MIXED DISCRETISATION METHODS FOR THE DISCONTINUOUS GALERKIN METHOD WITH ANALYTICAL TEST-FUNCTIONS

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Abstract. The idea of this paper is a framework for the Discontinuous Galerkin-methods for convection-diffusion-reaction-equations with improved test-functions. We develop an abstract theory for the stability and error-estimates for a mixed formulation. For diverse test-functions we apply our abstract theory.

Key words. Discontinuous Galerkin Methods, Stability, Error estimates, analytical solutions

AMS subject classifications. 65N12, 65N30

1. Introduction. Based on the idea for our future work we present a framework for the analytical test-function based on solving the adjoint-problem for the Discontinuous-Galerkin-methods in a mixed formulation (LDG-methods), confer [3].

First we derive an abstract theory for the stability and the error-estimates for an arbitrary test-function. In a second part we apply our results with respect to the analytical test-functions and get the improved results for the approximate solutions.

2. Mathematical model and mathematical equations. The mathematical model is based on a convection-diffusion-reaction equation. The model are transport and reaction process in a porous media, confer [1] and

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applied in our work [5].
The convection-diffusion-reaction-equation is given as

$$\partial_t R\, u \ +\ \nabla \cdot (\underline{v}\, u - a\, \nabla u) + \lambda\, R\, u = f \ \text{ in } \Omega_T \tag{2.1}$$

$$u(0) = u_0 \ \text{ on } \Omega \tag{2.2}$$

$$u = 0 \ \text{ on } \Gamma_T\ , \tag{2.3}$$

whereby the parameter $\underline{v}$ is a smooth velocity, with $\nabla \cdot \underline{v} = 0$, a is symmetric positive definite, bounded tensor given as diffusion and $\lambda \geq 0$ is the constant decay-rate. $R \geq 0$ is the constant retardation-factor. The definition for the domains are $\Omega_T = \Omega \times [0, T]$ where $T > 0$ and $\Omega \subset I\!R^d$ and d is the space-dimension.

In the next section we describe the weak formulation of our equations, confer [4].

2.1. Weak formulation for a mixed method. We use the notation $\underline{p} = a^{1/2}\ \nabla u$ and could reformulate in a mixed method. The diffusion-term is formulated in a mixed method for the further mixed discretisation methods. The solution is given by $u(x,t) \in C^2(\Omega) \times C^1([0,T])$ and $\underline{p}(x,t) \in (C^2(\Omega) \times C^1([0,T]))^d$ for the classical formulation

$$\partial_t Ru + \nabla \cdot \underline{v}\ u - \nabla \cdot a^{1/2}\ \underline{p} + R\lambda\ c = f\ ,\ \text{ in } \Omega_T \tag{2.4}$$

$$-a^{1/2}\ \nabla u + \underline{p} = 0\ ,\ \text{ in } \Omega_T \tag{2.5}$$

$$u(0) = u_0\ ,\ \text{ in } \Omega$$

$$u = 0\ ,\ \text{ on } \Gamma_T\ ,$$

We find the weak-formulation $u(x,t) \in L^2(H^1(\Omega), [0,T])$ and $\underline{p}(x,t) \in (L^2(H^1(\Omega), [0,T]))^d$

$$\int_\Omega \partial_t Ru\ \phi\ dx\ +\ \int_\Gamma (\underline{v} \cdot \underline{n}\ u)\ \phi\ ds\ -\ \int_\Omega u\ (\underline{v} \cdot \nabla\ \phi)\ dx \tag{2.6}$$

$$- \int_\Gamma (a^{1/2}\ \underline{p} \cdot \underline{n})\ \phi\ ds\ +\ \int_\Omega a^{1/2}\ \underline{p} \cdot \nabla \phi\ dx\ +\ \int_\Omega R\lambda\ u\ \phi\ dx\ =\ \int_\Omega f\ \phi\ dx\ ,$$

$$- \int_\Gamma u\ (\underline{n} \cdot a^{1/2}\ \underline{\chi})\ ds\ +\ \int_\Omega u\ (\nabla \cdot a^{1/2}\ \underline{\chi})\ dx\ +\ \int_\Omega \underline{p} \cdot \underline{\chi}\ dx\ =\ 0\ , \tag{2.7}$$

$$\int_\Omega u(0)\ \phi\ dx\ =\ \int_\Omega u_0\ \phi\ dx\ ,\ u = 0\ ,\ \text{on } \Gamma_T\ ,$$

for all $\phi \in H^1(\Omega)$ and for all $\underline{\chi} \in (H^1(\Omega))^d$.

