Part 2. Finite Volume Methods I

The finite volume method is a discretization method which is well suited for the numerical research of various types (elliptic, parabolic and hyperbolic for instance) of *conservation laws*. It is extensively used in several engineering fields, such as fluid mechanics, heat and mass transfer, or petrolium engineering.

Some features are equal to the Finite Element methods as *arbitrary geometries*, using structured grids, unstructured meshes and robust schemes. An additional feature is the local conservativity on the numerical fluxes, that the numerical flux is conserved from one discretization cell to its neighbour, i.e. the out-flux from one cell and the in-flux into a neighboring cell over a common edge are equal, see also Figure 0.2 below. This makes the finite volume scheme attractive, when modelling problems for which the flux is of importance.

The finite volume method is locally conservative because it is based on a balance, i.e., a local balance is written on each discretization cell, often called *control volume*. Therefore an integral formulation of the fluxes over the boundary of the control volumes can be obtained.

The grid of control volumes, is the dual grid of the triangulation, which can be either a Voronoi or Barycentric grid. In a Voronoi grid, the edges are perpendicular to the edges of the triangulation and intersect them at their midpoints, where as in a barycentric grid, the edges go from the intersection of the median lines of the triangles of the triangulation to the midpoint of the edges of the triangulation.

FIGURE 0.1. Example for a triangle with barycentric grid on the left and Voronoi grid on the right.

From here on we will only consider the Voronoi grid. The out-flux from one cell K over a common edge with a neighboring cell L is equal to the in-flux into L over this edge.

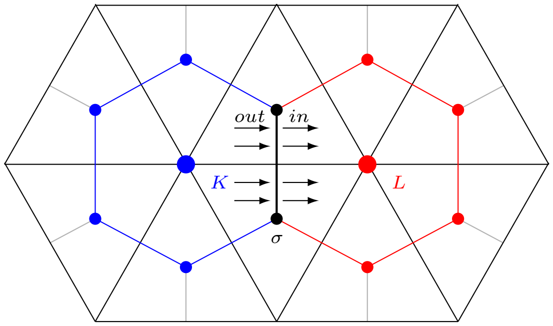

FIGURE 0.2. Flux from a cell K into a neighboring cell L over a common edge σ, where the out-flux from K and the in-flux into L over the common edge σ are equal.

1. Examples for Finite Volume Methods

1.A. **Transport Equation.** Consider the linear transport equation.

$$\begin{aligned} u_t(x,t) + \operatorname{div}(vu)(x,t) &= 0 \quad \text{for all } x \in \mathbb{R}^2, t \in \mathbb{R}^+ \\ u(x,0) &= u_0(x) \quad \text{for all } x \in \mathbb{R}^2, \end{aligned} \tag{1.1}$$

where u_t is the derivative in times, $u \in C^2(\mathbb{R}^2 \times \mathbb{R}^+)$, $u_0 \in L^\infty(\mathbb{R}^2)$, and $v = (v_1, v_2)$ is a vector with $v_1(x,t), v_2(x,t) \in C^2(\mathbb{R}^2 \times \mathbb{R}^+)$.

$\mathcal{T}$ is a mesh of $\mathbb{R}^2$ consisting of polynomial bounded convex subsets and $K \in \mathcal{T}$ is a control volume. We integrate (1.1) over K. Hence we have the following balance equation

$$\int_K u_t(x,t)\,dx + \int_{\partial K} v(x,t) \cdot n_K(x)\, u(x,t)\, ds = 0 \quad \text{for all } t \in \mathbb{R}^+, \tag{1.2}$$

where n_K denotes the normal vector to ∂K. We integrate over (1.2) over (t_n, t_{n+1}).

$$\begin{aligned} \int_K \int_{t_n}^{t_{n+1}} u_t(x,t)\,dt\,dx + \int_{t_n}^{t_{n+1}} \int_{\partial K} v(x,t) \cdot n_K(x)\, u(x,t)\, ds\, dt &= 0 \\ \int_K \big(u^{n+1}(x) - u^n(x)\big)\,dx + \int_{t_n}^{t_{n+1}} \int_{\partial K} v(x,t) \cdot n_K(x)\, u(x,t)\, ds\, dt &= 0 \end{aligned} \tag{1.3}$$

