GAFA, Geom. funct. anal.
Vol. 16 (2006) 767 – 837
1016-443X/06/040767-71
DOI 10.1007/s00039-006-0574-7
ONLINE FIRST: August 2006

GAFA Geometric And Functional Analysis

AN ANOMALY FORMULA FOR RAY–SINGER METRICS ON MANIFOLDS WITH BOUNDARY

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Abstract. Using the heat kernel, we derive first a local Gauss–Bonnet–Chern theorem for manifolds with a non-product metric near the boundary. Then we establish an anomaly formula for Ray–Singer metrics defined by a Hermitian metric on a flat vector bundle over a Riemannian manifold with boundary, not assuming that the Hermitian metric on the flat vector bundle is flat nor that the Riemannian metric has product structure near the boundary.

0 Introduction

Let X be a compact m-dimensional smooth manifold, not necessarily oriented, with boundary $\partial X = Y$ and canonical embedding $j : Y \to X$, and let F be a flat complex vector bundle over X, equipped with its flat connection ∇^F. We denote by $H^\bullet(X,F) = \bigoplus_{p=0}^m H^p(X,\, F)$ the (absolute) de-Rham cohomology of X with coefficients in F. For a finite dimensional vector space E, set $\det E := \Lambda^{\max} E$, and denote by $(\det E)^{-1} := \det E^*$ the dual line. Then it is customary to call the complex line,

$$\det H^\bullet(X,F) = \bigotimes_{p=0}^{m} \big(\det H^p(X,F)\big)^{(-1)^p}, \tag{0.1}$$

the determinant of the cohomology of F.

Next, choose a Hermitian metric h^F on F and a smooth Riemannian metric g^{TX} on TX. By Hodge–de Rham theory, the de Rham cohomology $H^\bullet(X,F)$ is canonically isomorphic to the kernel of an appropriately chosen Laplacian. Hence the chosen metrics induce a canonical L^2- metric $h^{H^\bullet(X,F)}$ on $H^\bullet(X,F)$. Then the Ray–Singer metric $\|\ \|^{\mathrm{RS}}_{\det H^\bullet(X,F)}$ on $\det H^\bullet(X,F)$

Keywords and phrases: Ray–Singer analytic torsion, anomaly formula, characteristic classes
AMS *Mathematics Subject Classification*: 58J52, 58J28, 58J35

is defined as the product of the metric induced on $\det H^\bullet(X,F)$ by $h^{H^\bullet(X,F)}$ with the Ray–Singer analytic torsion [RS], see §4.1.

If $Y=\emptyset$ and h^F is flat, then it is known that $\| \ \|^{\mathrm{RS}}_{\det H^\bullet(X,F)}$ does not depend on g^{TX}. Indeed, the celebrated Cheeger–Müller theorem [Ch], [Mü1] tells us that in this case the Ray–Singer metric can be identified with the so called Reidemeister metric which is a topological invariant of the flat bundle F [Mi]. Müller [Mü2] extended his result to the case where $m=\dim X$ is odd and only the metric induced on $\det F$ is required to be flat.

It is natural to ask whether the Ray–Singer metric will be independent of the chosen Hermitian metric on F even without any assumption of flatness. Bismut and Zhang [BiZ1] generalized this discussion to arbitrary flat vector bundles with arbitrary Hermitian metrics and showed that, in even dimensions, the independence ceases to hold. Instead, there is an anomaly formula: the logarithms of two Ray–Singer metrics differ by a local term involving new secondary invariants.

There are also various extensions to the equivariant case, cf. [LR], [Lü], [BiZ2]. Recently, Bismut and Goette [BiG], [Go] obtained a family version of the Bismut–Zhang theorem under the assumption that there exists a fiberwise Morse function for the fibration in question, which generalizes all the above results.

Now consider X with $Y\neq\emptyset$. This case was studied in [LR] and [Lü], [V], [H], under the assumption that h^F is flat and that g^{TX} has product structure near the boundary. Dai and Fang [DF] were the first to study this problem with h^F flat but without assuming a product structure for g^{TX} near Y; they used methods completely different from ours. In [LüS], Lück and Schick computed some examples to show that the boundary term does not vanish in general. We learned about this problem from [DF] which was very helpful for us even though we investigate the problem from a rather different perspective. As an interesting by-product, our reasoning also leads to a local Gauss–Bonnet–Chern theorem [Ch1,2], see Theorem 3.2 below, for manifolds without a product structure near the boundary. As a notable feature of our proof, we derive the boundary contribution from the fundamental solution of the model problem.

Let us now describe the geometric setting in greater detail. Let $\| \ \|_{\det F}$ be the metric on the line bundle $\det F$ induced by h^F, and let g^{TY} be the metric on TY induced by g^{TX}. Denote by ∇^{TX} and ∇^{TY} the Levi–Civita connection on (TX,g^{TX}) and (TY,g^{TY}), respectively, with curvature R^{TX} and R^{TY}. Let $o(TX)$ be the orientation bundle of TX. Let $e(TX,\nabla^{TX})$,

$e(TY,\nabla^{TY})$ be the associated representatives of the Euler class of TX, TY in Chern–Weil theory, such that $e(TX,\nabla^{TX})$ is an $o(TX)$-valued closed m-form on X. To account for the boundary, we introduce in (1.17), (1.19) certain forms on Y, $e_b(Y,\nabla^{TX})$ and $B(\nabla^{TX})$, such that $e_b(Y,\nabla^{TX})$ is an $o(TY)$-valued $m-1$-form on Y. If m is odd, then $e(TX,\nabla^{TX})=0$ and $e_b(Y,\nabla^{TX})=\frac{1}{2}e(TY,\nabla^{TY})$.

