## 2.A. **Linear algebra.**

2.A.1. *Linear equation system.* $A \in K^{I \times I}$ and $b \in K^I$

$$Ax = b, \quad \sum_{\beta \in I} a_{\alpha\beta} x_\beta = b_\alpha \quad \text{for all } \alpha \in I \tag{2.1}$$

solvable for regular $A$.

**Theorem 2.1.** If we have $A \in K^{I \times I}$, then the following properties are equivalent:

(i) $A$ is regular,
(ii) $\operatorname{rank} A = \#I$
(iii) $\det A \neq 0$
(iv) $Ax0$ has only the trivial solution $x = 0$
(v) $Ax = b$ is for all $b$ uniquely solvable.
(vi) $\ker A = 0$.

2.A.2. *Eigenvalues and eigenvectors.* We have $A \in K^{I \times I}$ where $K = \mathbb{R} or K = \mathbb{C}$. The spectrum of a matrix $A$ is $\sigma(A) = \{\lambda \in \mathbb{C} : \det(A - \lambda I) = 0\}$, $\lambda \in \sigma(A)$ is the eigenvalue and $Ae = \lambda e$ for $e \in K^I$ is the eigenvector.

**Definition 2.1.** Two matrices $A, B \in K^{I \times I}$ are equivalent if we have a regular matrix $T$ such that $A = T^{-1} A T$.

The Spectral radius $\rho(A)$ of a matrix $A$ is the absolute maximal eigenvalue

$$\sigma(A) = \max\{|\lambda| : \lambda\ in \sigma(A)\}.$$

The maximum-norm $\|\cdot\|_\infty$ is defined by

$$\|x\|_\infty = \max\{|x_\alpha| : \alpha \in I\}$$

The Euclidian norm $\|\cdot\|_2$ is defined by

$$\|x\|_2 = (\sum_{\alpha \in I} |x_\alpha|^2)^{1/2}.$$

The Matrix-norm

$$|||A||| = \sup\{\frac{\|Ax\|}{\|x\|} : 0 \neq x \in K^I\},$$

where for example $\|A\|_2$ is the spectral-norm and $\|A\|_\infty$ is the size-sum-norm.

**Lemma 2.2.** If $\|\cdot\|$ is a matrix-norm, we have $|\lambda| \leq \|A\|$ for all eigenvalues $\lambda$ of $A$, and $\rho(A) \leq \|A\|$ for all matrices $A$.

Positive definite matrices

**Definition 2.2.** We have $\langle\ cdot, \cdot\rangle$ as the euclidian scalar product for $K^I$ and $AK^{I \times I}$, then $A$ is positive definite, if $A$ is hermitic ($A = A^H := A^T$) and $\langle Ax, x\rangle > 0$ for all $x \in \mathbb{K}^I$.

2.B. **Iterative methods.** We deal with the linear equation

$$Ax = b,\ A \in \mathbb{K}^{I\times I},\ b \in K^I \tag{2.2}$$

where $A$ is regular. An iterative method, that produces from the initial value $x^0$, the iterates $x^1, x^2, x^3, \dots$ is given as

$$x^{m+1} = \Phi(x^m, b); \tag{2.3}$$

**Definition 2.3.** A linear iteration method is a linear mapping

$$\Phi : \mathbb{K}^I \times \mathbb{K}^I \to \mathbb{K}^I. \tag{2.4}$$

The next iterates are done with the initial value $x^0 \in \mathbb{K}^I$ and are denoted by

$$x^m(x^0, b) :\ x^0(y, b) = y^* x^{m+1}(y, b) = \Phi(x^m(y, b), b). \tag{2.5}$$

2.B.1. *Fixpoints.*

**Definition 2.4.** $x^* = x*(b)$ is a fixpoint of the iteration method $\Phi$, if (2.5) $x^* = \Phi(x^*, b)$.

2.B.2. *Consistency.*

**Definition 2.5.** The iteration method $\Phi$ is consistent for (2.2), if for all $b \in K^I$ it is fulfilled, that the solution of $Ax = b$ is a fixpoint with $b$.

2.B.3. *Convergence.*

**Definition 2.6.** An consistent iteration method $\Phi$ is convergent, if for all $b \in \mathbb{K}^I$, there exists an independent unique limit $x^*(b)$, from the initial value $x^0 = y \in \mathbb{K}^I$.

2.B.4. *Linear iteration method.*

**Definition 2.7.** An iteration method $\Phi$ is linear, if $\Phi(x, b)$ is in $x$ and $b$ linear and we obtain

$$\Phi(x, b) = Mx + Nb, \tag{2.6}$$

where $M$ is the iteration matrix of the iteration $\Phi$.

