^{\ell-1})\Big),$$

where $y^0 = x$.

For $\ell \in \mathbb{N}$ let us calculate $H(\cdot, y^\ell) - H(\cdot, y^{\ell-1})$. Since $B_2^{-n}(\xi, y^\ell) = B_2^{-n}(\xi, y^{\ell-1})$ for $n \geq \sigma_\ell$ we have

$$\begin{aligned} H(\xi, y^\ell) - H(\xi, y^{\ell-1}) =& \sum_{n \in \mathbb{Z}} \gamma^n \Big[\Phi\big(B_2^{-n}(\xi, y^\ell)\big) - \Phi\big(B_2^{-n}(\xi, y^{\ell-1})\big)\Big] \\ =& \sum_{n < \sigma_\ell} \gamma^n \Big[\Phi\big(B_2^{-n}(\xi, y^\ell)\big) - \Phi\big(B_2^{-n}(\xi, y^{\ell-1})\big)\Big] \\ =& \gamma^{\sigma_\ell} \sum_{m=1}^{\infty} \gamma^{-m} \Big[\Phi\Big(B_2^m\big(B^{-\sigma_\ell}(\xi, y^\ell)\big)\Big) - \Phi\Big(B_2^m\big(B_2^{-\sigma_\ell}(\xi, y^{\ell-1})\big)\Big)\Big]. \end{aligned}$$

Now for $m \leq \sigma_\ell$, denoting $\hat{x}_l = 2^{\sigma_\ell} x (\mathrm{mod} 1)$, we have

$$B_2^m(B_2^{-\sigma_\ell}(\xi, y^\ell)) = \frac{\hat{x}_l}{2^m} + \frac{\overline{y}_{\sigma_\ell}}{2^m} + \cdots + \frac{\overline{y}_{\sigma_\ell - (m-1)}}{2},$$

$$B_2^m(B_2^{-\sigma_\ell}(\xi, y^{\ell-1})) = \frac{\hat{x}_l}{2^m} + \frac{\overline{x}_{\sigma_\ell}}{2^m} + \cdots + \frac{\overline{y}_{\sigma_\ell - (m-1)}}{2},$$

and similar formulas for $m > \sigma_\ell$ where in the last summands components of the dyadic expansion of ξ enter. Hence

$$\Phi\big(B_2^m(B^{-\sigma_\ell}(\xi, y^\ell))\big) - \Phi\big(B_2^m(B^{-\sigma_\ell}(\xi, y^{\ell-1}))\big) = (-1)^{(1-\overline{y}_{\sigma_\ell})}\, 2^{-m} \Phi'\big(B_2^m(B^{-\sigma_\ell}(\xi, x \wedge y))\big),$$

and therefore, by $\kappa = \frac{1}{2\gamma}$, and since S is known to be constant in x,

$$H(\xi, y^\ell) - H(\xi, y^{\ell-1}) = (-1)^{(1-\overline{y}_{\sigma_\ell})} \sum_{m=1}^\infty \kappa^m \Phi'\big(B_2^m(B^{-\sigma_\ell}(\xi, x\wedge y))\big) = S(B_1^{-\sigma_\ell}(\xi, x\wedge y), 0).$$

This again implies the formula

$$H(\xi, y) - H(\xi, x) = \sum_{\ell=1}^\infty (-1)^{(1-\overline{y}_{\sigma_\ell})}\, \gamma^{\sigma_\ell}\, S(B_1^{-\sigma_\ell}(\xi, x\wedge y), 0),$$

which is just the claimed one. □

5.1. **Positivity properties of increments of H.** In this subsection we shall deduce positivity properties of the increments $H(\xi, y) - H(\xi, x)$ for $\xi \in [0,1]$, and x, y belonging to the macroscopic set $\{y > x + \frac{1}{2}\}$. This will be crucial for obtaining the existence of a local time for H, in a dual way to proving the smoothness of the SBR measure in section 4. Thereby, the dual representation formula of Proposition 5.3 will replace the formula of Proposition 3.4. Let us first use the representation formulas for S from section 3 to describe increments of H in more detail. For this purpose, for $x, y \in [0,1]$ with dyadic representations $(\overline{x}_n)_{n\in\mathbb{N}}, (\overline{y}_n)_{n\in\mathbb{N}}$ define