Let $\Omega(X,o(TX)),\Omega(Y,o(TX))$ be the $o(TX)$-valued C^∞ forms on X,Y. The algebraic mapping cone of $j^*:\Omega(X,o(TX))\to\Omega(Y,o(TX))$ is defined as the following object: we put $\Omega^p(X,Y,o(TX)):=\Omega^p(X,o(TX))\oplus\Omega^{p-1}(Y,o(TX))$ and define, for $(\sigma_1,\sigma_2)\in\Omega^p(X,Y,o(TX))$, the differential by $d(\sigma_1,\sigma_2):=(d^X\sigma_1, j^*\sigma_1-d^Y\sigma_2)$; then the complex $(\Omega^\bullet(X,Y,o(TX)),d)$ calculates the relative cohomology $H^\bullet(X,Y,o(TX))$ [BoT, p. 78]. For $(\sigma_1,\sigma_2)\in\Omega(X,Y,o(TX))$, $\sigma_3\in\Omega(X)$, we define a nonsingular pairing

$$\int_{(X,Y)}(\sigma_1,\sigma_2)\wedge\sigma_3:=\int_X\sigma_1\wedge\sigma_3-\int_Y\sigma_2\wedge j^*\sigma_3\,;\tag{0.2}$$

this induces the Poincaré duality $H^\bullet(X,Y,o(TX))\times H^\bullet(X,\mathbb{R})\to\mathbb{R}$. By Theorem 1.9, the relative differential form

$$E(TX,\nabla^{TX}):=\big(e(TX,\nabla^{TX}),e_b(Y,\nabla^{TX})\big)\tag{0.3}$$

is closed in $\Omega(X,Y,o(TX))$ and defines the relative Euler class of TX, and $E(TX):=[E(TX,\nabla^{TX})]\in H^m(X,Y,o(TX))$ does not depend on the choice of g^{TX}.

Next let (g_0^{TX},h_0^F) and (g_1^{TX},h_1^F) be two couples of metrics on (TX,F). We will use the subscripts $0,1$ to distinguish various objects associated with these metrics. In Theorem 1.9, we show that there exists a canonical class

$$\widetilde{E}\big(TX,\nabla_0^{TX},\nabla_1^{TX}\big)\in\Omega^{m-1}\big(X,Y,o(TX)\big)/d\Omega\big(X,Y,o(TX)\big)$$

such that

$$d\widetilde{E}\big(TX,\nabla_0^{TX},\nabla_1^{TX}\big)=E\big(TX,\nabla_1^{TX}\big)-E\big(TX,\nabla_0^{TX}\big)\,.\tag{0.4}$$

It is obvious from (0.4) that $\widetilde{E}$ defines the secondary relative Euler class of TX in the spirit of Chern–Simons theory.

Let $\theta(F,h_1^F)=\mathrm{Tr}[(h_1^F)^{-1}\nabla^F h_1^F]$ be the closed 1-form (cf. Definition 4.4), which measures the variation of the metric $\|\ \|_{\det F,1}$ on $\det F$ with respect to the induced flat connection on $\det F$; $\theta(F,h_1^F)$ vanishes if the metric $\|\ \|_{\det F,1}$ is flat.

The main result of this paper generalizes [BiZ1, Thm. 0.1] to manifolds with boundary; it reads as follows:

Theorem 0.1. *Let (g_0^{TX},h_0^F), (g_1^{TX},h_1^F) be two couples of metrics on TX and F. Then*

$$\log\left(\left(\frac{\|\ \|^{\mathrm{RS}}_{\det H^\bullet(X,F),1}}{\|\ \|^{\mathrm{RS}}_{\det H^\bullet(X,F),0}}\right)^2\right) = (-1)^m \int_{(X,Y)} \log\left(\left(\frac{\|\ \|_{\det F,1}}{\|\ \|_{\det F,0}}\right)^2\right) E(TX,\nabla_0^{TX})$$
$$+\int_{(X,Y)} \widetilde{E}(TX,\nabla_0^{TX},\nabla_1^{TX})\wedge\theta(F,h_1^F) + \mathrm{rk}(F)\left[\int_Y B(\nabla_1^{TX}) - \int_Y B(\nabla_0^{TX})\right]. \tag{0.5}$$

Especially, if $h_0^F = h_1^F$ is flat, then (0.5) reads

$$\log\left(\left(\frac{\|\ \|^{\mathrm{RS}}_{\det H^\bullet(X,F),1}}{\|\ \|^{\mathrm{RS}}_{\det H^\bullet(X,F),0}}\right)^2\right) = \mathrm{rk}(F)\left[\int_Y B(\nabla_1^{TX}) - \int_Y B(\nabla_0^{TX})\right]. \tag{0.6}$$

In fact, in [Ch, Cor. 3.29] Cheeger states (what is also implicit in the proof of [RS, Thm. 7.3]) that the left-hand side of (0.6) depends only on the germ of g_i^{TX} restricted to Y, a crucial fact for Cheeger's proof of the Ray–Singer conjecture [Ch, §4]. In subsection 4.5, we find that our results coincide with the results in [LüS, App. A] where Lück and Schick computed some examples to show that the left-hand side of (0.6) does not vanish in general. Note that (0.6) is different from the corresponding boundary contribution in [DF, Thm. 1.1, or §2.2 and §4.4].

Note that in Theorem 0.1, we do not assume that h_j^F $(j=0,1)$ is flat, nor do we impose any restriction on the metrics g_0^{TX} and g_1^{TX}. To derive the (somewhat mysterious) local contribution from the boundary (i.e. to calculate the asymptotics of the left-hand side in Theorem 4.5), we only do the rescaling of the Clifford variable along Y and, inspired by [BiL, §13 d), e)], we use a special trivialization of the vector bundles adapted to the boundary geometry, in order to get a manageable limiting boundary value problem. Finally, we introduce two extra Grassmann variables and a strange rescaling.

In this paper, we work out in detail the results associated to the absolute boundary condition on X. In [BrüM2], we will give the corresponding results for the relative boundary condition on X, and will apply it to derive the gluing formula for the analytic torsion in this general setting. Our results were announced in [BrüM1].

This paper is organized as follows: In section 1, we construct the secondary classes for manifolds with boundary which appear in Theorem 0.1. In section 2, we express the secondary relative Euler class from section 1 in terms of the Mathai–Quillen current. In section 3, we establish a local Gauss–Bonnet–Chern theorem for manifolds with boundary which serves as a model problem for our main result. In section 4, we establish the anomaly formula for the Ray–Singer metric under the additional assumption (4.30), and in section 5 we prove Theorem 0.1 in general.

Some notation. If the vector space $V = V^+ \oplus V^-$ is $\mathbb{Z}_2$ graded under the involution τ, and if $A, B \in \mathrm{End}(V)$, then we denote by $[A, B]$ the supercommutator of A and B, and put

$$\mathrm{Tr}_s[A] := \mathrm{Tr}[\tau A]\,.$$

We will denote by $i(\cdot)$ and $w(\cdot)$, respectively, the operators induced by the interior and exterior product on forms. For $I = \{i_1, \dots, i_k\} \subset \{1, \dots, m\}$, and $\{e_i\}$ an orthogonal basis of V with dual basis $\{e^i\}$, we write $w(e^I) = w(e^{i_1}) \circ \cdots \circ w(e^{i_k})$ and $i(e_I) = i(e_{i_1}) \circ \cdots \circ i(e_{i_k})$. Indices α, β, γ will always run from 1 to $m-1$, and indices i, j, k will always run from 1 to m.