Then semi-discretization of (1.3) by implicit Euler yields

$$\begin{aligned} \int_K \big(u^{n+1}(x) - u^n(x)\big)\,dx + \triangle t \int_{\partial K} v(x,t_{n+1}) \cdot n_K(x)\, u^{n+1}(x)\, ds &= 0 \\ \frac{1}{\triangle t} \int_K \big(u^{(n+1)}(x) - u^{(n)}(x)\big)\,dx + \int_{\partial K} v(x,t_{n+1}) \cdot n_K(x)\, u^{n+1}(x)\, ds &= 0, \end{aligned} \tag{1.4}$$

for all $n \in \mathbb{N}, K \in \mathcal{T}$, where $\triangle t = t_{n+1} - t_n$. Let $\mathcal{E}_K$ be the set of edges of ∂K, and $n_{K,\sigma}$ the normal vector on σ pointing outward from K. Consider

$$\int_{\partial K} v(x,t_{n+1}) \cdot n_K\, u^{n+1}(x)\, ds = \sum_{\sigma \in \mathcal{E}_K} \int_\sigma v(x,t_{n+1}) \cdot n_{K,\sigma}\, u^{n+1}(x)\, ds. \tag{1.5}$$

We decide the discretized flux with the up-winding scheme

$$F_{K,\sigma}^{n+1} = \begin{cases} v_{K,\sigma}^{n+1}\, u_K^{n+1} & \text{if } v_{K,\sigma}^{n+1} \geq 0 \\ v_{K,\sigma}^{n+1}\, u_L^{n+1} & \text{if } v_{K,\sigma}^{n+1} < 0, \end{cases} \tag{1.6}$$

where $v_{K,\sigma}^{n+1} = \int_\sigma v(x,t_{n+1}) \cdot n_{K,\sigma}\, ds$. Then, full discretization of equation (1.4) yields

$$\frac{|K|}{\triangle t}\,(u_K^{n+1} - u_K^n) + \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+1} = 0, \tag{1.7}$$

for all $n \in \mathbb{N}$, $K \in \mathcal{T}$, where $u^n(x) = u_K^n$ for all $x \in K$, $|K|$ is the measure of K, $u_k^{(0)} = \frac{1}{|K|} \int_K u_0(x)\,dx$.

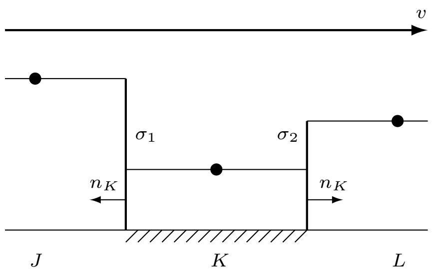

FIGURE 1.1

1.B. **Stationary Diffusion Equation.** Consider the stationary diffusion equation

$$(1.8)\qquad \begin{aligned} -\triangle u &= f \quad \text{on } \Omega =]0,1[\times]0,1[\\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

on a regular mesh $\mathcal{T}$. Integrating (1.8) over a control volume and using Gauss theorem with $-\int_K \nabla u \, dx = \int_{\partial K} \nabla u \cdot n \, ds$ yields

$$\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u(x) \cdot n_{K,\sigma} \, ds = \int_K f(x) \, dx.$$

We approximate the flux $\int_\sigma \nabla u(x) \cdot n_{K,\sigma} \, ds$ by the finite difference approximation

$$F_{K,\sigma} = |\sigma| \, \frac{u_L - u_K}{d_{K,L}},$$

where $d_{K,L} := |x_L - x_K|$ is the distance between x_L and x_K and L is the neighboring control volume with common edge σ. Then there holds the linear system of equations

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} = |K| \, f_K \quad \text{for all } K \in \mathcal{T},$$

where $f_K := \frac{1}{|K|} \int_K f(x) \, dx$.

2. The One-dimensional Elliptic Problem

The formulation is done for the one-dimensional problem and we introduce the error estimates for the finite volume methods.

Definition 2.1 (Admissible one-dimensional mesh)**.** An admissible mesh of $(0,1)$, denoted by $\mathcal{T}$ is given by a family $(K_i)_{i=1,...,N}$ of control volumes with $N \in \mathbb{N}_0$, such that $K_i = (x_{i-1/2} x_{i+1/2})$ and a family $(x_i)_{i=0,...,N+1}$ of nodes such that

$$\begin{aligned} 0 = x_0 = x_{1/2} < x_1 < x_{3/2} < ... < x_{i-1/2} < x_i < \\ < x_{i+1/2} < ... < x_{N-1/2} < x_N < x_{N+1/2} = x_{N+1} = 1. \end{aligned}$$

We have $h_i := |K_i| = x_{i+1/2} - x_{i-1/2}$.

