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BRIEF COMMUNICATIONS

Continuity and Asymptotic Behavior of Integral Kernels Related to Schrödinger Operators on Manifolds

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The continuity of integral kernels related to Schrödinger operators (the kernel of the heat equation, Green's function) plays an important role in the study of different properties of quantum systems. In the Euclidean case, it was shown in [1] that sufficiently general functions of the Schrödinger operator have continuous integral kernels for scalar potentials belonging to the Kato class. In [2], this result was generalized to operators with nontrivial vector potential of the magnetic field; in this case, arbitrary domains in the Euclidean space were admitted as configuration spaces. Simultaneously, in view of several problems in mesoscopy and gravitation quantization [3, 4], it seems to be of interest to study the integral kernels related to the Schrödinger operators on Riemann manifolds. In this paper, we presents several results concerning the existence and continuity of integral kernels for different operators generated by the Schrödinger operator on a manifold. In addition, we give several estimates for Green's function for the cases in which its arguments are far from one another or, conversely, close to one another.

Let X be a ν-dimensional manifold of bounded geometry. By $d(x,y)$ we denote the geodesic distance between points $x,y\in X$; by $B(x,r)$ we denote an open ball of radius r centered at x; and by D we denote the set $\{(x,y)\in X\times X,\ x=y\}\subset X\times X$. We consider a 1-form A on X with smooth coefficients; this form determines the connection in a trivial linear bundle over X; by Δ_A we denote the corresponding Bochner Laplacian (for $A=0$, we obtain $\Delta_A=\Delta$, which is the Beltrami–Laplace operator). The method used to prove our results requires several (rather weak) restrictions on the scalar potentials under study; in what follows, we assume that this potential belongs to the class $\mathcal{P}(X)$ of real functions U on X with the properties

$$U_+ := \max(U,0)\in L^{p_0}_{\mathrm{loc}}(X) \qquad \text{and} \qquad U_- := \max(-U,0)\in \sum_{i=1}^{n} L^{p_i}(X),$$

where $2\le p_i\le\infty$ for $\nu\le 3$ and $\nu/2<p_i\le\infty$ for $\nu\ge 4$, $0\le i\le n$. By $H_{A,U}$ we denote the operator acting on $C_0^\infty(X)$ according to the rule $H_{A,U}\phi=-\Delta_A\phi+U\phi$. Using the results obtained in [5], we can prove the following assertion.

Proposition. *The operator $H_{A,U}$ introduced above is lower semibounded and essentially self-adjoint in $L^2(X)$.*

We preserve the same notation for the closure of $H_{A,U}$. This operator will be called the *Schrödinger operator* with vector potential A and scalar potential U; we denote its spectrum and the resolvent set by $\sigma(H_{A,U})$ and $\rho(H_{A,U})$, respectively.

The Schrödinger operator generates other operators closely related to it: the Schrödinger semigroup

$$P_{A,U}(t) := e^{-tH_{A,U}}, \qquad t > 0;$$

the resolvent

$$R_{A,U}(\zeta) := (H_{A,U} - \zeta)^{-1}, \qquad \zeta \in \rho(H_{A,U});$$

and powers of the resolvent defined for $\alpha > 0$ and $\operatorname{Re}\zeta < \inf\sigma(H_{A,U})$ by the rule

$$R^{\alpha}_{A,U}(\zeta) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-tH_{A,U}} e^{t\zeta} t^{\alpha-1}\, dt.$$

In particular, for a positive integer α, we obtain the usual powers of the resolvent defined for all $\zeta \in \rho(H_{A,U})$.

We are interested in the existence problem for integral kernels of the operators listed above and of operators related to them, as well as in the properties of these kernels such as continuity, integrability, etc. To describe these properties, we introduce the class

$$\mathcal{K}(\alpha, p), \qquad \alpha \in [0, \nu), \quad 1 \le p \le \infty,$$

consisting of kernels $K(x, y)$ locally integrable on $X \times X$ and satisfying the following conditions:

$$|K(x, y)| \le \max\left(\frac{c}{d(x, y)^{\alpha}}, 1\right) \qquad \text{for some} \quad c > 0,$$

$$\max\Bigl(\sup_{x\in X} \|\chi_{X\setminus B(x,r)} K(x, \,\cdot\,)\|_p, \sup_{x\in X} \|\chi_{X\setminus B(x,r)} K(\,\cdot\,, x)\|_p\Bigr) < \infty \qquad \text{for any} \quad r > 0.$$

The main results describing the global properties of integral kernels are contained in the following theorem.

Theorem 1. (1) *The operators* $R^{\alpha}_{A,U}(\zeta)$ *are integral operators. If* $G^{(\alpha)}_{A,U}(x, y; \zeta)$ *denotes their kernels, then the* $G^{(\alpha)}_{A,U}$ *are continuous in* $X \times X \setminus D$ *if one of the following conditions is satisfied*:

(a) $\operatorname{Re}\zeta$ *is sufficiently close to* $-\infty$;
(b) α *is integer and* $\zeta \in \rho(H_{A,U})$.

Moreover, for $\alpha > \nu/2$, *the kernels* $G^{(\alpha)}_{A,U}$ *are continuous in* $X \times X$ *and bounded.*

(2) *The kernel* $G^{(\alpha)}_{A,U}(x, y; \zeta)$ *belongs to* $\mathcal{K}(\lambda, p)$, *where* $1 \le p \le \infty$ *is arbitrary,* $\lambda = \nu - 2\alpha$ *for* $\alpha < \nu/2$, λ *is an arbitrary positive number for* $\alpha = \nu/2$, *and* $\alpha = 0$ *for* $\alpha > \nu/2$.