Acknowledgment. We thank Jean-Michel Bismut for very useful and inspiring conversations. We are indebted to a referee for his very careful reading of the manuscript, and also for very detailed comments. Our collaboration was partially supported by Deutsche Forschungsgemeinschaft through SFB 288 and the European Commission through the Research Training Network "Geometric Analysis". The final revision was done while the second author was visiting Nankai Institute of Mathematics during July and August 2005. He would like to thank Weiping Zhang for hospitality.

1 Secondary Classes for Manifolds with Boundary

In this section we define the secondary classes for manifolds with boundary which appear in Theorem 0.1.

This section is organized as follows. In subsection 1.1, we recall the definition of the Berezin integral [BiZ1, §3a)]. In subsection 1.2, we describe the geometric setting, especially the connection from [Bi1, Def. 1.6]. In subsection 1.3, we state a technical result which we prove in subsection 1.4. In subsection 1.5, we construct the secondary classes for a fibration of manifolds with boundary.

1.1 The Berezin integral. For $\mathbb{Z}_2$-graded algebras $\mathcal{A}, \mathcal{B}$ with identity, we introduce the $\mathbb{Z}_2$-graded tensor product $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}$ and define $\mathcal{A} := \mathcal{A} \mathbin{\widehat{\otimes}} I$, and $\widehat{\mathcal{B}} := I \mathbin{\widehat{\otimes}} \mathcal{B}$, and we write $\wedge := \widehat{\otimes}$ such that $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B} = \mathcal{A} \wedge \widehat{\mathcal{B}}$. Let E and V be finite dimensional real vector spaces of dimension n and l, respectively. Assume that E is Euclidean and oriented, with oriented orthonormal basis $\{e_i\}_{i=1}^n$ and dual basis $\{e^i\}_{i=1}^n$ with respect to the Euclidean metric h^E, and denote by ΛE^* the exterior algebra of E^*. Then the Berezin integral we use is the linear map

$$\int^B : \Lambda V^* \wedge \widehat{\Lambda E^*} \to \Lambda V^*, \qquad \alpha \wedge \hat{\beta} \mapsto c_B\, \beta(e_1, \dots, e_n)\alpha\,, \tag{1.1}$$

where the normalizing constant is given by $c_B := (-1)^{n(n+1)/2}\pi^{-n/2}$. More generally, for any Euclidean space E with orientation line $o(E)$, the Berezin integral maps $\Lambda V^* \wedge \widehat{\Lambda E^*}$ into $\Lambda V^* \otimes o(E)$.

In the sequel, we no longer assume that E is oriented. Let C be an antisymmetric endomorphism of E. We identify C with the element of $\widehat{\Lambda E^*}$ given by

$$\dot{C} = \frac{1}{2} \sum_{1\leq i,j\leq n} \langle e_i, Ce_j\rangle \widehat{e^i} \wedge \widehat{e^j}. \tag{1.2}$$

This being understood, the Pfaffian $\mathrm{Pf}[C/2\pi]$ of $C/2\pi$ is defined by

$$\int^B \exp\left(\frac{-\dot{C}}{2}\right) = \mathrm{Pf}\left[\frac{C}{2\pi}\right] \in o(E)\,. \tag{1.3}$$

Clearly, $\mathrm{Pf}[C/2\pi]$ vanishes if n is odd.

1.2 The canonical connection of a fibration. Let W be a smooth manifold with boundary V. Let S be another manifold and let $\pi_W : W \to S$ be a fibration with m-dimensional compact fiber X, which induces a fibration $\pi_V : V \to S$ with compact fiber Y such that $Y = \partial X$. Let TX, TY be the corresponding tangent bundles along the fibers, and let $T^HW \subset TW$ be a horizontal subbundle such that $T^HW \oplus TX = TW$, with projections P^{T^HW} and P^{TX}. For $U \in TS$, we denote by $U^H \in T^HW$ its horizontal lift.

Let g^{TX} be a metric on TX and denote by g^{TY} the metric on TY induced by g^{TX}. Let N be the (orientable) normal bundle to Y in X which we identify with the orthogonal complement of TY in TX.

Definition 1.1. The canonical metric connection ∇^{TX} on $(TX \to W, g^{TX})$ is defined by the following properties:

a) On each fiber X, ∇^{TX} restricts to the Levi–Civita connection of (TX, g^{TX}).
b) If $U \in TS$, then

$$\nabla^{TX}_{U^H} = \mathcal{L}_{U^H} + \frac{1}{2}(g^{TX})^{-1}(\mathcal{L}_{U^H} g^{TX})\,. \tag{1.4}$$

We recall a simple result stated in [BiGS1, Thm. 1.2] (cf. [Bi3, Thms. 1.1 and 1.2]) the proof of which can also be found in [BGV, Prop. 10.2].

Theorem 1.2. *If g^{TW} is a metric on TW with Levi–Civita connection ∇^{TW} such that g^{TW} induces the given metric g^{TX} on TX, and T^HW is the orthogonal bundle to TX in TW with respect to g^{TW}, then*

$$\nabla^{TX} = P^{TX}\nabla^{TW}. \tag{1.5}$$

On V, we denote by P^{TY} and P^N the orthogonal projections from $TX|_V$ onto TY and N respectively. Set $\nabla^{TY} := P^{TY}\nabla^{TX|V}$ and let R^{TY} be the curvature of ∇^{TY}.