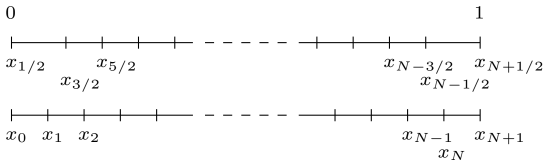

Figure 2.1

2.A. **Example of finite volume method for the one-dimensional Poisson Equation.** Consider the following differential equation

$$\begin{aligned} -u_{xx}(x) &= f(x) \quad x \in (0,1),\ f \in C(\mathbb{R}) \\ u(0) &= 0,\ u(1) = 0 \end{aligned} \tag{2.1}$$

We can write equation (2.1) in the conservative form

$$\operatorname{div}(F) = f \quad \text{with } F = -u_x.$$

The discrete unknowns are denoted by u_i, $i = 1, ..., N$ and are approximations of u in the cell K_i. Equation (2.1) is integrated over each K_i and yields

$$-\big(u_x(x_{i+1/2}) - u_x(x_{i-1/2})\big) = \int_{K_i} f(x)\,dx \quad i = 1, ..., N. \tag{2.2}$$

The approximation of $u_x(x_{i+1/2})$ is given by the differential quotient

$$F_{i+1/2} = \frac{u_{i+1} - u_i}{h_{i+1/2}}$$

and $h_{i+1/2} := x_{i+1} - x_i$. The approximation is consistent, if $u \in C^2([0,1])$. Thus, we have by applying Taylor expansion to $u(x_i)$ and $u(x_{i+1})$ in $x_{i+1/2}$

$$\begin{aligned} F_{i+1/2} &= \frac{u_{i+1} - u_i}{h_{i+1/2}} = \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} + O(h) \\ &= \frac{u(x_{i+1}) - u(x_i)}{h_{i+1/2}} + O(h) = u_x(x_{i+1/2}) + O(h) = \overline{F}_{i+1/2} + O(h), \end{aligned} \tag{2.3}$$

where $\overline{F}_{i+1/2}$ is the exact flux.

Remark: The Dirichlet boundary condition is taken into account by using the values imposed at the boundaries to compute the fluxes on these boundaries

$$F_{i+1/2} = \frac{u_{i+1} - u_i}{h_{i+1/2}}, \quad F_{1/2} = \frac{u_1}{h_{1/2}}, \quad F_{N+1/2} = \frac{u_N}{h_{N+1/2}}.$$

Then the numerical scheme looks like $Au = b$, where $u = (u_1, ..., u_N)^T$, $b = (b_1, ..., B_N)^T$ and

$$\begin{aligned} (Au)_i &= -\Big(\frac{u_{i+1} - u_i}{h_{i+1/2}} - \frac{u_i - u_{i-1}}{h_{i-1/2}}\Big) = -\big(F_{i+1/2} - F_{i-1/2}\big) \\ b_i &= \int_{K_i} f(x)\, dx. \end{aligned} \tag{2.4}$$

2.B. **Convergence Theorem and error-estimate for the one-dimensional Poisson Problem.** Consider equation (2.1), i.e.,

$$\begin{aligned} -u_{xx}(x) = f(x) \quad x \in (0,1),\ f \in C(\mathbb{R}) \\ u(0) = 0,\ u(1) = 0 \end{aligned}$$

Theorem 2.1 (Finite volume error estimate)**.** Let $f \in C^1([0,1])$ and $u \in C^2([0,1])$ be the unique solution of the problem (2.1). Let $\mathcal{T} = (K_j)_{j=1,\dots,N}$ be an admissible mesh in the sense of Definition 2.1. Thus, there exists a constant $c \geq 0$, only depending on u, such that

$$\sum_{i=0}^{N} \frac{(e_{i+1} - e_i)^2}{h_{i+1/2}} \leq C^2 h^2 \quad \text{and } |e_i| \leq Ch \text{ for } i = \{1, ..., N\} \tag{2.5}$$

where $e_i = u(x_i) - u_i$, $e_0 = e_{N+1} = 0$, $h = \max_{i \in \{0,\dots,N\}}\{h_{i+1/2}\}$, and where $h_{i+1/2} = x_{i+1} - x_i$.