$$\alpha_1 = \inf\{k \geq 1 : x_k \wedge y_k \neq 0\}, \quad \alpha_{n+1} = \inf\{k > \alpha_n : \overline{x}_k \wedge \overline{y}_k \neq 0\},\ n \in \mathbb{N},$$

and for $\ell \in \mathbb{N}$ let

$$R_\ell = \sup\{k \in \mathbb{N} : \alpha_k \leq \sigma_\ell\}.$$

Corollary 5.4. *Let $\xi, x, y \in [0,1]$. Then we have*

$$\begin{aligned} H(\xi, y) - H(\xi, x) &= \sum_{\ell=1}^\infty \gamma^{\sigma_\ell} (-1)^{(1-\overline{y}_{\sigma_\ell})} [\frac{\kappa}{1-\kappa} + 2\kappa^{\sigma_\ell} \sum_{p=1}^{R_\ell} \kappa^{-\alpha_p}] \\ &+ \sum_{\ell=1}^\infty (\frac{1}{2})^{\sigma_\ell} (-1)^{(1-\overline{y}_{\sigma_\ell})} [S(\xi, 0) - S(0,0)] \\ &= \sum_{\ell=1}^\infty \gamma^{\sigma_\ell} (-1)^{(1-\overline{y}_{\sigma_\ell})} [\frac{\kappa}{1-\kappa} + 2\kappa^{\sigma_\ell} \sum_{p=1}^{R_\ell} \kappa^{-\alpha_p}] + (y-x)[S(\xi,0) - S(0,0)]. \end{aligned} \tag{5.4}$$

Proof: According to Proposition 5.3 we have to calculate $S(B_1^{-\sigma_\ell}(\xi, x\wedge y), 0)$ for $\xi, x, y \in [0,1], \ell \in \mathbb{N}$ using the formulas provided by Corollaries 3.6, 3.7. To exploit them, we first note that the dyadic sequence associated with $B^{-\sigma_\ell}(\xi, x\wedge y)$ is given by $(\overline{z}_{\sigma_\ell}, \overline{z}_{\sigma_{\ell-1}}, \cdots, \overline{z}_2, \overline{z}_1, \overline{\xi}_0, \overline{\xi}_{-1}, \cdots)$, where $\overline{z}_k = \overline{x}_k \wedge \overline{y}_k, k \in \mathbb{N}$. We therefore can write

$$\begin{aligned} S(B_1^{-\sigma_\ell}(\xi, x\wedge y), 0) &= \frac{\kappa}{1-\kappa} + 2\kappa^{\sigma_\ell} \sum_{p=1}^{R_\ell} \kappa^{-\alpha_p} + 2\sum_{m=1}^\infty \kappa^{\sigma_\ell + \tau_m^0 + 1} \\ &= \frac{\kappa}{1-\kappa} + 2\kappa^{\sigma_\ell} \sum_{p=1}^{R_\ell} \kappa^{-\alpha_p} + \kappa^{\sigma_\ell}(S(\xi, 0) - S(0,0)). \end{aligned}$$

Now insert this formula into the representation of Proposition 5.3, and observe that $\gamma\kappa = \frac{1}{2}$ to obtain the first equation. For the second, observe that the second term on the right hand side of the first equation simply equals $(y-x)[S(\xi,0) - S(0,0)]$, by the definition of $\sigma_k, k \in \mathbb{N}$. □

Our main goal in this subsection is to show positivity of the right hand side of (5.4) on the macroscopic set $\{y > x + \frac{1}{2}\}$. By Corollary 3.7, on this set the second term is clearly positive, provided $\xi \neq 0$. We will therefore be occupied with the first term. We first note how the condition $y > x + \frac{1}{2}$ translates into the dyadic representations of x and y. In fact, it just says that

$$\sigma_1 = 1, \quad \overline{x}_1 = 0, \overline{y}_1 = 1, \quad , \sigma_2 \geq 2, \quad \overline{y}_{\sigma_2} = 1, \overline{x}_{\sigma_2} = 0. \tag{5.5}$$