Theorem 1.3 [Bi3, Thm. 1.9]. *Set $T^HV := (T^HW \oplus N) \cap TV$. Then ∇^{TY} is the canonical connection on TY defined by the data (π_V, T^HV, g^{TY}).*

Proof. Let g^{TV} be the metric on TV induced by g^{TW} in Theorem 1.2; then T^HV is the orthogonal bundle to TY in TV with respect to g^{TV}. Also on V, $TW = T^HV \oplus TY \oplus N$. If P^{TV} is the orthogonal projection $TW|_V \to TV$ and if ∇^{TV} is the Levi–Civita connection on (TV, g^{TV}), then clearly $\nabla^{TV} = P^{TV}\nabla^{TW}$. Thus

$$\nabla^{TY} = P^{TY}P^{TX}\nabla^{TW} = P^{TY}P^{TV}\nabla^{TW} = P^{TY}\nabla^{TV}, \tag{1.6}$$

and Theorem 1.3 follows from (1.5). □

For later use, it is useful to extend the splitting $TX|_V = TY \oplus N$ to a neighborhood of V. We suppose that W is compact. Let e_{n} be the inward pointing unit normal at any boundary point. Then there exists $\varepsilon > 0$ such that for any $y \in V$, $\exp_y^X(ue_{\mathsf{n}}(y)) \in W$, the geodesic in the fiber X with respect to g^{TX}, is well defined for $u \in [0, \varepsilon)$; we use this map to identify $V \times [0, \varepsilon)$ with $\mathcal{U}_\varepsilon$, a neighborhood of V in W; we denote by $\pi_\varepsilon : V \times [0, \varepsilon) \to V$ the natural projection. Then there exists a family of metrics, $g^{TY}(x_m)$, on TY such that

$$g^{TX}(y, x_m) = dx_m^2 + g^{TY}(x_m)\,, \quad (y, x_m) \in V \times [0, \varepsilon)\,. \tag{1.7}$$

Let ${}^1TX = \pi_\varepsilon^* TY, {}^2TX = \pi_\varepsilon^* N$, then ${}^1TX$ and ${}^2TX$ are smooth orthogonal sub-bundles of $TX|_{\mathsf{U}_\varepsilon}$, and ${}^2TX$ is the trivial line bundle with the frame $e_m = \partial/\partial x_m$; and for $x = (y, x_m) \in \mathcal{U}_\varepsilon$, ${}^1TX_x, {}^2TX_x$ are obtained by parallel transport of T_yY, N_y with respect to the connection ∇^{TX} along the geodesic $[0,1] \ni u \mapsto (y, ux_m)$. For $j = 1, 2$, let $P^{{}^jTX}$ be the orthogonal projection operator from $TX|_{\mathsf{U}_\varepsilon}$ onto ${}^jTX$, and denote by $\nabla^{{}^jTX}$ the connection on ${}^jTX$ given by $\nabla^{{}^jTX} = P^{{}^jTX}\nabla^{TX}$. Then the restriction of $\nabla^{{}^1TX}$ to V coincides with ∇^{TY}. Next let ${}^{sp}\nabla^{TX} = \nabla^{{}^1TX} \oplus \nabla^{{}^2TX}$ be the direct sum connection on $TX = {}^1TX \oplus {}^2TX$ with curvature ${}^{sp}R^{TX}$ (where “sp” refers to “split”), and set

$$A := \nabla^{TX} - {}^{sp}\nabla^{TX} = P^{{}^1TX}\nabla^{TX}P^{{}^2TX} + P^{{}^2TX}\nabla^{TX}P^{{}^1TX}. \tag{1.8}$$

Then A is a 1-form on $\mathcal{U}_{\varepsilon_0}$ taking values in the skew-adjoint endomorphisms of TX which exchange ${}^1TX$ and ${}^2TX$. By construction,

$$\begin{aligned} {}^{sp}\nabla^{TX} e_m = 0\,, &\qquad \nabla^{TX}_{e_m} e_m = 0 \quad \text{on} \quad \mathcal{U}_\varepsilon\,, \\ A(e_m) = 0\,, &\qquad {}^{sp}R^{TX}|_V = R^{TY} \quad \text{on} \quad V\,. \end{aligned} \tag{1.9}$$

Moreover, A defines the second fundamental form of Y which measures the deviation from a metric product structure near the boundary: if $y_0 \in V$ and $Z_1, Z_2 \in T_{y_0}Y$, we choose extensions $\widetilde{Z}_1, \widetilde{Z}_2$ to vector fields on $\mathcal{U}_\varepsilon$ by extending first to Y and then trivially to $V\times[0,\varepsilon)$; then we get at y_0

$$\big\langle A(Z_1)Z_2,e_m\big\rangle=\big\langle\nabla^{TX}_{Z_1}\widetilde{Z}_2,e_m\big\rangle=-\big\langle\nabla^{TX}_{e_m}\widetilde{Z}_1,Z_2\big\rangle=-\tfrac{1}{2}(\mathcal{L}_{e_m}g^{TX})(Z_1,Z_2)\,. \tag{1.10}$$

In particular, $A_{y_0}(Z_1)Z_2$ is symmetric for $Z_1,Z_2\in T_{y_0}Y$. From (1.8), and (1.9), we get the Gauss–Codazzi equation for the second fundamental form (cf. [C, §6.3]): If $y_0\in V$, $Z_1,Z_2\in T_{y_0}Y$, $W_1,W_2\in T_{y_0}X$, then

$$\begin{aligned}\big\langle(R^{TX}_{y_0}-{}^{sp}R^{TX}_{y_0})(W_1,W_2)Z_1,Z_2\big\rangle_{g^{TX}}&=\big\langle A^2_{y_0}(W_1,W_2)Z_1,Z_2\big\rangle_{g^{TX}}\\&=\big\langle A^2_{y_0}(P^{TY}W_1,P^{TY}W_2)Z_1,Z_2\big\rangle_{g^{TX}}\,.\end{aligned} \tag{1.11}$$

For $U\in TS$, let $U^{H,V}\in T^HV\subset TV$ be the lift of U, and $U^{H,N}_{(y,x_m)}=U^H_{(y,x_m)}-U^{H,V}_y\in TX$ for $(y,x_m)\in V\times[0,\varepsilon)$, then $U^{H,N}\in N$ on V. Moreover, $[U^{H,V},e_m]=0$, thus by (1.7), $(\mathcal{L}_{U^{H,V}}g^{TX})e_m=0$. By (1.4), (1.7) and (1.9), on V

$$A(U^{H,V})e_m=\nabla^{TX}_{U^H-U^{H,N}}e_m=\nabla^{TX}_{U^H}e_m=\mathcal{L}_{U^{H,N}}e_m+\tfrac{1}{2}(g^{TX})^{-1}(\mathcal{L}_{U^{H,N}}g^{TX})e_m\,. \tag{1.12}$$

1.3 An identity for $e_{\mathfrak{b}}$. If ω is a smooth section of $\Lambda(T^*W)$ we identify ω with the section $\omega\,\widehat{\otimes}\,1$ of $\Lambda(T^*W)\,\widehat{\otimes}\,\Lambda\widehat{(T^*W)}$; $\widehat{\omega}$ will denote the section $1\,\widehat{\otimes}\,\widehat{\omega}$ of $\Lambda(T^*W)\,\widehat{\otimes}\,\Lambda\widehat{(T^*W)}$ as before. We will apply the Berezin integral from subsection 1.1 to $\Lambda(T^*W)\widehat{\otimes}\Lambda\widehat{(T^*X)}$ and $\Lambda(T^*V)\widehat{\otimes}\Lambda\widehat{(T^*Y)}$, and, for convenience, we will denote this operation by $\int^{B_X}$ and $\int^{B_Y}$, respectively.