Proof. There exists a unique vector $u = (u_1, ..., u_N)^T \in \mathbb{R}^N$ solution to (2.1). We multiply (2.4) with u_i and summing for $i = 1, ..., N$ gives

$$\sum_{i=1}^{N} (Au)_i \cdot u_i = \frac{u_1^2}{h_{1/2}} + \sum_{i=2}^{N-1}\Big(\frac{(u_{i+1} - u_i)^2}{h_{i+1/2}}\Big) + \frac{u_N^2}{h_{N+1/2}} = \sum_{i=1}^{N} h_i f_i \cdot u_i = \sum_{i=1}^{N} b_i \cdot u_i,$$

where $f_i = \frac{1}{h_i}\int_{K_i} f(x)\, dx$. If $f_i = 0$, then $u_i = 0$ due to uniqueness and existence. We integrate the equation $-u_{xx} = f$ over K_i, as above in Definition 2.1, i.e.,

$$(Au)_i = -(F_{i+1/2} - F_{i-1/2}) = h_i A_i = b_i.$$

The exact and the discrete flux yield

$$\begin{aligned} -\overline{F}_{i+1/2} + \overline{F}_{i-1/2} = \int_{K_i} f(x)\, dx &= h_i f_i \\ -F_{i+1/2} + F_{i-1/2} &= h_i f_i, \end{aligned}$$

respectively. Subtracting both equations and defining $G_{i+1/2} := \overline{F}_{i+1/2} - F_{i+1/2}$, we have $G_{i+1/2} - G_{i-1/2} = 0$. The consistency of the flux yields

$$\overline{F}_{i+1/2} = F_{i+1/2} + R_{i+1/2},$$

as in equation (2.3), where $R_{i+1/2} \leq c_i h$. We can write the error e_i as

$$e_i = u(x_i) - u_i \quad \text{for } i = 1, \dots N \text{ with } e_0 = e_{N+1} = 0.$$

Using equation (2.3), we have

$$G_{i+1/2} = u_x(x_{i+1/2}) - \frac{u_{i+1} - u_i}{h_{i+1/2}} = \frac{u(x_{i+1}) - u(x_i)}{h_{i+1/2}} = \frac{e_{i+1} - e_i}{h_{i+1/2}} + R_{i+1/2} \quad i = 1, \dots, N.$$

Multiplying $G_{i+1/2} - G_{i-1/2}$ with e_i and summing up from $i = 1$ to N, there holds

$$\begin{aligned} \sum_{i=1}^{N} \frac{e_{i+1} - e_i}{h_{i+1/2}}\, e_i - \sum_{i=1}^{N} \frac{e_i - e_{i-1}}{h_{i-1/2}}\, e_i &= -\sum_{i=1}^{N} R_{i+1/2}\, e_i + \sum_{i=1}^{N} R_{i-1/2}\, e_i \\ \Leftrightarrow \quad \sum_{i=0}^{N} \frac{e_{i+1} - e_i}{h_{i+1/2}}\, e_i - \sum_{i=1}^{N+1} \frac{e_i - e_{i-1}}{h_{i-1/2}}\, e_i &= -\sum_{i=0}^{N} R_{i+1/2}\, e_i + \sum_{i=1}^{N+1} R_{i-1/2}\, e_i. \end{aligned}$$

Thus we have that

$$\sum_{i=0}^{N} \frac{(e_{i+1} - e_i)^2}{h_{i+1/2}} \le Ch \sum_{i=0}^{N} |e_{i+1} - e_i|,$$

since

$$\sum_{i=0}^{N} \frac{e_{i+1} - e_i}{h_{i+1/2}}\, e_i - \sum_{i=1}^{N+1} \frac{e_i - e_{i-1}}{h_{i-1/2}}\, e_i = \frac{e_1^2}{h_{1/2}} + \sum_{i=1}^{N} \frac{(e_{i+1} - e_i)^2}{h_{i+1/2}} + \frac{e_N^2}{h_{1/2}}.$$

Using Cauchy-Schwarz inequality, there holds

$$Ch \sum_{i=0}^{N} |e_{i+1} - e_i| \le Ch \Big(\sum_{i=0}^{N} \frac{|e_{i+1} - e_i|^2}{h_{i+1/2}} \Big)^{1/2} \underbrace{\Big(\sum_{i=0}^{N} h_{i+1/2} \Big)^{1/2}}_{=1}.$$

Thus we have

$$\sum_{i=1}^{N} \frac{(e_{i+1} - e_i)^2}{h_{i+1/2}} \le C^2 h^2.$$

□

Remark: The error estimates did not use the discrete maximum-principle (which means: $f_i \ge 0$ for all $i = 1, \dots, N$ implies $u_i \ge 0$ for all $i = 1, \dots, N$), but coercivity is used.[13]

Comparison with finite difference method: With the same notations as in finite volume method, we consider u_i as an approximation of $u(x_i)$

$$-\frac{1}{h_i} \Big(\frac{u_{i+1} - u_i}{h_{i+1/2}} - \frac{u_i - u_{i-1}}{h_{i-1/2}} \Big) = f_i = f(x_i).$$

But this is not a consistent approximation of $-u_{xx}(x_i)$ in the finite difference sense, since the difference quotient does not test to zero, if $h \to 0$.

$$r = Au - b \quad \text{with } r = (r_1, \dots, r_N)^T \text{ and } b_i = f(x_i).$$

Applying Taylor expansion, we get

$$u(x_{i+1}) = u(x_i) + h_{i+1/2} u_x(x_i) + \frac{1}{2} h_{i+1/2}^2 u_{xx}(x_i) + O(h^3)$$

and

$$r_i = -\frac{1}{h_i} \frac{h_{i+1/2} - h_{i-1/2}}{2} u_{xx}(x_i) + O(h^3),$$

which in general does not tend to zero, if $h \to 0$.