Note also that for $\overline{z}_k = \overline{x}_k \wedge \overline{y}_k, k \in \mathbb{N}$, since $\sigma_k, k \in \mathbb{N}$, mark the components m for which $\overline{x}_m \neq \overline{y}_m$, we have

$$\overline{z}_{\sigma_k} = 0, \;\; k \in \mathbb{N}, \tag{5.6}$$

while $\alpha_p, p \in \mathbb{N}$, just mark the components m for which $\overline{x}_m = \overline{y}_m = 1$. As a consequence of (5.5), the first two summands of the first series on the right hand side of (5.4) are positive. It further follows from (5.5) and (5.6) that

$$\alpha_1 \geq 2, \quad \alpha_{R_\ell} \leq \sigma_\ell - 1. \tag{5.7}$$

Lemma 5.5. *Let $\kappa \in [\frac{1}{2}, 1[, \ell, N \in \mathbb{N}$, with $\ell > N$, and assume that $\sigma_1 = 1, \sigma_2 = 2, \cdots, \sigma_N = N$. Then*

$$\frac{\kappa}{1-\kappa} + 2\kappa^{\sigma_\ell} \sum_{p=1}^{R_\ell} \kappa^{-\alpha_p} \leq \frac{\kappa}{1-\kappa} \frac{3 + \kappa - 2\kappa^{2([\frac{\sigma_\ell - \sigma_N - 2}{2}]+1)}}{1+\kappa}.$$

Proof: Define

$$Q_\ell = \inf\{k \in \mathbb{N} : \alpha_k > \sigma_\ell\}, \quad \ell \in \mathbb{N}.$$

With this notation, we may write

$$\begin{aligned}
\kappa^{\sigma_\ell} \sum_{p=1}^{R_\ell} \kappa^{-\alpha_p} &\leq \kappa^{\sigma_\ell} \sum_{k=\sigma_N+1}^{\ell} \sum_{m=\alpha_{Q_{k-1}}}^{\alpha_{R_k}} \kappa^{-m} 1_{\{\sigma_{k-1} < Q_{k-1} \leq R_k < \sigma_k\}} \\
&\leq \kappa^{\sigma_\ell} \sum_{k=\sigma_N+1}^{\ell} \sum_{m=\sigma_{k-1}+1}^{\sigma_k - 1} \kappa^{-m} 1_{\{\sigma_k - \sigma_{k-1} \geq 2\}} \\
&= \frac{\kappa}{1-\kappa} \kappa^{\sigma_\ell} \sum_{k=\Sigma_N+1}^{\ell} \kappa^{-\sigma_k} (1 - \kappa^{\sigma_k - \sigma_{k-1} - 1}) 1_{\{\sigma_k - \sigma_{k-1} \geq 2\}} \\
&= \frac{\kappa}{1-\kappa} \kappa^{\sigma_\ell} \sum_{k=\sigma_N+1}^{\ell} [\kappa^{-\sigma_k} - \kappa^{-\sigma_{k-1}-1}] 1_{\{\sigma_k - \sigma_{k-1} \geq 2\}} \\
&\leq \frac{\kappa}{1-\kappa} \kappa^{\sigma_\ell} \sum_{k=\sigma_N+1}^{\ell} \kappa^{-\sigma_{k-1}-1} [\frac{1}{\kappa} - 1] 1_{\{\sigma_k - \sigma_{k-1} \geq 2\}} \\
&= \kappa^{\sigma_\ell} \sum_{k=\sigma_N+1}^{\ell} \kappa^{-\sigma_{k-1}-1} 1_{\{\sigma_k - \sigma_{k-1} \geq 2\}} \\
&= \sum_{p=1}^{\ell-1} \kappa^{\sigma_\ell - \sigma_p - 1} 1_{\{\sigma_{p+1} - \sigma_p \geq 2\}} \\
&\leq \frac{\kappa}{1-\kappa^2} (1 - \kappa^{2([\frac{\sigma_\ell - \sigma_N - 2}{2}]+1)}).
\end{aligned}$$

Adding $\frac{\kappa}{1-\kappa}$ to the doubled estimate just obtained evidently provides the claimed estimate. □

Lemma 5.5 allows the following estimate for the remainder of the first term in (5.4).