3. Discretisation method with Discontinuous-Galerkin methods.

3.1. Broken Sobolev Spaces. We introduce in the following notation the multi-dimensional case of the Discontinuous-Galerkin methods.

We use the triangulation $\mathcal{K}_h$ for $h > 0$ for the domain Ω. We have for each sub-domain $K \in \mathcal{K}_h$ a Lipschitz boundary. The adjacent elements of $\mathcal{K}_h$ could be lie on an edge or a face. $\mathcal{E}_h^i$ is the set of all interior boundaries e of $\mathcal{K}_h$ and $\mathcal{E}_h^b$ is the set of all exterior boundaries e of $\Gamma = \partial\Omega$, whereby $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$. The exterior boundaries could be imposed as Dirichlet- $\mathcal{E}_h^1$ on Γ^1, Neumann-boundary or inflow- and outflow boundaries $\mathcal{E}_h^2$ on Γ^2 .

We define the broken Sobolev-space by:

$$H^l(\mathcal{K}_h) = \{ v \ \in \ L^2(\Omega) \ : \ v|_K \in H^l(K) \ \ \forall K \in \mathcal{K}_h \} \ . \tag{3.1}$$

whereby $l \geq 0$ is the order of the Sobolev-space and we have the the H_l-norm:

$$||v||_{H^l(\mathcal{K}_h)} = \left(\sum_{K \in \mathcal{K}_h} ||v||^2_{H^l(K)} \right)^{1/2} \ . \tag{3.2}$$

We have the following scalar-products :

$$(u, \phi)_S = \int_S u \ \phi \ ds \ , \ < \underline{p}, \underline{q} >_S = \sum_{i=1}^d (p_i, q_i)_S \ , \tag{3.3}$$

We have then the jumps across the edge $e = \partial K_1 \cap \partial K_2$:

$$[v] = (v|_{K_2})|_e - (v|_{K_1})|_e \ . \tag{3.4}$$

We also have the averages on the interfaces

$$\{v\} = \frac{(v|_{K_2})|_e + (v|_{K_1})|_e}{2} \ . \tag{3.5}$$

and the following conventions for the boundary Γ

$$\{v\} = v|_e \ . \tag{3.6}$$

$$[v] = \begin{cases} 0 \ , & e \in \mathcal{E}_h^D \\ v \ , & e \in \mathcal{E}_h^N \end{cases} \tag{3.7}$$

For further transformation we use the identity for the jumps:

$$[uv] = [u]\{v\} + \{u\}[u] \ . \tag{3.8}$$

Therefore the fluxes are given by

$$h_{conv}(u) = \{u \, \underline{v} \, \underline{n}\} \tag{3.9}$$
$$h_{diff}(\underline{w}) = (a^{1/2} \, \underline{n} \, \{u\}, \{a^{1/2} \, \underline{p} \cdot \underline{n}\})^t \tag{3.10}$$

whereby we have $\underline{w} = (u, \underline{p})^t$ and our formulation are the central fluxes and we get the formulation in the following bilinear-forms.

$$A(u, \phi) = - \sum_{K \in \mathcal{K}_h} (u, \underline{v} \cdot \nabla \phi)_K + \sum_{e \in \mathcal{E}_h} (h_{conv}(u), [\phi])_e \; , \tag{3.11}$$
$$B(\underline{p}, \phi) = \sum_{K \in \mathcal{K}_h} < a^{1/2} \, \underline{p}, \nabla \phi >_K - \sum_{e \in \mathcal{E}_h} (h_{diff}(\underline{p}), [\phi])_e \; , \tag{3.12}$$
$$B^T(u, \underline{\chi}) = \sum_{K \in \mathcal{K}_h} (u, \nabla \cdot a^{1/2} \, \underline{\chi})_K - \sum_{e \in \mathcal{E}_h} < h_{diff}(u), [\underline{\chi}] >_e \; , \tag{3.13}$$
$$C(u, \phi) = (R \, \lambda \, u, \phi) \; , \; D(\underline{p}, \underline{\chi}) = < \underline{p}, \underline{\chi} > \; , \; F(\phi) = (f, \phi) \; . \tag{3.14}$$