Example: Let $f = 1$, then consider $(0, 1)$ and[14]

$$h_i = h \quad \text{for even } i, \quad h_i = \frac{h}{2} \quad \text{for odd } i, \quad x_i = \frac{x_{i+1/2} - x_{i-1/2}}{2} \quad \text{for } i = 1, \dots, N.$$

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[13] But where is coercivity used??

[14] The example needs corrections or further explanation.

Then there holds

$$r_i = -\frac{1}{4} \quad \text{for even } i, \quad r_i = \frac{1}{2} \quad \text{for odd } i.$$

Hence $\sup\{|r_i|,\ i = 1, \ldots, N\} \not\to 0$ for $h \to 0$.

A consistent finite difference method is given by

$$\frac{4}{h_{i-1} + 2h_i + h_{i+1}} \left(-\frac{u_{i+1} - u_i}{h_{i+1/2}} + \frac{u_i - u_{i-1}}{h_{i-1/2}} \right) = f(x_i).$$

3. ELLIPTIC PROBLEMS IN TWO OR THREE DIMENSIONS

We discuss the discretization of elliptic problems in several space dimensions by the finite volume methods. The one-dimensional case is easily generalised to non-uniform rectangular, or parallel-epipetic[15] meshes. For a generalization of control volumes, the definition of the schemes (and proofs) require some assumptions, which define an "admissible mesh".

The proofs are based on the discrete "Poincaré inequality" for the Dirichlet- and Neumann-boundary conditions. The error estimate between the finite volume approximated solutions and the C^2- and H^1-regular exact solutions is also proven.

Consider the following elliptic equation

$$\begin{aligned} -\operatorname{div}(A\,\nabla u(x)) + \operatorname{div}(vu)(x) + b\,u(x) &= f(x) \\ u(x) &= g(x), \end{aligned} \tag{3.1}$$

for all $x \in \partial\Omega$. Assumptions:

(i) Ω is an open bounded polynomial set of $\mathbb{R}^d$ with $d = 2, 3$.
(ii) $b \ \geq 0$.
(iii) $v \in C^1(\overline{\Omega})$, $\operatorname{div} v \geq 0$.
(iv) $g \in C(\partial\Omega)$ such that there exists $\tilde{g} \in H^1(\Omega)$ with $\overline{\gamma}(\tilde{g}) = g$ on $\partial\Omega$, where $\overline{\gamma}$ denotes the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$.
(v) A is coercive.

One can show by the Lax-Milgram theorem and under the assumptions above, that there exists a unique solution $u \in H^1(\Omega)$ of problem (3.1). For a proof, see [EGH].

The solution satisfies $u = w + \tilde{g}$, where $\tilde{g} \in H^1(\Omega)$, such that $\gamma(\tilde{g}) = g$ and w is a unique function of $H_0^1(\Omega)$.

$$\begin{aligned} &\int_\Omega \nabla w \cdot \psi(x) + \operatorname{div}(vw)(x)\,\psi(x) + b\,w(x)\,\psi(x)\,dx \\ &\qquad = \int_\Omega -\nabla\tilde{g}(x)\cdot\nabla\psi(x) - \operatorname{div}(v\tilde{g})(x)\,\psi(x) - b\tilde{g}(x)\,\psi(x) + f(x)\,\psi(x)\,dx \end{aligned} \tag{3.2}$$

for all $\psi \in H_0^1(\Omega)$.