Lemma 5.6. *Let $\kappa \in [\frac{1}{2}, 1[, \ell, N \in \mathbb{N}$, with $\ell > N$, and assume that $\sigma_1 = 1, \sigma_2 = 2, \cdots, \sigma_N = N$. Then*

$$\begin{aligned}
&|\sum_{k=\ell}^{\infty} \gamma^{\sigma_k} (-1)^{(1-\bar{y}_{\sigma_k}}[\frac{\kappa}{1-\kappa} + 2\kappa^{\sigma_\ell} \sum_{p=1}^{R_\ell} \kappa^{-\alpha_p}]| \\
&\quad \leq \frac{\kappa}{1-\kappa} \sum_{k=\sigma_\ell}^{\infty} \gamma^{\sigma_\ell} \frac{3+\kappa - 2\kappa^{2([\frac{\sigma_\ell - \sigma_N - 2}{2}]+1)}}{1+\kappa} \\
&\quad \leq \frac{\gamma^{\sigma_\ell}}{1-\gamma} \frac{\kappa}{1-\kappa} \frac{3+\kappa}{1+\kappa}.
\end{aligned}$$

Proof: Combine Lemma 5.5 with an obvious estimate for geometric series. □

We next provide a range of roughness parameters for which the first term on the right hand side of (5.4) is positive.

Lemma 5.7. *There exists $\gamma_0 \geq 0.668 > \frac{2}{3}$ such that for $\gamma \in I =]\frac{1}{\sqrt{2}}, \gamma_0]$ the expression J defined by*

$$\begin{aligned}
J =& \sum_{\ell=1}^{\infty} \gamma^{\sigma_\ell} (-1)^{(1-\bar{y}_{\sigma_\ell})} \Big[\frac{\kappa}{1-\kappa} + 2\kappa^{\sigma_\ell} \sum_{p=1}^{R_\ell} \kappa^{-\alpha_p}\Big] \\
=& \sum_{\ell=1}^{\infty} \gamma^{\sigma_\ell} (-1)^{(1-\bar{y}_{\sigma_\ell})} \Big[\frac{1}{2\gamma-1} + 2\sum_{p=1}^{R_\ell} (\frac{1}{2\gamma})^{\sigma_\ell - \alpha_p}\Big]
\end{aligned}$$

satisfies $J > 0$.

Proof: Using Lemma 5.6, we can write

$$\begin{aligned}
J \;=\;& \sum_{\ell=1}^{\infty} \gamma^{\sigma_\ell} (-1)^{(1-\bar{y}_{\sigma_\ell})} \Big[\frac{1}{2\gamma-1} + 2\sum_{p=1}^{R_\ell} (\frac{1}{2\gamma})^{\sigma_\ell - \alpha_p}\Big] \\
=\;& \gamma \frac{1}{2\gamma-1} + \gamma^{\sigma_2} \Big[\frac{1}{2\gamma-1} + 2\sum_{p=1}^{R_2} (\frac{1}{2\gamma})^{\sigma_2 - \alpha_p}\Big] \\
+\;& (-1)^{(1-\bar{y}_{\sigma_3})} \gamma^{\sigma_3} \Big[\frac{1}{2\gamma-1} + 2\sum_{p=1}^{R_3} (\frac{1}{2\gamma})^{\sigma_3 - \alpha_p}\Big] + (-1)^{(1-\bar{y}_{\sigma_4})} \gamma^{\sigma_4} \Big[\frac{1}{2\gamma-1} + 2\sum_{p=1}^{R_2} (\frac{1}{2\gamma})^{\sigma_4 - \alpha_p}\Big] \\
+\;& \sum_{\ell=5}^{\infty} \gamma^{\sigma_\ell} (-1)^{(1-\bar{y}_{\sigma_\ell})} \Big[\frac{1}{2\gamma-1} + 2\sum_{p=1}^{R_\ell} (\frac{1}{2\gamma})^{\sigma_\ell - \alpha_p}\Big] \\
\geq\;& \gamma \frac{1}{2\gamma-1} + \gamma^{\sigma_2} \Big(\frac{1}{2\gamma-1} + 2\sum_{p=1}^{R_2} (\frac{1}{2\gamma})^{\sigma_2 - \alpha_p}\Big) - \gamma^{\sigma_3} \Big(\frac{1}{2\gamma-1} + 2\sum_{p=1}^{R_2} (\frac{1}{2\gamma})^{\sigma_3 - \alpha_p}\Big) \\
&- \frac{\gamma^{\sigma_\ell}}{(2\gamma-1)(1-\gamma)} \frac{6\gamma+1}{2\gamma+1}.
\end{aligned} \tag{5.8}$$

We now distinguish four cases.

(1) Case 1: $\sigma_2 = 5$ and $\sigma_3 \geq 6$. Then (5.8) becomes

$$J \geq \frac{1}{2\gamma-1} \Big(\gamma + \gamma^5 - \frac{\gamma^6}{(1-\gamma)} \frac{6\gamma+1}{2\gamma+1}\Big) > 0$$

for $\gamma < 0.702$.