Now the weak formulation has the following compact forms :
Find $u(t) \in V$ and $\underline{p}(t) \in W$ such that,

$$\begin{aligned} (R \, \dot{u}, \phi) + A(u, \phi) + B(\underline{p}, \phi) + C(u, \phi) &= F(\phi) \; , \; \phi \in V \; , \; t > 0 \; , \\ D(\underline{p}, \underline{\chi}) + B^T(u, \underline{\chi}) &= 0 \; , \; \underline{\chi} \in W \; , \; t > 0 \; , \\ (u(0), \phi) &= (u_0, \phi) \; , \; \phi \in V \; , \; t = 0. \end{aligned}$$

3.2. Discrete Formulation for local spaces. We based our stability and error-indicator to a local space $\mathcal{Q}(K)$ with arbitrary functions :

$$\mathcal{Q}(\mathcal{K}_h) = \{v \; \in \; L^2(\Omega) \; : \; v|_K \in \mathcal{F}(K) \; \; \forall K \in \mathcal{K}_h\} \; , \tag{3.15}$$

whereby $\mathcal{F}$ is finite dimensional space (e.g. polynomials, exponential functions, etc.).

We have to find the unknowns $\underline{p}_h(t) \in (L_2(\mathcal{Q}(\mathcal{K}_h), [0, T]))^d$ and $u_h(t) \in L_2(\mathcal{Q}(\mathcal{K}_h), [0, T])$ as follows :

$$\begin{aligned} (\dot{u}_h, \phi) + \hat{A}(u_h, \phi) + \hat{B}(\underline{p}_h, \phi) + \hat{C}(u_h, \phi) &= \hat{F}(\phi) \; , \; \phi \in V_h \; , \; t > 0 \; , \\ \hat{D}(\underline{p}_h, \underline{\chi}) + \hat{B}^T(u_h, \underline{\chi}) &= \hat{G}(\underline{\chi}) \; , \; \underline{\chi} \in W_h \; , \; t > 0 \; , \\ (u(0), \phi) &= (u_0, \phi) \; , \; \phi \in V_h \; , \; t = 0 \; , \end{aligned}$$

whereby

$$\hat{A}(u_h, \phi) = -\sum_{K \in \mathcal{K}_h} (u_h, \underline{v} \cdot \nabla \phi)_K + \sum_{e \in \mathcal{E}_h} (\hat{h}_{conv}(u_h), [\phi])_e \,, \tag{3.16}$$

$$\hat{B}(\underline{p}_h, \phi) = \sum_{K \in \mathcal{K}_h} < a^{1/2} \, \underline{p}_h, \nabla \phi >_K - \sum_{e \in \mathcal{E}_h} (\hat{h}_{diff}(\underline{p}_h), [\phi])_e \,, \tag{3.17}$$

$$\hat{B}^T(u_h, \underline{\chi}) = \sum_{K \in \mathcal{K}_h} (u_h, \nabla \cdot a^{1/2} \, \underline{\chi})_K - \sum_{e \in \mathcal{E}_h} < \hat{h}_{diff}(u_h), [\underline{\chi}] >_e \,, \tag{3.18}$$

whereby the bilinear-forms $C = \hat{C}$, $D = \hat{D}$, $F = \hat{F}$ are equal and the fluxes are defined as

$$\hat{h}_{conv}(u_h) = \begin{cases} \{u_h \, \underline{v} \, \underline{n}\} & \text{central differences} \\ \{u_h \, \underline{v} \, \underline{n}\} - \frac{|\underline{v} \, \underline{n}|}{2}[u_h] & \text{upwind} \end{cases} \tag{3.19}$$

