Overlapping Operator-Splitting Methods and Applications to Stiff Differential Equations

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Contribution of the Overlapping Operator-Splitting Methods

The overlapping method is an extension to the iterative operator-splitting method. The efficiency of considering overlapping methods instead of standard methods occurs in decompositions into simpler problems. Theoretically the overlapping method can be treated as an abstract Cauchy problem. The applications are stiff linear and nonlinear parabolic partial differential equations.

Let us consider the semi-discretized linear differential equation in a Banach space $\mathbf{X}$:

$$\frac{dc(t)}{dt} = A\,c(t) + B\,c(t), \quad 0 < t \leq T, \; c(0) = c_0,$$

where $A, B : \mathbf{X} \to \mathbf{X}$ are given linear operators representing spatial discretized operators, e.g. convection terms.
The following algorithm iterates with a fixed splitting discretization step size τ. On the time interval $[t^n, t^{n+1}]$, we solve the following subproblems consecutively for $i = 1, 3, \dots 2m+1$ and $j = 1, 3, \dots 2m+1$. In this notation, i represents the iteration index for the temporal splitting and j represents the iteration index for the spatial splitting.

$$\frac{dc_{i,j}(t)}{dt} = A|_{\overline{\Omega}_1} c_{i,j}(t) \;+\; A|_{\overline{\Omega}_2} c_{i,j-1}(t) \;+\; B|_{\overline{\Omega}_1} c_{i-1,j}(t) \;+\; B|_{\overline{\Omega}_2} c_{i-1,j-1}(t), \tag{1}$$
with $c_{i,j}(t^n) = c^n$,

$$\frac{dc_{i+1,j}(t)}{dt} = A|_{\overline{\Omega}_1} c_{i,j}(t) \;+\; A|_{\overline{\Omega}_2} c_{i,j-1}(t) \;+\; B|_{\overline{\Omega}_1} c_{i+1,j}(t) \;+\; B|_{\overline{\Omega}_2} c_{i-1,j-1}(t), \tag{2}$$
with $c_{i+1,j}(t^n) = c^n$,

$$\frac{dc_{i,j+1}(t)}{dt} = A|_{\overline{\Omega}_1} c_{i,j}(t) \;+\; A|_{\overline{\Omega}_2} c_{i,j+1}(t) \;+\; B|_{\overline{\Omega}_1} c_{i+1,j}(t) \;+\; B|_{\overline{\Omega}_2} c_{i-1,j-1}(t), \tag{3}$$
with $c_{i,j+1}(t^n) = c^n$,

$$\frac{dc_{i+1,j+1}(t)}{dt} = A|_{\overline{\Omega}_1} c_{i,j}(t) \;+\; A|_{\overline{\Omega}_2} c_{i,j+1}(t) \;+\; B|_{\overline{\Omega}_1} c_{i+1,j}(t) \;+\; B|_{\overline{\Omega}_2} c_{i+1,j+1}(t), \tag{4}$$
with $c_{i+1,j+1}(t^n) = c^n$,

where $c_{0,0}(t), c_{1,0}(t)$ and $c_{0,1}(t)$ are fixed functions, e.g. $c_{0,0}(t) = c_{1,0}(t) = c_{0,1}(t) = 0$, for each iteration. c^n is the known split approximation at the time level $t = t^n$, cf. [Farago and Geiser 2005]. The boundary conditions are Neumann conditions, that are embedded in the equations. We have the domain Ω with $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \Omega_{1,2}$ and the restriction to each operator, i.e. $A|_{\Omega_i}, B|_{\Omega_i}$ with $i = 1, 2$ for two subdomains. The coupling is done at the intermediate boundaries $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \Gamma_{\Omega_1,\Omega_2}$ ($\Omega_{1,2} = \emptyset$, i.e. mass zero) for the non-overlapping method, or at the overlapping set ($\Omega_{1,2} \neq \emptyset$, i.e. of mass 1) for the overlapping method.

Theorem 1. *Let us consider the linear operator equation in a Banach space* $\mathbf{X}$*:*

$$\frac{dc(t)}{dt} = A_1\,c(t) + A_2\,c(t) + B_1\,c(t) + B_2\,c(t), \quad 0 < t \leq T,$$
$$c(0) = c_0,$$

where $A_1, A_2, B_1, B_2, A_1 + A_2 + B_1 + B_2 : \mathbf{X} \to \mathbf{X}$ *are given linear operators being generators of the* C_0*-semigroup and* $c_0 \in \mathbf{X}$ *is a given element. Then the iteration process (1)–(4) is convergent with a convergence rate of one.*
We obtain the iterative result: $\|e_{i,j}(t)\| \leq K\tau_n \|e_{i-1,j-1}(t)\| + \mathcal{O}(\tau_n^2)$*, where* $\tau_n = t^{n+1} - t^n$ *and* $t \in [t^n, t^{n+1}]$.

Example: Partial differential equation

We deal with the time-dependent 2D equation:

$$\partial_t u(x,y,t) = Au + Bu\,, \; \text{ in } \Omega = [-1,1] \times [-1,1],$$
$$Au = u_{xx} + u_{yy}, \; Bu = -4(1+y^2)e^{-t}e^{x+y^2},$$
$$u(x,y,0) = e^{x+y^2}, \; \text{ on } \Omega,$$
$$u(x,y,t) = e^{-t}e^{x+y^2}, \; \text{ on } \partial\Omega \times [-1,1],$$

with exact solution

$$u(x,y,t) = e^{-t}e^{x+y^2}.$$

We choose the time interval $[0,1]$ and again use finite differences for the space with $\Delta x = 2/19$.

We define our operators by spatial splitting as: $A_1 = A|_{\Omega_1 = [-1,0] \times [-1,1]}$ and $A_2 = A|_{\Omega_2 = [0,1] \times [-1,1]}$.

Numerical results for the overlapping iterative operator-splitting method (BDF3 with $\tau = 10^{-1}$).

Iteration steps	Number of splitting partitions	Max-error
1	2	2.7183e+000
4	2	2.5147e+000
10	1	6.8750e-001
20	1	8.7259e-002
30	1	5.3147e-003

Graphical results,
left : starting solution (decoupled domains)
right : solution after 20 iterations (smoothed domains)