2.2 Lecture 3: Continuated Finite Difference Methods and Finite Element Methods

see also Lecture Notes (Carstensen, Bartels, Geiser (WS06) and Danielle Boffi SS05)

2.2.1 Repetition of Finite Difference Methods

Recapitulation of the finite difference methods:

To design derivatives with difference quotients, we have introduced finite difference for structured grids. By the way they are hard to extend to general domains and boundary condition, for such problems we introduce the finite element methods.

We have the following settings:

Setting $x_j := jx$, $i = \{1, N\}$, we use the slope of secants to approximate u, the forward difference is given as:

$$u'(x_j) \approx \partial^+ u(x_j) := \frac{u(x_{j+1}) - u(x_j)}{\Delta x} \tag{2.3}$$

where we have a first order accuracy.

See last lecture.

2.2.2 Finite Difference Method for Diffusion equation (Crank-Nicolson and second order derivative)

As mathematical model we assume, that our diffusion equation models a motionless liquid filling in a tube and that a chemical substance is diffusing through the liquid. We assume by the Ficks laws of diffusion, that the rate of motion is proportional to the concentration gradient, so that in- and outflowing mass of the concentration is conserved.

Therefore we have the following equation:

$$\begin{aligned}
u_t &= k u_{xx} \text{ in } (0,1) \times (0,T), && (2.4)\\
u(x,0) &= k u_0(x) \; x \in (0,1) \times (0,T), && (2.5)\\
u(0,t) &= u(1,t) = 0 \; t \in (0,T), && (2.6)\\
& && (2.7)
\end{aligned}$$

For the spatial discretization we apply the forward and backward differences given as:

$$|\partial_x^+ \partial_x^- - u_{xx}| \leq C(\Delta x)^2 |u|_{C^4([0,1])}. \tag{2.8}$$

See the outline of the proof:

$$u(x + \Delta x) = u(x) + u_x(x)\Delta x + 1/2 u_{xx}(x)\Delta x^2 + 1/6 u_{xxx}(x)\Delta x^3 \tag{2.9}$$
$$+O(\Delta x^4),$$
$$u(x - \Delta x) = u(x) - u_x(x)\Delta x + 1/2 u_{xx}(x)\Delta x^2 - 1/6 u_{xxx}(x)\Delta x^3 \tag{2.10}$$
$$+O(\Delta x^4),$$

We subtract the 2 equations and by reordering we obtain:

$$\frac{u(x + \Delta x) - 2u(x) + u(x - \Delta x)}{\Delta x^2} = u_{xx}(x) + O(\Delta x^4), \tag{2.11}$$

where $O(\Delta x^4) = C(\Delta x^2)|u|_{C^4([0,1])}$.

For the time-discretization we apply the θ-method and for the spatial discretization we apply the differential quotient for a second derivative :

The discretization scheme is given for any $\theta \in [0, 1]$ as :

$$\partial_t^- U_j^{n+1} = \partial_x^+ \partial_x^- (\theta U_j^{n+1}$$
$$+(1 - \theta)U_j^n), j = 1, \dots, M - 1, n = 0, 1, \dots, N - 1, \tag{2.12}$$
$$U_j^0 = u_0(x_j), j = 0, \dots, M, \tag{2.13}$$
$$U_0^{n+1} = U_M^{n+1} = 0 \; n = 1, \dots, N. \tag{2.14}$$

For $\theta = 0$ we obtain the forward Euler method, for $\theta = 1$ we obtain the backward Euler method and for $\theta = 0.5$ we obtain the Crank-Nicolson method, which is of second order in time.

Proof for the Crank-Nicolson Scheme see page 19, (Lecture Notes Bartels, Carstensen.)

2.2.3 Finite Element Methods

In the last sections we discuss the finite difference method as a tool for the numerical approximation of partial differential equations for structured grids. However, when the domain is complicated, e.g., has a curved boundary, the implementation can become very tedious.

Moreover, in order to prove convergence of the method we assume regularity, which is not guaranteed in general.

Here an alternative formulation with weak derivatives and Lebesgues integration theory helps to design the finite element methods.

We will concentrate to the Laplace operator which is essential in the design of approximation schemes for the diffusion and the wave equation. Here we want to design numerical schemes that are convergent even if there is no classical solution, see

$$-\Delta u = f \text{ in } \Omega \ , \tag{2.15}$$

$$u = u_D \text{ on } \Gamma_D \ , \tag{2.16}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N = \partial\Omega \backslash \Gamma_D \ , \tag{2.17}$$

In the subsection we discuss the weak formulation and the introduction to finite element methods.

2.2.4 Introduction Finite Element Methods

see Lecture notes of Danielle Boffi SS05 and Carstensen, Bartels, Geiser WS06.