Let $\{e_i\}_{i=1}^m$ be an orthonormal frame of (TX,g^{TX}) and let $\{e^i\}$ be the corresponding dual frame of T^*X. Set

$$\dot{R}^{TX}:=\tfrac{1}{2}\sum_{1\le i,j\le m}\big\langle e_i,R^{TX}e_j\big\rangle\widehat{e^i}\wedge\widehat{e^j}\in\Lambda^2(T^*W)\widehat{\otimes}\Lambda^2\widehat{(T^*X)}\,. \tag{1.13}$$

Let $j:V\hookrightarrow W$ be the natural embedding. We only consider orthonormal frames $\{e_i\}_{i=1}^m$ of TX with the property that near the boundary V, $e_m=:e_{\mathfrak{n}}$ is the inward-pointing unit normal at any boundary point. We will use Greek indices to specify the induced frame of TY, such that $\{e_\alpha\}_{\alpha=1}^{m-1}$ denotes a local orthonormal frame for TY.

Let $o(TX)$ be the orientation bundle of TX, which is a flat real line bundle on X [BoT, p. 88]. To compare the Berezin integrals $\int^{B_X}$ and $\int^{B_Y}$, we identify $o(TX)|_Y$ with $o(TY)$ as follows: as explained in [BoT, p. 88], we can view $o(TX)$ as the real line bundle $\det(T^*X)$ on the fiber X, with transition functions in $\{\pm1\}$; then we identify the section $(-e^m)\wedge\sigma\in\det(T^*X)|_Y$ with $\sigma\in\det(T^*Y)$ on Y. Consequently, for $\gamma\in\Lambda(T^*W)$ and $\beta\in\Lambda\widehat{(T^*Y)}$

on Y, we have

$$\int^{B_X} \gamma\beta \wedge \widehat{e^m} = \pi^{-1/2} \int^{B_Y} \gamma\beta \,. \tag{1.14}$$

(Assume $W = X$. If X is orientable and $(-e^m) \wedge e^1 \wedge \cdots \wedge e^{m-1}$ defines the orientation of X, then (1.14) means that $e^1 \wedge \cdots \wedge e^{m-1}$ is the induced orientation on Y, such that the Stokes theorem, $\int_X d\gamma = \int_Y \gamma$, holds for $\gamma \in \Omega(X)$. In general, if $\gamma \in \Omega(X, \widehat{\Lambda^m(T^*X)})$ such that $\gamma = \gamma_1 \wedge \widehat{e^m}$ on Y, then (1.14) means that $\int_X d \int^{B_X} \gamma = \pi^{-1/2} \int_Y \int^{B_Y} \gamma_1$; this equation holds even if X is not orientable.)

On V, set

$$\begin{aligned}
\dot{S} &:= \tfrac{1}{2} j^* \nabla^{TX} \widehat{e^m} = -\tfrac{1}{2} \sum_{\beta=1}^{m-1} \langle e_{\mathsf{n}}, A e_\beta \rangle \, \widehat{e^\beta} && \in T^*V \widehat{\otimes} \widehat{\Lambda^1(T^*Y)} \,, \\
j^* \dot{R}^{TX} &:= \tfrac{1}{2} \sum_{1 \le i,j \le m} \langle e_i, j^* R^{TX} e_j \rangle \widehat{e^i} \wedge \widehat{e^j} && \in \Lambda^2(T^*V) \widehat{\otimes} \widehat{\Lambda^2(T^*X)} \,, \\
\dot{R}^{TX}|Y &:= \tfrac{1}{2} \sum_{1 \le \gamma,\delta \le m-1} \langle e_\gamma, j^* R^{TX} e_\delta \rangle \widehat{e^\gamma} \wedge \widehat{e^\delta} && \in \Lambda^2(T^*V) \widehat{\otimes} \widehat{\Lambda^2(T^*Y)} \,, \\
\dot{R}^{TY} &:= \tfrac{1}{2} \sum_{1 \le \gamma,\delta \le m-1} \langle e_\gamma, R^{TY} e_\delta \rangle \widehat{e^\gamma} \wedge \widehat{e^\delta} && \in \Lambda^2(T^*V) \widehat{\otimes} \widehat{\Lambda^2(T^*Y)} \,.
\end{aligned} \tag{1.15}$$

By (1.8) and (1.11), we have

$$\dot{R}^{TY} = \dot{R}^{TX}|Y - 2\dot{S}^2. \tag{1.16}$$

The Chern–Weil forms $e(TX, \nabla^{TX}) := \mathrm{Pf}[R^{TX}/2\pi]$ and $e(TY, \nabla^{TY}) := \mathrm{Pf}[R^{TY}/2\pi]$ are closed and $e(TX, \nabla^{TX})$ is an $o(TX)$-valued m-form on W which represents the Euler class of TX. On V, we further introduce

$$\begin{aligned}
e_b(V, \nabla^{TX}) &:= (-1)^{m-1} \int^{B_Y} \exp\left(-\frac{1}{2} \big(\dot{R}^{TX}|Y \big) \right) \sum_{k=0}^{\infty} \frac{\dot{S}^k}{2\Gamma\big(\frac{k}{2}+1\big)} \,, \\
\mu(\nabla^{TX}) &:= \sqrt{\pi} \int^{B_X} \exp\left(-\frac{1}{2} j^* \dot{R}^{TX} \right) \sum_{k=1}^{\infty} \frac{\dot{S}^{k-1}}{2\Gamma\big(\frac{k}{2}+1\big)}, \\
B(\nabla^{TX}) &:= -\int_0^1 \frac{du}{u} \int^{B_Y} \exp\left(-\frac{1}{2} \dot{R}^{TY} - u^2 \dot{S}^2 \right) \sum_{k=1}^{\infty} \frac{(u\dot{S})^k}{2\Gamma\big(\frac{k}{2}+1\big)} \,.
\end{aligned} \tag{1.17}$$

(There is a misprint in the definition of $B(\nabla^{TX})$ in [BrüM1, (14)], the right-hand side must be multiplied by -1.) Then $e_b(V, \nabla^{TX}), B(\nabla^{TX})$ (resp. $\mu(\nabla^{TX})$) are $m-1$ (resp. m)-forms on V with values in $o(TY)$. If