3.A. **Structured Meshes.** We can generalize the one-dimensional scheme. Rectangular meshes for the Laplace-operator

$$\begin{aligned} -\triangle u(x,y) = f(x,y) \quad &\text{for all } (x,y) \in \Omega \\ u(x,y) = 0 \quad &\text{for all } (x,y) \in \partial\Omega. \end{aligned} \tag{3.3}$$

Let

$$\mathcal{T} = (K_{i,j})_{\substack{i = 1,\dots,N_1, \\ j = 1,\dots,N_2}}$$

be an admissible mesh of $(0,1)\times(0,1)$ with the assumptions above, i.e., let $N_1, N_2 \in \mathbb{N}$, $h_1, \dots, h_{N_1} > 0$, $k_1, \dots, k_{N_2} > 0$ such that

$$\sum_{i=1}^{N_1} h_i = 1 \quad \sum_{i=1}^{N_2} k_i = 1.$$

Furthermore, let $h_0 = h_{N_1+1} = 0$ and $k_0 = k_{N_2+1} = 0$. For $i = 1, \dots, N_1$, let

$$x_{1/2} = 0, \quad x_{i+1/2} = x_{i-1/2} + h_i, \quad x_{N_1+1/2} = 1$$

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[15] What means epipetic?

and for $j = 1, \dots, N_2$

$$y_{1/2} = 0, \quad y_{j+1/2} = y_{j-1/2} + k_j, \quad x_{N_2+1/2} = 1.$$

Then, $K_{i,j}$ is defined by

$$K_{i,j} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}].$$

Now, we can apply the finite volume scheme to equation (3.3) and get

$$\begin{aligned} &-\int_{y_{j-1/2}}^{y_{j+1/2}} u_x(x_{i+1/2}, y)\,dy + \int_{y_{j-1/2}}^{y_{j+1/2}} u_x(x_{i-1/2}, y)\,dy \\ &+ \int_{x_{i-1/2}}^{x_{i+1/2}} u_y(x, y_{i-1/2})\,dx - \int_{x_{i-1/2}}^{x_{i+1/2}} u_y(x, y_{i+1/2})\,dx = \int_{K_{i,j}} f(x,y)\,dx\,dy. \end{aligned}$$

Applying the numerical fluxes in the same way as in the one-dimensional case, we get

$$F_{i+1/2,j} - F_{i-1/2,j} + F_{i,j+1/2} - F_{i,j-1/2} = h_{i,j}\, f_{i,j} \tag{3.4}$$

for all $(i,j) \in \{1, \dots, N_1\} \times \{1, \dots, N_2\}$, where $h_{i,j} = h_i k_j$ and $f_{i,j}$ is the mean value of f on $K_{i,j}$.

$$\begin{aligned} F_{i+1/2,j} &= -\frac{k_j}{h_i + 1/2}(u_{i+1,j} - u_{i,j}) \\ F_{i,j+1/2} &= -\frac{h_i}{k_j + 1/2}(u_{i,j+1} - u_{i,j}) \\ u_{0,j} &= u_{N_1+1,j} = u_{i,0} = u_{i,N_2+1} = 0, \end{aligned}$$

for $i = 1, \dots, N_1$, and $j = 1, \dots, N_2$.

3.B. **General meshes and schemes (unstructured meshes).** For the generalization of the meshes and the schemes, we concentrate on the convection-diffusion equation. The advantage of finite volumes using non-structured meshes is simple; because of the bounding integrals, which also fit in the non-structured case and stability. We obtain a robust scheme for any admissible mesh. Therefore the use of a non-structured meshes allows for the computation of a solution for any shape of the physical domain.

Definition 3.1 (Admissible mesh)**.** Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$ with $d = 2, 3$. An admissible finite volume mesh of Ω, denoted by $\mathcal{T}$, is given by a family of control volumes, which are open subsets of $\overline{\Omega}$ containing hyper-planes of $\mathbb{R}^d$, denoted by $\mathcal{E}$ (edges or sides of the control volumes), which are strictly positive $(d-1)$-dimensional measures, and a family of points of Ω denoted by $\mathcal{P}$ satisfying

(i) the closure of the union of all control volumes is $\overline{\Omega}$,
(ii) for any $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that

$$\partial K = \overline{K} \setminus K = \bigcup_{\sigma \in \mathcal{E}_K} \overline{\sigma},$$

(iii) for any $K, L \in \mathcal{T}$ with $K \neq L$ either the $(d-1)$-dimensional Lebesgue measure of $\overline{K} \cap \overline{L}$ is zero or $\overline{K} \cap \overline{L} = \sigma$ for some $\sigma \in \mathcal{E}$ also denoted by $K|L$,
(iv) the family $\mathcal{P} = (x_K)_{K \in \mathcal{T}}$ is such that for $x_K \in \overline{K}$ and $x_{K'} \in \overline{K}'$, the straight line from x_K to $x_{K'}$ intersects σ, where $\sigma = K|K' \neq \emptyset$ and $K \neq K'$,
(v) for any $\sigma \in \mathcal{E}$, with $\sigma \in \partial\Omega$, let K be the control volume such that $\sigma \in \mathcal{E}_K$.