$$\hat{h}_{diff}(\underline{w}_h) = (a^{1/2} \, \underline{n} \, \{u_h\}, \{a^{1/2} \, \underline{p}_h \cdot \underline{n}\})^t + C_{diff}[(u_h, \underline{p}_h)^t] \tag{3.20}$$

whereby the flux-matrix C_{diff} is given as

$$C_{diff} = \begin{pmatrix} 0 & -\underline{c}^T \\ \underline{c} & 0 \end{pmatrix} \tag{3.21}$$

whereby $\underline{c} = (c_{1,2}, \ldots, c_{1,d+1})^T$, and $c_{1,i} = c_{1,i}((\underline{w}_h|_{K_2})|_e, \underline{w}_h|_{K_1})|_e$ is locally Lipschitz.

We denote the anti-symmetry for the bilinear-form $\hat{B}$.

LEMMA 3.1. *We have the Bilinear-form $\hat{B}(\underline{p}_h, u_h)$ as defined in equation (3.17) and we have the solutions $u_h(t) \in V_h$, $\underline{p}_h(t) \in W_h$. We assume for the diffusive flux the central flux. For such assumption we have anti-symmetry of the bilinear-form $\hat{B}$:*

$$\hat{B}(\underline{p}_h, u_h) = -\hat{B}^T(u_h, \underline{p}_h) \tag{3.22}$$

4. Stability of the scheme. We will concentrate us to the multi-dimensional case and proof the stability for arbitrary test-functions.

We derive the stability from the given bilinear-forms (3.16), (3.16) and (3.16).

We add the equations (3.16) and (3.16) and get the following results:

$$E_h(\underline{w}_h, \underline{\psi}) := (R \, \dot{u}_h, \phi) + \hat{A}(u_h, \phi) + \hat{B}(\underline{p}_h, \phi) \tag{4.1}$$
$$+\hat{C}(u_h, \phi) + \hat{D}(\underline{p}_h, \underline{\chi}) + \hat{B}^T(u_h, \underline{\chi}) \,,$$

whereby $\underline{w}_h = (u_h, \underline{p}_h)^T$ and $\underline{\psi} = (\phi, \underline{\chi})^T$.

Applying the lemma 3.1 we get :

$$E_h(\underline{w}_h, \underline{w}_h) \;=\; (R\, \dot{u}_h, u_h) + \Theta_C(\underline{w}_h, \underline{w}_h) + < \underline{p}_h, \underline{p}_h > + \lambda(u_h, u_h)\ .$$

whereby $\Theta_C(\underline{w}_h, \underline{w}_h)$ is given as

$$\Theta_C(\underline{w}_h, \underline{w}_h) = \sum_{e \in \mathcal{E}_h} < [\underline{w}_h]\ ,\ C\ [\underline{w}_h] >_e\ , \tag{4.2}$$

for C we have

$$C = \begin{pmatrix} c_{1,1} & -\underline{c}^T \\ \\ \underline{c} & 0 \end{pmatrix}\ , \tag{4.3}$$

and $c_{1,1} = 0$ for central differences for the convective flux or $c_{1,1} = \frac{|\underline{v}\ \underline{n}|}{2}$ for the upwind scheme for the convective flux. We get $c_{1,i} = 0$ for a 5 point central difference scheme for the diffusion flux, for $i = 2, \ldots, d+1$ and $c_{1,i} = \frac{a_{i-1}^{1/2}}{2}$ for a upwind scheme for the diffusive flux for $i = 2, \ldots, d+1$.

We applying the definitions we get the following stability-result :

Corollary 4.1. *We have the stability for the full-discrete form with the solutions $u_h \in V$ and $\underline{p}_h \in W$ such that*

$$\begin{aligned} R\,\frac{1}{2}||u_h(T)||^2_{L^2(\Omega)} + \int_0^T\ \Theta_C(\underline{w}_h, \underline{w}_h)\ dt + \int_0^T\ ||\underline{p}_h||^2_{(L^2(\Omega))^d}\ dt \\ + \int_0^T R\,\lambda\ ||u_h||^2_{L^2(\Omega)}\ dt \leq \frac{1}{2}||u_h(0)||^2_{L^2(\Omega)} + \int_0^T\ ||f||^2_{L^2(\Omega)}\ dt\ . \end{aligned} \tag{4.4}$$

In the next section derive the abstract error estimates.