$\dim X = m$ is odd, then by (1.16) and (1.17),

$$\begin{aligned} e_b(V, \nabla^{TX}) &= \int^{B_Y} \exp\Big(-\frac{1}{2}(\dot{R}^{TX}|Y)\Big) \sum_{k=0}^{\infty} \frac{\dot{S}^{2k}}{2\Gamma(k+1)} \\ &= \frac{1}{2}\int^{B_Y} \exp\Big(-\frac{1}{2}\dot{R}^{TY}\Big) = \frac{1}{2} e(TY, \nabla^{TY}), \qquad (1.18) \\ B(\nabla^{TX}) &= \int_0^1 \int^{B_Y} \exp\Big(-\frac{1}{2}\dot{R}^{TY}\Big) \Big(\exp(-u^2\dot{S}^2) - 1\Big) \frac{du}{2u} \\ &= \int^{B_Y} \exp\Big(-\frac{1}{2}\dot{R}^{TY}\Big) \sum_{k=1}^{\infty} \frac{(-\dot{S}^2)^k}{4k\Gamma(k+1)}. \qquad (1.19) \end{aligned}$$

Now we present the main result of this subsection; its proof will be given in the next subsection.

Theorem 1.4. *On V we have*

$$d^V e_b(V, \nabla^{TX}) = j^* e(TX, \nabla^{TX}), \qquad (1.20a)$$

$$d^V B(\nabla^{TX}) = \mu(\nabla^{TX}). \qquad (1.20b)$$

REMARK 1.5. If $\dim X = m$ is even, then by (1.32), (1.33) and (1.34), $e_b(V, \nabla^{TX})$ represents the Chern–Simons class $\widetilde{e}(V, \nabla^{TX}, {}^{sp}\nabla^{TX})$.

With the notation of subsection 1.4 (cf. (1.36), (1.40)), we get in general

$$B\big(\nabla^{TX}\big) = (-1)^m \int_0^1 \frac{ds}{s} \Big[e_b\big(V, \nabla_s^{TX}\big) - \tfrac{1}{2}\int^{B_Y} \exp\Big(-\tfrac{1}{2}\big(\dot{R}_s^{TX}|Y\big)\Big)\Big]. \quad (1.21)$$

1.4 Proof of Theorem 1.4.

Set $\widetilde{W} = W \times \mathbb{R}$, $\widetilde{j} := j \times \mathrm{id}_{\mathsf{R}}$, $\pi_{\widetilde{W}} := \pi_W \times \mathrm{id}_{\mathsf{R}} : \widetilde{W} \to S \times \mathbb{R}$. Let $p_W : \widetilde{W} \to W$, $p_V : \widetilde{V} := V \times \mathbb{R} \to V$, $p_{\mathsf{R}} : \widetilde{W} \to \mathbb{R}$ be the natural projections. We denote by $\widetilde{X}$ the fibers with boundary $\widetilde{Y}$ of the fibration $\pi_{\widetilde{W}}$. We will denote by $T\widetilde{X}$ the tangent bundle along the fiber of $\pi_{\widetilde{W}}$.

We only need to prove Theorem 1.4 on compact subsets of S, thus we may assume S compact. We use the notation in subsections 1.2, 1.3. Let g_0^{TX} be a smooth metric on TX such that on $V \times [0, \varepsilon)$,

$$g_0^{TX}(y, x_m) = dx_m^2 + g^{TY}(0). \qquad (1.22)$$

Let $g^{T\widetilde{X}}$ be a smooth metric on the bundle $T\widetilde{X}$ over $\widetilde{W}$ such that for $(x, s) \in W \times [0, 1]$, with $g_1^{TX} := g^{TX}$,

$$g^{T\widetilde{X}}(x, s) = (1-s) g_0^{TX} + s g_1^{TX}. \qquad (1.23)$$

We extend $T^H W$ on $W \times \{0\}$ to $\widetilde{T}^H W$ on $\widetilde{W}$ such that on $V \times [0, \varepsilon) \times \mathbb{R}$, for $U \in TS$, $y \in V$, $x_m \in [0, \varepsilon)$, $s \in [0, 1]$,

$$U^H(y, x_m, s) = (1-s) U^{H,V}(y) + s U^H(y, x_m). \qquad (1.24)$$

Set $T^H \widetilde{W} = \widetilde{T}^H W \oplus p_{\mathsf{R}}^* T\mathbb{R}$.

Now we apply the construction in subsection 1.2 to this new fibration $(\pi_{\widetilde{W}}, T^H\widetilde{W}, g^{T\widetilde{X}})$. Thus let $\nabla^{T\widetilde{X}}$ be the canonical connection on $T\widetilde{X}$ with curvature $R^{T\widetilde{X}}$, and let $\widetilde{A}, \dot{\widetilde{S}}$ be the tensors on $V \times \mathbb{R} = \partial W \times \mathbb{R}$ defined by (1.8) and (1.15). Let ∇^{TX}_s be the connection on TX induced by the restriction of $\nabla^{T\widetilde{X}}$ on $W \times \{s\}$ and denote its curvature by R^{TX}_s. By (1.4) and (1.23), $\nabla^{T\widetilde{X}}_{\partial/\partial s} e_{\mathsf{n}} = 0$ on $\partial W \times [0,1]$. Thus for $(y,s) \in \partial W \times \mathbb{R}$ and $s \in [0,1]$, we get from (1.10), (1.12), (1.23) and (1.24),

$$\widetilde{A}(y,s) = sA(y)\,, \qquad \dot{\widetilde{S}}(y,s) = s\dot{S}(y)\,. \tag{1.25}$$

Proposition 1.6. *If* $\dim X$ *is odd, then on* $V \times \mathbb{R}$,

$$d^{V\times \mathsf{R}} \int^{B_X} \widehat{e^m} \wedge \exp\Big(-\tfrac{1}{2}\widetilde{j}^* \dot{R}^{T\widetilde{X}}\Big) = \int^{B_X} 2\dot{\widetilde{S}} \wedge \exp\Big(-\tfrac{1}{2}\widetilde{j}^* \dot{R}^{T\widetilde{X}}\Big)\,. \tag{1.26}$$