The first normal form is given as

$$x^{m+1} = Mx^m + Nb,\ m \geq 0,\ b \in \mathbb{K}^I.$$

The second normal form is given as

$$x^{m+1} = Mx^m - N(Ax^m - b),\ m \geq 0,$$

where the second normal form is consistent because of $M = I - NA$. The third normal form is given as

$$W(x^{m+1} - x^m) = Ax^m - b),$$

where $W$ is the matrix of the third normal form.

2.B.5. *Iteration-error.* is defined by $e^m = x^m - x$.

2.B.6. *Defect.* of the $m$-th iteration is given by

$$d^m = Ax^m - b.$$

2.B.7. Iteration method (2.6) is convergent if we have

$$\rho(M) < 1.$$

FIGURE 3.1. damped Jacobian, Jacobian iteration undamped

## 3. CLASSICAL ITERATION EFFECTS

We discuss smoothing effects of the classical iterations. We apply the Jacobi-iteration and get the transparent results. The system $L_\ell u_\ell = f_\ell$ can be written as

$$D_\ell u_\ell = B_\ell u_\ell + f_\ell$$
$$B_\ell ? D_\ell - L_\ell,$$

where $D_\ell$ is the diagonal matrix $2h_\ell^{-2}I$ and we consider the simple model problem (1.1). One could write

$$e_\ell^{j+1} = D_\ell^{-1}(B_\ell u_\ell^j) + f_\ell, \tag{3.1}$$

where $j$ denotes the $j$-th iterate. An equivalent formulation

$$u_\ell^{j+1} = u_\ell^j - D_\ell^{-1} \underbrace{(L_\ell u_\ell^j - f_\ell)}_{\text{defect}} \tag{3.2}$$

correction $u_\ell^{j+1} - u_\ell = e_\ell^{j+1}$ is obtained from the defect (or residual). For the reason of less convergence the damped Jacobian-iteration:

$$u_\ell^{j+1} = u_\ell^j - \Phi D_\ell^{-1}(L_\ell u_\ell^j - f_\ell) \tag{3.3}$$

with $\Phi \in (0, 1)$. Concrete we have

$$D_\ell^{-1} = \frac{1}{2} h_\ell^{-1} I.$$

We write (**??**) as

$$u_\ell^{j+1} = u_\ell^j - \omega h_\ell^2 (L_\ell u_\ell^j - f_\ell)$$

with $\omega \in (0, \frac{1}{2})$. For the analysis we have the eigenvectors

$$e_\ell^\mu = \sqrt{2h_\ell}(\sin(v\mu\pi b_\ell))_{v=1}^{n_\ell}$$

for $\mu = 1, \ldots, u_\ell$. The eigenvalues of $L_\ell$ are

$$\lambda_\ell^\mu = 4h_\ell^{-2} \sin^2(\mu\pi \frac{h_\ell}{2})$$

Then

$$L_\ell e_\ell^\mu = \lambda_\ell^m u e_\ell \tag{3.4}$$

since $M_\ell = I - \omega h_\ell^2 L_\ell$ (iteration matrix), we have the eigenvalues

$$\lambda_\mu(\omega) = 1 - 4\omega \sin^2(\mu\pi \frac{h_\ell}{2}).$$

The maximum absolute value is $1 \leq \mu \leq n_\ell = h_\ell^{-1} - 1$. $\rho(\mu_\ell)$ is taken at $\mu = 1$.

$$\rho()$$

The smoothing effect of the damped Jacobian iterations yield the error

$$u_\ell - u_{\ell,\,exact} = \sum_\mu \beta_\mu e_\ell^\mu.$$

[19] Iteration (**??**) serves as a smoothing iteration. Some obtain the smoothing part of the two- or multigrid-iterations:
We replace $M_\ell, N_\ell$ by $S_\ell, T_\ell$ (for notation purposes only)

$$u_\ell^{j+1} = S_\ell u_\ell^j + T_\ell f_\ell. \tag{3.5}$$

For the damped Jacobian the matrices are

$$S_\ell = I - \omega h_\ell^2 L_\ell, \quad \omega h_\ell^2 I. \tag{3.6}$$

The notation is given as

$$S_\ell(u_\ell, f_\ell) : \; u_\ell^{j+1} = S_\ell(u_\ell^j, f_\ell).$$

The $v-$th iterate of $S_\ell$ is given as

$$u_j^{\ell+1} = S_\ell^v(u_\ell^j, f_\ell).$$

[19] picture: $16_0 6_0 01$