5. Abstract Error-Estimates. The error estimator for the multi-dimension for the convection-diffusion-reaction equation is based on our former stability assumptions.

The error-estimator for multi-dimension for the convection-diffusion-reaction term is given as

Theorem 5.1. *The error-estimates is given as follows :*

If $u, \underline{p}$ and $u_h, \underline{p}_h$ are respective solutions of (2.4) and (2.5) and $\mathcal{P}_h$ is the L^2-projection $\mathcal{P}_h : L^2(V) \to V_h$ and $\mathcal{P}_h : L^2(W) \to W_h$. We get the *error-estimates such that*

$$R\,\frac{1}{2}||u(T)-u_h(T)||^2_{L^2(\Omega)} \;+\int_0^T \;||\underline{p}-\underline{p}_h||^2_{L^2(\Omega)^d}\,dt \tag{5.1}$$
$$+\int_0^T \;\Theta_C(\underline{w}-\underline{w}_h,\underline{w}-\underline{w}_h)\,dt \;+\;\int_0^T\;\frac{R\,\lambda}{2}\;||u-u_h||^2_{L^2(\Omega)}\,dt$$
$$\leq c\int_0^T \;\big(\;||\mathcal{P}_h(\dot u(t))-\dot u(t)||^2_{L^2(\Omega)} \;+\;\Theta_C(\mathcal{P}_h(\underline{w})-\underline{w},\mathcal{P}_h(\underline{w})-\underline{w})$$
$$+\sum_{e\in\mathcal{E}_h}\big(\;||\{\mathcal{P}_h(u)-u)\}||^2_{L^2(e)}+||\{\mathcal{P}_h(\underline{p})-\underline{p}\}||^2_{(L^2(e))^d}\;\big)$$
$$+\sum_{K\in\mathcal{K}_h}\alpha(h_K)^{-2}(||\mathcal{P}_h(u)-u||^2_{L^2(K)}\;+\;||\mathcal{P}_h(\underline{p})-\underline{p}||^2_{(L^2(K))^d}\;)\;\big)\,dt$$

whereby c is a constant, independent from t. The function $\alpha(h_K)$ depend from the test-spaces and are specified in the application in the next sections.

6. Application of the Error-Estimates.

6.1. Application for standard Polynomial-space. Our first application for the discontinuous finite element space is the polynomial spaces is defined as

Let $\Omega\subset I\!R^d$ be a polynomial domain and let K_h be a regular finite element partition of Ω. We define

$$\mathcal{D}_r(\mathcal{K}_h)=\{v\;\in\;L^2(\Omega)\;:\;v|_K\in\mathcal{P}^r(K)\;\;\forall K\in\mathcal{K}_h\}\;. \tag{6.1}$$

whereby $\mathcal{P}^r$ is the set of polynomials of degree at most r on K.

THEOREM 6.1. *Error-Estimates for the polynomial space, where h is the diameter of the element. We have the estimates for the terms such that*

$$R\,\frac{1}{2}||u(T)-u_h(T)||^2_{L^2(\Omega)} \;+\int_0^T ||\underline{p}-\underline{p}_h||^2_{(L^2(\Omega))^d}\,dt \tag{6.2}$$
$$+\int_0^T \Theta_C(\underline{w}-\underline{w}_h,\underline{w}-\underline{w}_h)\,dt+\;R\,\lambda\int_0^T ||u-u_h||^2_{L^2(\Omega)}\,dt$$
$$\leq c\left(\;h^{2r}\;||u||^2_{H^{r+1}(\Omega)}\;+\;h^{2r}\;||\underline{p}||^2_{(H^{r+1}(\Omega))^d}\;\right)$$

whereby we get $\alpha(h_K)=h_K$.

6.2. Application for special function spaces with respect to local analytical solutions.

6.2.1. Local test-functions in space for one dimension. We derive the one-dimensional solutions for the local convection-reaction equation, given as adjoint problem

$$-v\,\partial_x \phi + R\,\lambda\,\phi = 0\;, \tag{6.3}$$

where by $\phi(0) = a_0$ is a constant.