Proof. The Bianchi identity asserts that $[\nabla^{T\widetilde{X}}, \dot{R}^{T\widetilde{X}}] = 0$ on $\widetilde{W}$. As the connection $\nabla^{T\widetilde{X}}$ preserves the metric $g^{T\widetilde{X}}$, we have, for any smooth section σ of $\Lambda(T^*\widetilde{W}) \widehat{\otimes} \widehat{\Lambda(T^*\widetilde{X})}$, $d^{\widetilde{W}} \int^{B_X} \sigma = \int^{B_X} \nabla^{T\widetilde{X}}\sigma$ (cf. [BGV, Prop. 1.50]). Thus with (1.15) we get (1.26). □

Lemma 1.7. *For* $s \in [0,1]$,

$$\begin{aligned}\widetilde{j}^* \dot{R}^{T\widetilde{X}}|_{V\times\{s\}} &= \dot{R}^{TX}|Y - 2(1-s^2)\dot{S}^2 \\ &\quad + s\sum_{\alpha=1}^{m-1} \big\langle e_\alpha, j^*R^{TX} e_m\big\rangle\, \widehat{e^\alpha} \wedge \widehat{e^m} + 2ds \wedge \dot{S} \wedge \widehat{e^m}.\end{aligned} \tag{1.27}$$

Proof. From (1.8), with our notation for supercommutators,

$$j^*R^{TX} = {}^{sp}R^{TX} + [{}^{sp}\nabla^{TX|V}, A] + A^2\,. \tag{1.28}$$

By (1.8), (1.23), and (1.25), we get, on $\partial W \times [0,1]$,

$$\widetilde{j}^*\nabla^{T\widetilde{X}} = {}^{sp}\nabla^{T\widetilde{X}|V\times\mathsf{R}} + sp_V^*A\,, \quad {}^{sp}\nabla^{T\widetilde{X}|V\times\mathsf{R}} = p_V^{*\,sp}\nabla^{TX|V} + ds \wedge \mathcal{L}_{\partial/\partial s}\,. \tag{1.29}$$

Thus

$$\widetilde{j}^* R^{T\widetilde{X}}|_{V\times\{s\}} = {}^{sp}R^{TX} + s^2 p_V^*A^2 + ds \wedge p_V^*A + sp_V^*[{}^{sp}\nabla^{TX|V}, A]\,. \tag{1.30}$$

From (1.15), (1.16), (1.28), and (1.30), we get (1.27). □

Proof of Theorem 1.4. Note that for $k \in \mathbb{N}$,

$$\int_0^1 (1-s^2)^k ds = \frac{1}{2}\int_{-1}^1 (1-s^2)^k ds = 2^{2k}\int_0^1 s^k(1-s)^k ds = \frac{2^{2k}(k!)^2}{(2k+1)!}\,. \tag{1.31}$$

Let $\delta_{1,s}, \mu_{1,s} \in C^\infty(W, \Lambda(T^*W) \otimes o(TX))$, $\delta_{2,s}, \mu_{2,s} \in C^\infty(V, \Lambda(T^*V) \otimes o(TX))$ be forms such that

$$e\big(T\widetilde{X}, \nabla^{T\widetilde{X}}\big)|_{W\times\{s\}} = \int^{B_X} \exp\Big(-\tfrac{1}{2}\dot{R}^{T\widetilde{X}}\Big)\Big|_{W\times\{s\}} = p_W^*\mu_{1,s} + ds \wedge p_W^*\delta_{1,s}\,,$$

$$\int^{B_X} \dot{\widetilde{S}} \exp\left(-\tfrac{1}{2}\widetilde{j}^* \dot{R}^{T\widetilde{X}}\right)\Big|_{V\times\{s\}} = p_V^* \mu_{2,s} + ds \wedge p_V^* \delta_{2,s}\,. \tag{1.32}$$

As $e(T\widetilde{X}, \nabla^{T\widetilde{X}})$ is closed on $\widetilde{W}$, (1.32) implies

$$d^W \delta_{1,s} = \tfrac{\partial}{\partial s} \int^{B_X} \exp\left(-\tfrac{1}{2}\dot{R}_s^{TX}\right). \tag{1.33}$$

If $\dim X$ *is even*, by (1.14), (1.27), and (1.31), we have on V

$$\begin{aligned}\int_0^1 ds j^* \delta_{1,s} &= \int_0^1 ds \int^{B_X} \exp\left(-\frac{1}{2}\dot{R}^{TX}|Y + (1-s^2)\dot{S}^2\right)(-\dot{S}\wedge \widehat{e^m})\\ &= -\frac{1}{\sqrt{\pi}}\int^{B_Y} \exp\left(-\frac{1}{2}\dot{R}^{TX}|Y\right)\sum_{k=0}^{\infty} \frac{2^{2k}k!}{(2k+1)!}\dot{S}^{2k+1} = e_b(V, \nabla^{TX})\,. \end{aligned}\tag{1.34}$$

From (1.33) and (1.34), we get (1.20a).

Applying (1.20a) on $W\times\mathbb{R}$, we get $d^{\widetilde{V}} e_b(\widetilde{V}, \nabla^{T\widetilde{X}}) = \widetilde{j}^* e(T\widetilde{X}, \nabla^{T\widetilde{X}})$, and by (1.17), the component in $\Lambda(T^*V)\otimes o(TX)$ of $e_b(\widetilde{V}, \nabla^{T\widetilde{X}})$ is $e_b(V, \nabla_s^{TX})$. Thus (1.32) implies

$$j^* \mu_{1,s} = d^V e_b\big(V, \nabla_s^{TX}\big)\,. \tag{1.35}$$

Thus by (1.16), (1.17), (1.25), (1.27), and (1.35),

$$\int_0^1 \frac{ds}{s} j^* \mu_{1,s} = d^V \int_0^1 \frac{ds}{s} e_b\big(V, \nabla_s^{TX}\big) = d^V B\big(\nabla^{TX}\big)\,. \tag{1.36}$$

But by (1.17), (1.27), (1.31) and (1.32), we have

$$\begin{aligned}\int_0^1 \frac{ds}{s} j^* \mu_{1,s} &= \int_0^1 ds \int^{B_X} \exp\left(-\frac{1}{2}\dot{R}^{TX}|Y + (1-s^2)\dot{S}^2\right)\\ &\qquad \times \left(-\frac{1}{2}\right)\sum_{\alpha=1}^{m-1}\left\langle e_\alpha, j^* R^{TX} e_m\right\rangle \widehat{e^\alpha}\wedge \widehat{e^m}\\ &= \int_0^1 ds \int^{B_X} \exp\left(-\frac{1}{2}j^* \dot{R}^{TX} + (1-s^2)\dot{S}^2\right)\\ &= \sqrt{\pi}\int^{B_X} \exp\left(-\frac{1}{2}j^*\dot{R}^{TX}\right)\sum_{k=0}^{\infty}\frac{\dot{S}^{2k}}{2\Gamma\big(k+\frac{3}{2}\big)} = \mu(\nabla^{TX})\,.\end{aligned}\tag{1.37}$$

This gives (1.20b) in the even dimensional case.