3.C. **Example for Grids.** Triangular mesh: Let Ω be an open bounded polygonal subset of $\mathbb{R}^2$. Let $\mathcal{T}$ be a family of an open triangular disjoint subset of Ω, such that two triangles having a common edge, also have two common vertices.
We assume all angles of the triangles are less than $\pi/2$. This is sufficient for the bisections to intersect the sides of each triangle. Defining the points $x_k \in K$ with either barycentric or Voronoi grid construction, we obtain an admissible mesh, also see Figure 0.1 and 0.2.

Remark:

(i) In the case of the elliptic operators, the finite volume scheme defined on such a grid using differential quotients is a 4-point scheme

$$\frac{1}{h_i^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}.$$

(ii) The consistency is only verified for the approximation of the fluxes, but this together with the conservation of the scheme, yields the convergence of the scheme.

3.D. **Existence and Uniqueness.** We prove existence and estimates of the solution. The estimates ensure stability of the scheme and will be obtained by using the discrete Poincaré inequality. Assumptions for the following Lemma

(i) $\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} + \sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma}\, u_\sigma + b\,|K|\,u_K = |K|\,f_K$ for all $K \in \mathcal{T}$,

(ii) $F_{K,\sigma} = -F_{L,\sigma}$ for all $\sigma \in \mathcal{E}_{\text{int}}$, i.e., if $\sigma = K|L$,

(iii) $F_{K,\sigma}\, d_{K,\sigma} = -|\sigma|(u_\sigma - u_K)$ for all $\sigma \in \mathcal{E}_K$, $K \in \mathcal{T}$ and $d_{K,\sigma} = |x_\sigma - x_K|$,

(iv) $u_\sigma = g(y_\sigma)$ for all $\sigma \in \mathcal{E}_{\text{ext}}$ with $\mathcal{E}_{\text{ext}} \subset \partial\Omega$.

Furthermore assuming $g = 0$ and $u_\mathcal{T} \in X(\mathcal{T})$ by $u_\mathcal{T}(x) = u_K$ for all $x \in K$ the flux

$$v_{K,\sigma} = \int_\sigma v(x) \cdot n_{K,\sigma} dx.$$

The value of u_σ has to be decided for interior edges by for example the upwinding scheme.

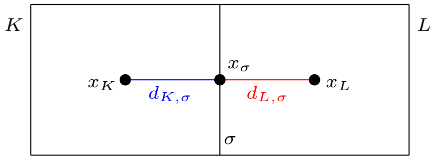

FIGURE 3.1

Definition 3.2. $H_0^1(\Omega)$ with $\|\cdot\|_{1,\mathcal{T}}$ is the discretization of $H_0^1(\Omega)$ with $\|\cdot\|_1$, where

$$\|u\|_1 = \sum_{i=0}^{1} \|\frac{\partial u}{\partial x}\|_{L^2} \quad \|u\|_{1,\mathcal{T}} = \Big(\sum_{\sigma\in\mathcal{E}} \tau_\sigma (D_\sigma u)^2\Big)^2,$$

and $\tau_\sigma = \frac{|\sigma|}{d_\sigma}$,

$$D_\sigma u = \begin{cases} |u_K - u_L| & \text{if } \sigma = K|L \in \mathcal{E}_{\text{int}} \\ |u_K| & \text{else.} \end{cases}$$

[16]

Lemma 3.1. Under the assumptions above, let $\mathcal{T}$ be an admissible mesh in the sense of Definition 3.1. Thus, there exists a unique solution $(u_K)_{K\in\mathcal{T}}$ to equation (3.1). We assume $g = 0$ and define $u_{\mathcal{T}} \in X(\mathcal{T})$, as in the assumptions above. For $x \in K$ and for any $K \in \mathcal{T}$, there holds

$$\|u_{\mathcal{T}}\|_{1,\mathcal{T}} \leq \operatorname{diam}(\Omega)\, \|f\|_{L^2(\Omega)},$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the discrete norm $H^1_0(\Omega)$, defined in Definition 3.2

Proof. [17] Step 1, Existence and uniqueness

Assume that $(u_K)_{K\in\mathcal{T}}$ satisfies the linear system with $g(y_\sigma) = 0$ for any $\sigma \in \mathcal{E}_{ext}$ and $f_K = 0$ for all $K \in \mathcal{T}$. We multiply assumption (i) above with u_K and sum over K.