(2) Case 2: $\sigma_2 = 4$ and $\sigma_3 \geq 5$. Similarly, using the first three terms of the series we get

$$J \geq \frac{1}{2\gamma - 1}\Big(\gamma + \gamma^4 - \frac{\gamma^5}{(1-\gamma)}\frac{6\gamma+1}{2\gamma+1}\Big) > 0$$

for $\gamma < 0.668$.

(3) Case 3: $\sigma_2 = 3$. We distinguish two subcases. First we use the first two terms of the series, then the first four terms of the series.
- Suppose $\sigma_3 \geq 5$. We then have

$$J \geq \frac{1}{2\gamma - 1}\Big(\gamma + \gamma^3 - \frac{\gamma^5}{(1-\gamma)}\frac{6\gamma+1}{2\gamma+1}\Big) > 0$$

for $\gamma < 0.681$.
- Suppose $\sigma_3 = 4, \sigma_4 = 5, \sigma_5 \geq 6$. Thus using the first four terms of the series, we obtain

$$J \geq \frac{1}{2\gamma - 1}\Big(\gamma + \gamma^3 - \gamma^4 - \gamma^5 - \frac{\gamma^6}{(1-\gamma)}\frac{6\gamma+1}{2\gamma+1}\Big) > 0$$

for $\gamma < 0.675$.

(4) Case 4: $\sigma_2 = 2$. In this case we consider subcases depending on the values of $\sigma_3, \sigma_4, \ldots, \sigma_7$. We distinguish seven of them.
- Suppose $\sigma_3 \geq 5$. Then

$$J \geq \frac{1}{2\gamma - 1}\Big(\gamma + \gamma^2 - \frac{\gamma^5}{(1-\gamma)}\frac{6\gamma+1}{2\gamma+1}\Big) > 0$$

for $\gamma < 0.697$.
- Suppose $\sigma_3 = 4, \sigma_4 = 5, \sigma_5 \geq 6$. In this case we have

$$J \geq \frac{1}{2\gamma - 1}\Big(\gamma + \gamma^2 - \gamma^4 - \frac{\gamma^5}{(1-\gamma)}\frac{6\gamma+1}{2\gamma+1}\Big) - \gamma^3 - \frac{\gamma^3}{2} > 0$$

for $\gamma < 0.674$.
- Suppose $\sigma_3 = 3, \sigma_4 \geq 6$. Here

$$J \geq \frac{1}{2\gamma - 1}\Big(\gamma + \gamma^2 - \gamma^3 - \frac{\gamma^6}{(1-\gamma)}\frac{6\gamma+1}{2\gamma+1}\Big) > 0$$

for $\gamma < 0.699$.
- Suppose $\sigma_3 = 3, \sigma_4 = 5$ and $\sigma_5 \geq 6$. In this case

$$J \geq \frac{1}{2\gamma - 1}\Big(\gamma + \gamma^2 - \gamma^3 - \gamma^5 - \frac{\gamma^6}{(1-\gamma)}\frac{6\gamma+1}{2\gamma+1}\Big) - \gamma^4(2\gamma - 1) > 0$$

for $\gamma < 0.673$.
- Suppose $\sigma_3 = 3, \sigma_4 = 4$ and $\sigma_5 \geq 6$. We have

$$J \geq \frac{1}{2\gamma - 1}\Big(\gamma + \gamma^2 - \gamma^3 - \gamma^4 - \frac{\gamma^6}{(1-\gamma)}\frac{6\gamma+1}{2\gamma+1}\Big) > 0$$

for $\gamma < 0.673$.

- Suppose $\sigma_3 = 3, \sigma_4 = 4, \sigma_5 = 5$ and $\sigma_6 \geq 7$. Then we have

$$J \geq \frac{1}{2\gamma - 1}\Big(\gamma + \gamma^2 - \gamma^3 - \gamma^4 - \gamma^5 - \frac{\gamma^7}{(1-\gamma)}\frac{6\gamma+1}{2\gamma+1}\Big) > 0$$

  for $\gamma < 0.682$.
- Suppose $\sigma_3 = 3, \sigma_4 = 4, \sigma_5 = 5, \sigma_6 = 6$ and $\sigma_7 \geq 7$. Then we have

$$J \geq \frac{1}{2\gamma - 1}\Big(\gamma + \gamma^2 - \gamma^3 - \gamma^5 - \gamma^6 - \frac{\gamma^7}{(1-\gamma)}\frac{6\gamma+1}{2\gamma+1}\Big) > 0$$

  for $\gamma < 0.669$.