We could solve the equation (6.3) exactly and get the solution with respect to the velocity $v \in I\!R$ and $v \neq 0$

$$\phi_{anal,i}(x) = a_0 \begin{cases} \exp(-\beta(x_{i+1/2} - x)) & v > 0 \\ \\ \exp(-\beta(x - x_{i-1/2})) & v < 0 \end{cases} \tag{6.4}$$

and and $\beta = \frac{R\,\lambda}{|v|}$ whereby

$$\phi_{new,i} = \phi_{anal,i}(x)\;, \tag{6.5}$$

whereby $x_{i-1/2} < x < x_{i+1/2}$. For this test-function we have one freedom-degree, so that we could use only a constant initial condition.

For linear initial-conditions we use the analytical test-function and multiply it with the standard-test-function of first order.

Such that

$$\phi_{new,i} = \phi_{stand,i}(x)\;\phi_{anal,i}(x)\;, \tag{6.6}$$

whereby the standard test-functions are given as polynomial-functions $\phi_{stand}(x) = \{1, x, x^2, \ldots\}$.

6.2.2. Error-estimates for the new test-functions with analytical test-functions. We derive in the following section the error-estimates for our analytical solutions.

The idea came from the infimum of the error between the exact and the numerical solution using analytical test-functions..

Our new test-space $\mathcal{F}$ is as follows

$$\mathcal{F}_r = \{\exp(-\beta x), x\;\exp(-\beta x), \ldots, x^r \exp(-\beta x)\} \tag{6.7}$$

whereby we assume a one-dimensional problem, and $\beta = \frac{R\lambda}{|v|}$ and we transform to the local coordinates to $0 < x < h$. We derive an error-estimates for the exponential space, whereby h is our grid-width. We have therefore a suboptimal estimates such that

THEOREM 6.2.

$$R \, \frac{1}{2} ||u(T) - u_h(T)||_{L^2(\Omega)} \; + \int_0^T \; ||\underline{p} - \underline{p}_h||_{L^2(\Omega)^d} \; dt \tag{6.8}$$
$$+ \int_0^T \; \big(\, \Theta_C(\underline{w} - \underline{w}_h, \underline{w} - \underline{w}_h) \, \big)^{1/2} \, dt + \; \frac{R \, \lambda}{2} \; \int_0^T ||u - u_h||_{L^2(\Omega)} \; dt$$
$$\leq \; c \; \int_0^T \; \big(\, h^r \; (\; ||u||_{H^{r+1}(\Omega)} \; + \; ||\underline{p}||_{((H^{r+1}(\Omega))^d} \;) \, \big) \; dt \; .$$

whereby c is a constant independent from t and we get $\alpha(h_K) = h$ for the estimation of the derivatives.

We get an optimal error-estimate for the pure convection-equation, i.e. $\underline{c} = (0, \ldots, 0)^T$. We have therefore an optimal estimates such that

THEOREM 6.3.

$$R \, \frac{1}{2} ||u(T) - u_h(T)||_{L^2(\Omega)} \; + \; \int_0^T \; \big(\, \Theta_C(\underline{w} - \underline{w}_h, \underline{w} - \underline{w}_h) \, \big)^{1/2} \; dt \tag{6.9}$$
$$+ \; \frac{R \, \lambda}{2} \; \int_0^T ||u - u_h||_{L^2(\Omega)} \; dt \; \leq \; c \; \int_0^T \; \big(\, h^{r+1/2} \; ||u||_{H^{r+1}(\Omega)} \, \big) \; dt \; .$$

whereby c is a constant independent from t and we get $\alpha(h_K) = 0$ and could skip the estimation of the derivatives.

7. Conclusions. We discuss a new discretisation method based on the local Discontinuous Galerkin method (LDG) with improved test-functions. We derive the stability and the error-estimates for the new discretisation method. The new test-functions are derived from the adjoint problem with respect to the standard test-functions. We introduce the error-estimates for the new test-functions. We get an sub-optimal error-estimates for the polynomial-space and an optimal error-estimates for the new analytical test-spaces.

In future works we would generalise our results for the different Discontinuous Galerkin methods.

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