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}\, u_K + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma}\, u_\sigma\, u_K + b \sum_{K\in\mathcal{T}} |K|\, u_K^2 = 0.$$

We reorder the sum and get

$$\sum_{\sigma\in\mathcal{E}} \tau_\sigma (D_\sigma u)^2 + \sum_{\sigma\in\mathcal{E}} v_\sigma \left(u_{\sigma,+} - u_{\sigma,-}\right) u_{\sigma,+} + b \sum_{K\in\mathcal{T}} |K|\, u_K^2 = 0,$$

where $u_{\sigma,-}$ is the downstream value on σ with respect to v,

$$u_{\sigma,-} = u_K \quad \text{if } v_{K,\sigma \leq \sigma},$$

and

$$v_\sigma = \Big| \int_\sigma v(x) \cdot n \, dx \Big|.$$

Then we have

$$\begin{aligned} \sum_{\sigma\in\mathcal{E}} v_\sigma u_{\sigma,+}(u_{\sigma,+} - u_{\sigma,-}) &= \frac{1}{2} \sum_{\sigma\in\mathcal{E}} v_\sigma \left((u_{\sigma,+} - u_{\sigma,-})^2 + (u_{\sigma,+} - u_{\sigma,-}) \right) \\ \sum_{\sigma\in\mathcal{E}} v_\sigma (u_{\sigma,+}^2 - u_{\sigma,-}^2) &= \int_\Omega (\operatorname{div} v)\, u_{\mathcal{T}^2}(x)\, dx \geq 0, \end{aligned}$$

since $\operatorname{div} v \geq 0$. Hence there holds

$$b\, \|u_{\mathcal{T}}\|^2_{L^2(\Omega)} + \|u_{\mathcal{T}}\|^2_{1,\mathcal{T}} = b \sum_{K\in\mathcal{T}} |K|\, u_K^2 + \sum_{\sigma\in\mathcal{E}} (D_\sigma u)^2\, \tau_\sigma \leq 0.$$

Thus we have existence.

Step 2, Estimates: $f \neq 0$ use step 1.

$$b\, \|u_{\mathcal{T}}\|^2_{1,\mathcal{T}} + \|u_{\mathcal{T}}\|^2_{1,\mathcal{T}} \leq \sum_{K\in\mathcal{T}} |K|\, f_K\, u_K.$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|u_{\mathcal{T}}\|^2_{1,\mathcal{T}} &\leq \Big(\sum_{K\in\mathcal{T}} |K|\, u_K^2 \Big)^{1/2} \Big(\sum_{K\in\mathcal{T}} |K|\, f_K^2 \Big)^{1/2} \\ &= \|u_{\mathcal{T}}\|_{L^2(\Omega)} \Big(\sum_{K\in\mathcal{T}} |K|\, f_K^2 \Big)^{1/2}. \end{aligned}$$

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[16] Definition of d_σ and $X(\mathcal{T})$?

[17] Proof of Lemma needs to be corrected.

Using the discrete Poincaré inequality we have

$$\|u_{\mathcal{T}}\|_{1,\mathcal{T}}^2 \leq \operatorname{diam}(\Omega)\, \|f\|_{L^2(\Omega)}$$

□

3.E. **Error Estimate.** We will now consider the error estimate for the multi-dimensional elliptic equation (3.3), i.e.,

$$-\triangle u(x,y) = f(x,y) \quad \text{for all } (x,y) \in \Omega$$
$$u(x,y) = 0 \quad \text{for all } (x,y) \in \partial\Omega.$$

Let $\mathcal{T} = (K_{i,j})_{i=1,\ldots,N_1\; j=1,\ldots,N_2}$ be an admissible mesh of $(0,1)\times(0,1)$ as above.

Proposition 3.2. Let $\Omega = (0,1)\times(0,1)$ and $f \in L^2(\Omega)$. Furthermore, let u be a unique solution of (3.3) under the assumptions made at the beginning of this section. Let $\xi > 0$ such that $h_i > \xi h$ and $k_j > \xi h$ for $i = 1,\ldots,N_1$ and $j = 1,\ldots,N_2$. Then there exists a unique discrete solution $(u_{i,j})_{i=1,\ldots,N_1,\, j=1,\ldots,N_2}$ and a constant $c > 0$ only depending on u, Ω, ξ such that

$$\sum_{i,j} \frac{(e_{i+1,j} - e_{i,j})^2}{h_{i+1/2}}\, k_j + \sum_{i,j} \frac{(e_{i,j+1} - e_{i,j})^2}{k_{i+1/2}}\, h_i \leq Ch^2 \tag{3.5}$$

and

$$\sum_{i,j} (e_{i,j})^2\, h_i k_j \leq Ch^2,$$

where $e_{i,j} = u(x_{i,j}) - u_{i,j}$.

Proof. Apply in x and y direction the one-dimensional case, using a structured grid and decoupling the directions.