$\square$

5.2. **The relationship between $\chi, \hat{\chi}$ and Lebesgue measure.** Let us next define analogues of the measures ρ and $\hat{\rho}$ describing the spatial distribution of the increments of S with respect to the two coordinates. For the first version, the increments of H are taken with respect to the second variable x which is "dual" to ξ. We define the image measure of λ^3 on $[0,1]^3$ by the mapping $[0,1]^3 \ni (x,y,\xi) \mapsto H(\xi,y) - H(\xi,x) \in \mathbb{R}$, namely for Borel sets $A \subset \mathbb{R}$ define

$$\chi(A) = \lambda^3\Big(\big\{(x,y,\xi) \in [0,1]^3 : H(\xi,x) - H(\xi,y) \in A\big\}\Big).$$

Further, denote by $\hat{\chi}$ the measure χ, restricted to $\{\frac{1}{2} < |y-x| \leq 1\}$, and by $\check{\chi}$ the measure $\hat{\chi}$, restricted to $\{\overline{\xi}_0 = 1\}$. Note that $\check{\chi}$ is macroscopic in both variables.

To prove existence of a local time, we shall use smoothness properties of the density of the measure $\hat{\chi}$. We will deduce them via a Fourier analytic criterion as before. Similarly to section 4.2, in this subsection we first address the scaling properties of H with respect to increments in the second variable as to establish an analogous relationship between χ and $\hat{\chi}$.

In the following formula and using arguments dual to those given in subsection 4.2, χ is shown to be a weighted average of expansions measured by $\hat{\chi}$.

Proposition 5.8. *For Borel sets A on the real line, we have*

$$\chi(A) = \sum_{n=0}^{\infty} 2^{-n}\hat{\chi}(\gamma^{-n}A).$$

Proof: By definition of B and H, we have

$$\begin{aligned}
\chi(A) &= \sum_{n=0}^{\infty} \lambda^3\Big(\Big\{(\xi,x,y) : H(\xi,x) - H(\xi,y) \in A, 2^{-(n+1)} < |y-x| \leq 2^{-n}\Big\}\Big) \\
&= \sum_{n=0}^{\infty} \lambda^3\Big(\Big\{(\xi,x,y) : H(\xi,x) - H(\xi,y) \in A, \frac{1}{2} < |B_2^{-n}(\xi,y) - B_2^{-n}(\xi,x)| \leq 1\Big\}\Big) \\
&= \sum_{n=0}^{\infty} \lim_{\epsilon \to 0} \frac{1}{\lambda^4(D_\epsilon)}\lambda^4\Big(\Big\{(\xi,\eta,x,y) : H(\xi,x) - H(\eta,y) \in A, \\
&\qquad \frac{1}{2} < |B_2^{-n}(\xi,y) - B_2^{-n}(\xi,x)| \leq 1, |\xi - \eta| \leq \epsilon\Big\}\Big),
\end{aligned}$$

where for $\epsilon > 0$ we let $D_\epsilon = \{(\xi,\eta,x,y) : |\xi - \eta| \leq \epsilon\}$. We make use of invariance of the baker transform across the Lebesgue measure (see (2.2)) to estimate term n of the preceding series, then, for $n \geq 0$, by definition of the baker transform B

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\lambda^4(D_\epsilon)}\lambda^4\Big(\Big\{(\xi,\eta,x,y): H(\xi,x)-H(\xi,y)\in A,\\
&\qquad\qquad\qquad\qquad \frac{1}{2}<|B_2^{-n}(\xi,y)-B_2^{-n}(\xi,x)|\le 1, |\xi-\eta|\le\epsilon\Big\}\Big)\\
&=\lim_{\epsilon\to 0}\frac{1}{\lambda^4(D_\epsilon)}\lambda^4\Big(\Big\{(\xi,\eta,x,y): H(B^n(\xi,x))-H(B^n(\eta,y))\in A,\\
&\qquad\qquad\qquad\qquad \frac{1}{2}<|y-x|\le 1, |B_1^n(\xi,x)-B_1^n(\eta,y)|\le\epsilon\Big\}\Big)\\
&=\lim_{\epsilon\to 0}\frac{\lambda^4(D_{2^{-n}\epsilon})}{\lambda^4(D_\epsilon)}\frac{1}{\lambda^4(D_{2^{-n}\epsilon})}\lambda^4\Big(\Big\{(\xi,\eta,x,y): H(B^n(\xi,x))-H(B^n(\eta,y))\in A,\\
&\qquad\qquad\qquad\qquad \frac{1}{2}<|y-x|\le 1, |\xi-\eta|\le 2^{-n}\epsilon\Big\}\Big)\\
&=2^{-n}\lambda^3\Big(\Big\{(\xi,x,y): H(B^n(\xi,x))-H(B^n(\xi,y))\in A, \frac{1}{2}<|y-x|\le 1\Big\}\Big).
\end{aligned}$$