(3) *If* f *is a Borel function on* $\sigma H_{A,U}$ *and satisfies the condition* $f(\xi) \le b(1 + |\xi|)^{-\alpha}$ *with some* $b > 0$ *and* $\alpha > \nu/2$, *then the operator* $f(H_{A,U})$ *has an integral kernel* $F(x, y)$ *continuous in* $X \times X$ *and satisfying the uniform estimate* $|F(x, y)| \le Cb$, *where the constant* $C > 0$ *depends only on* α *but not on* b.

(4) *The operator* $P_{A,U}(t)$ *has an integral kernel* $P_{A,U}(x, y, t)$ *continuous and uniformly bounded in* $X \times X \times (0, \infty)$.

(5) *The spectral projection operator corresponding to any bounded Borel subset of the real axis has a continuous uniformly bounded integral kernel.*

(6) *The eigenfunctions of the operator* $H_{A,U}$ *are bounded and continuous.*

(7) *For integer k, the mapping $\zeta \mapsto G^{(k)}_{A,U}(x,y\,;\zeta)$ is holomorphic for all $x,y \in X$ if $k > \nu/2$ and for $x \ne y$ otherwise. In addition,*

$$\frac{\partial G^{(k)}_{A,U}(x,y\,;\zeta)}{\partial \zeta} = kG^{(k+1)}_{A,U}(x,y\,;\zeta), \qquad k = 1,2,\dots\,.$$

The asymptotic behavior of Green's function near the diagonal $x = y$ is also of some interest. More precisely, in perturbation theory methods based on the "contraction–dilatation" procedure [6] (in particular, in the zero-radius potential methods), Green's function is represented in the form

$$G_{A,U}(x,y\,;\zeta) = F(x,y) + G^{\mathrm{reg}}_{A,U}(x,y\,;\zeta),$$

where the function $G^{\mathrm{reg}}_{A,U}(x,y\,;\zeta)$ must be continuous on the entire space $X \times X$. In this case, $F(x,y)$ is called the singularity of Green's function on the diagonal. The problem of obtaining this representation in the context of the theory mentioned above is nontrivial only for the dimensions $\nu = 2$ and $\nu = 3$, because, for $\nu = 1$, one can assume that the singularity is zero, and, for $\nu > 3$, the singularity already depends on the spectral parameter. In this case, it is necessary to define the energy operator in a space with an indefinite metric, which is beyond the framework of the standard interpretation of quantum mechanics.

The preceding theorem gives only an upper bound for the singularity. A more detailed description is contained in the following theorem.

Theorem 2. (1) *Suppose that X is a two-dimensional space. Then, for any $A \in [C^\infty(X)]^2$ and $U \in \mathcal{P}(X)$, the singularity of Green's function $G_{A,U}$ on the diagonal coincides with a similar singularity of Green's function of the Beltrami–Laplace operator, i.e.,*

$$G_{A,U}(x,y\,;\zeta) = \frac{1}{2\pi}\log\frac{1}{d(x,y)} + G^{\mathrm{reg}}_{A,U}(x,y\,;\zeta),$$

where the second term is continuous in $X \times X$.

(2) *Suppose that X is three-dimensional, $A \in [C^\infty(X)]^3$, and $U, V \in \mathcal{P}(X)$. If the condition $U - V \in L^p_{\mathrm{loc}}(X)$ holds for some $p > 3$, then the singularities of Green's functions $G_{A,U}$ and $G_{A,V}$ coincide, i.e., the difference $G_{A,U}(x,y\,;\zeta) - G_{A,V}(x,y\,;\zeta)$ is continuous in $X \times X$ for any $\zeta \in \rho(H_{A,U}) \cap \rho(H_{A,V})$.*

In particular, the singularity of Green's function G_U of the operator $-\Delta + U$ (i.e., $A = 0$) for $U \in \mathcal{P}(X) \cap L^p_{\mathrm{loc}}(x)$ with any $p > 3$ coincides with the singularity of Green's function of the Beltrami–Laplace operator:

$$G_U(x,y\,;\zeta) = \frac{1}{4\pi d(x,y)} + G^{\mathrm{reg}}_U(x,y\,;\zeta),$$

where the second term is continuous in $X \times X$.

In the three-dimensional case, the singularity of Green's function depends, in general, on both the vector and scalar potentials and need not coincide with the singularity of Green's function of the Laplacian if the conditions given in item (2) of the preceding theorem are not satisfied. The corresponding examples can easily be constructed already in the case $X = \mathbb{R}^3$. Thus, Green's function $G(x,y\,;\zeta)$ for the Schrödinger operator with Coulomb potential $H = -\Delta + q/|x|$ (we note that $q/|x| \notin L^3_{\mathrm{loc}}(\mathbb{R}^3)$) has the following asymptotics near the charge:

$$G(x,0\,;\zeta) = \frac{1}{4\pi|x|} + \frac{q\log|x|}{4\pi} + k(x),$$

where k is a continuous function. The dependence of Green's function on the vector potential of the magnetic field appears already in the example of the Landau operator

$$H = \left(i\frac{\partial}{\partial x_1} - \pi\xi x_2\right)^2 + \left(i\frac{\partial}{\partial x_2} + \pi\xi x_1\right)^2 - \frac{\partial^2}{\partial x_3^2}, \qquad \xi \neq 0,$$

for which Green's function can be represented in the form

$$G(x, y\,;\,\zeta) = \frac{e^{i\pi\xi(x_1y_2 - x_2y_1)}}{4\pi|x-y|} + G^{\mathrm{reg}}(x, y\,;\,\zeta),$$

where the second term is continuous in $X \times X$.

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