Now we apply Lemma 5.2 to transform term n of the preceding chain of equations into

$$\lambda^3\big(\big\{(\xi,x,y):\gamma^n(H(\xi,x)-H(\xi,y))\in A, \frac{1}{2}<|y-x|\le 1\big\}\big)=\hat{\chi}(\gamma^{-n}A).$$

Altogether, this implies the desired equation. □

5.3. **The existence of local time.** In this subsection we will show that $\mathcal{T}(\cdot)-\int_0^\cdot S(\xi,z)\mathrm{d}z$ possesses a local time. For this purpose we shall use arguments such as those for the absolute continuity of the SBR measure, starting with the Fourier analytic criterion. For this purpose, denote by β the law of the map $[0,1]^2\ni(\xi,x)\mapsto H(\xi,x)=\mathcal{T}(x)-\int_0^x S(\xi,z)\mathrm{d}z$. We consider the Fourier transforms of the marginals β_ξ, $\xi\in[0,1]$. Let

$$\phi_\xi(u)=\int_{\mathbb{R}}\exp(iuy)\beta_\xi(\mathrm{d}y),\quad u\in\mathbb{R}.$$

By the integral transform, we have

$$\phi_\xi(u)=\int_0^1\exp\big(iuH(\xi,x)\big)\mathrm{d}x,\quad u\in\mathbb{R},\xi\in[0,1].$$

To prove absolute continuity of β_ξ, we have to prove that ϕ_ξ is square integrable on $\mathbb{R}$. Therefore, to prove that $H(\xi,\cdot)$ possesses a local time, it will be sufficient to show

$$\begin{aligned}
\int_0^1\int_{\mathbb{R}}|\phi_\xi(u)|^2\mathrm{d}u d\xi &= \int_{\mathbb{R}}\int_{[0,1]^3}\exp\Big(iu\big(H(\xi,y)-H(\xi,x)\big)\Big)d\xi\mathrm{d}x\mathrm{d}y\mathrm{d}u\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\exp(iux)\chi(\mathrm{d}x)\mathrm{d}u<\infty.
\end{aligned}$$

By macroscopic properties of χ, the verification of this criterion will boil down to a similar one with $\hat{\chi}$ replacing χ.

Theorem 5.9. *Let γ_0 be given by Lemma 5.7. Then for all $\gamma\in[\frac{1}{2},\gamma_0]$ and for a.a. $\xi\in[0,1]$ the function*

$$x\mapsto H(\xi,x)$$

has an absolutely continuous law with respect to the Lebesgue measure with a square integrable density. In particular, $H(\xi,\cdot)$ possesses a square integrable local time for a.e. $\xi\in[0,1]$.

Proof: Applying Proposition 5.8 and the integral transformation theorem we first obtain

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp(iux)\chi(\mathrm{d}x)\mathrm{d}u &= \sum_{n=0}^{\infty} 2^{-n}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp(iuy)\hat{\chi}(\gamma^{-n}\mathrm{d}y)\mathrm{d}u \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{n=0}^{\infty} 2^{-n}\exp(iu\gamma^n y)\hat{\chi}(\mathrm{d}y)\mathrm{d}u \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{n=0}^{\infty} \kappa^n\exp(iuy)\hat{\chi}(\mathrm{d}y)\mathrm{d}u \\ &= \frac{1}{1-\kappa}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp(iuz)\hat{\chi}(\mathrm{d}z)\mathrm{d}u.\end{aligned}$$

Now recall that by definition of $\hat{\chi}$ and corollary 3.7 we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \exp(iuz)\hat{\chi}(\mathrm{d}z)\mathrm{d}u = \int_{\mathbb{R}}\int_{\{\frac{1}{2}<|x-y|\}}\int_{[0,1]} \exp(iu(H(\xi,x)-H(\xi,y)))d\xi\mathrm{d}x\mathrm{d}y\mathrm{d}u.$$

The positivity of the integral and Fubini's theorem allow us to integrate in u first, giving

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\{\frac{1}{2}<|x-y|\}}\int_{[0,1]} \exp(iu(H(\xi,y)-H(\xi,x)))d\xi\mathrm{d}x\mathrm{d}y\mathrm{d}u \\ &= \lim_{N\to\infty}\int_{-N}^{N}\int_{\{\frac{1}{2}<|x-y|\}}\int_{[0,1]} \exp(iu(H(\xi,y)-H(\xi,x)))d\xi\mathrm{d}x\mathrm{d}y\mathrm{d}u \\ &= \lim_{N\to\infty}\int_{\{\frac{1}{2}<|x-y|\}}\int_{[0,1]} \frac{2\sin(N(H(\xi,y)-H(\xi,x))}{H(\xi,y)-H(\xi,x)}d\xi\mathrm{d}x\mathrm{d}y.\end{aligned}$$

For the last equation we use that the function $x\mapsto\frac{\sin(Nx)}{x}$ is pair. We now use Corollary 5.4 and Lemma 5.7 to show that the denominator of the integrand in the preceding equation is bounded away from 0 for the claimed parameter range. Since the numerator of the integrand is bounded in N, we conclude that the entire integral is bounded in N, and the claim follows. □

Remark. A simpler argument than the one following Lemma 5.7 gives the result of Theorem 5.9 for the smaller parameter range $\gamma\le\frac{2}{3}$. Recall that for $\xi,x,y\in[0,1]$ we have

$$H(\xi,y)-H(\xi,x)=\mathcal{T}(x)-\mathcal{T}(y)-(y-x)S(\xi,0)).$$

Now note that by definition for any $x,y\in[0,1]$ we have

$$\mathcal{T}(y)-\mathcal{T}(x)\le\mathcal{T}(y)\le\frac{1}{2(1-\gamma)}.$$

Recall from Corollary 3.7 that for $\xi\in[0,1]$ we have $S(\xi,0)\ge\frac{1}{2\gamma-1}$. Hence for $y>\frac{1}{2}+x$

$$(y-x)S(\xi,0)-(\mathcal{T}(y)-\mathcal{T}(x))\ge\frac{1}{2}\frac{1}{2\gamma-1}-\frac{1}{2(1-\gamma)}=\frac{1}{2}[\frac{1}{2\gamma-1}-\frac{1}{\gamma-1}]=\frac{1}{2}\frac{2-3\gamma}{(2\gamma-1)(1-\gamma)},$$

which is positive for $\gamma<\frac{2}{3}$. If the roles of x and y are switched, a similar argument applies.□

Let us give a geometric interpretation of the local time we just obtained. For this purpose let $l_{(\xi,x,v)}$ be the motion through $(\xi,x,\mathcal{T}(\xi,x))$ along the stable fiber described by $S(\xi,x)$, given by the solutions of the family of ODE

$$\frac{d}{dr}l_{(\xi,x,v)}(r)=S(\xi,r),\quad\text{with}\quad l_{(\xi,x,v)}(x)=v,\quad v\in\mathbb{R}^2.$$

Then for $\xi, x, y \in [0,1]$, $x < y$, the vertical distance between the stable fibers through the points $(\xi, y, \mathcal{T}(y))$ and $(\xi, x, \mathcal{T}(x))$ is given by

$$\begin{aligned} l_{(\xi,y,\mathcal{T}(y))}(y) - l_{(\xi,x,\mathcal{T}(x))}(y) &= \mathcal{T}(y) - l_{(\xi,x,\mathcal{T}(x))}(y) \\ &= \mathcal{T}(y) - \mathcal{T}(x) - \big(l_{(\xi,x,\mathcal{T}(x))}(y) - l_{(\xi,x,\mathcal{T}(x))}(x) \big) \\ &= \mathcal{T}(y) - \mathcal{T}(x) - \int_x^y S(\xi, z) \mathrm{d}z \\ &= H(\xi, y) - H(\xi, x). \end{aligned}$$

Therefore Theorem 5.9 just expresses that measuring occupation by the position of the different displacements on the stable fibers of our dynamical system leads to smooth measures.

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