If $\dim X$ *is odd*, then $e_b(V, \nabla^{TX})$ is closed on V by (1.18). From (1.17), (1.25), (1.27) and (1.32), as in (1.37)

$$\sqrt{\pi}\int_0^1 \tfrac{ds}{s}\mu_{2,s} = \sqrt{\pi}\int_0^1 ds \int^{B_X} s\dot{S}\exp\Big(-\tfrac{1}{2}\dot{R}^{TX}|Y + (1-s^2)\dot{S}^2\Big)$$
$$\times\ \tfrac{-1}{2}\sum_{\alpha=1}^{m-1}\big\langle e_\alpha, j^*R^{TX}e_m\big\rangle\,\widehat{e^\alpha}\wedge\widehat{e^m} \tag{1.38}$$
$$= \sqrt{\pi}\int_0^1 ds \int^{B_X} \exp\Big(-\tfrac{1}{2}j^*\dot{R}^{TX} + (1-s^2)\dot{S}^2\Big)\, s\dot{S} = \mu\big(\nabla^{TX}\big)\,.$$

By (1.16), (1.25) and (1.27), at $s\in[0,1]$,

$$\dot{R}^{T\widetilde{X}}|\widetilde{Y} = \dot{R}_s^{TX}|Y = \dot{R}^{TY} + 2s^2\dot{S}^2. \tag{1.39}$$

Observe that by (1.14), $\sqrt{\pi}\int^{B_X}\widehat{e^m}\exp\big(-\frac{1}{2}\dot{R}^{T\widetilde{Y}}\big) = \int^{B_Y}\exp\big(-\frac{1}{2}\dot{R}^{TY}\big) = e(TY,\nabla^{TY})$ is closed. We combine the identity (1.39) with (1.14), (1.19), (1.26), and (1.27) to find

$$\sqrt{\pi}\int_0^1 \tfrac{ds}{s}\mu_{2,s} = \sqrt{\pi}\int_0^1 \tfrac{ds}{s}d^V\int^{B_X}\tfrac{1}{2}\widehat{e^m}\Big[\exp\Big(-\tfrac{1}{2}j^*\dot{R}_s^{TX}\Big) - \exp\Big(-\tfrac{1}{2}\dot{R}^{T\widetilde{Y}}\Big)\Big]$$
$$= \int_0^1 \tfrac{ds}{s}d^V\int^{B_Y}\tfrac{1}{2}\Big[\exp\Big(-\tfrac{1}{2}(\dot{R}_s^{TX}|Y)\Big) - \exp\Big(-\tfrac{1}{2}\dot{R}^{TY}\Big)\Big] = d^V B\big(\nabla^{TX}\big)\,. \tag{1.40}$$

The proof of Theorem 1.4 is complete. □

1.5 The secondary classes for manifolds with boundary. Let $\Omega(W,o(TX))$, $\Omega(V,o(TX))$ be the $o(TX)$-valued C^∞ forms on W,V. The algebraic mapping cone of $j^*:\Omega(W,o(TX))\to\Omega(V,o(TX))$ is defined as the following object: we put

$$\Omega^p\big(W,V,o(TX)\big) = \Omega^p\big(W,o(TX)\big)\oplus\Omega^{p-1}\big(V,o(TX)\big)\,, \tag{1.41}$$

and define the differential by

$$d(\sigma_1,\sigma_2) = (d^W\sigma_1, j^*\sigma_1 - d^V\sigma_2)\,; \tag{1.42}$$

then the complex $(\Omega(W,V,o(TX)),d)$ calculates the relative cohomology $H^*(W,V,o(TX))$ (cf. [BoT, p. 78]). Now we define the *relative Euler form* of TX associated to ∇^{TX}

$$E(TX,\nabla^{TX}) := \big(e(TX,\nabla^{TX}), e_b(V,\nabla^{TX})\big)\in\Omega^m\big(W,V,o(TX)\big)\,. \tag{1.43}$$

Let g_0^{TX}, g_1^{TX} be two metrics on TX, and let T_0^HW, T_1^HW be two horizontal sub-bundles of TW. Let $(g_s^{TX})_{s\in\mathbb{R}}$ be a family of metrics connecting g_0^{TX} and g_1^{TX}, with a family $(T_s^HW)_{s\in\mathbb{R}}$ of horizontal sub-bundles. We will use the subscript s to distinguish the corresponding objects, like g_s^{TY}, ∇_s^{TX}, ∇_s^{TY}, etc.

For the fibration $\pi_{\widetilde{W}}:\widetilde{W}=W\times\mathbb{R}\to S\times\mathbb{R}$ in subsection 1.3. Let $T^H(W\times\mathbb{R})_s = T_s^HW\oplus T\mathbb{R}$, and denote by $g^{T\widetilde{X}}$ the metric on $T\widetilde{X}$ induced by g_s^{TX}. Then by (1.4), $\nabla^{T\widetilde{X}} = \nabla_s^{TX} + ds \wedge \big(\frac{\partial}{\partial s} + \frac{1}{2}(g_s^{TX})^{-1}\frac{\partial}{\partial s}g_s^{TX}\big)$, thus

$$R^{T\widetilde{X}} = p_W^* R_s^{TX} + ds \wedge \big(\tfrac{\partial}{\partial s}\nabla_s^{TX} - \tfrac{1}{2}\big[\nabla_s^{TX}, (g_s^{TX})^{-1}\tfrac{\partial}{\partial s}g_s^{TX}\big]\big)\,. \tag{1.44}$$

Definition 1.8. We define

$$\begin{aligned}\widetilde{e}\big(TX, \nabla_s^{TX}\big) &:= \int_0^1 ds\, i\big(\tfrac{\partial}{\partial s}\big)\, e\big(T\widetilde{X}, \nabla^{T\widetilde{X}}\big) \quad \in \Omega^{m-1}\big(W, o(TX)\big)\,,\\ \widetilde{e_b}\big(V, \nabla_s^{TX}\big) &:= \int_0^1 ds\, i\big(\tfrac{\partial}{\partial s}\big)\, e_b\big(V\times\mathbb{R}, \nabla^{T\widetilde{X}}\big) \quad \in \Omega^{m-2}\big(V, o(TX)\big)\,,\\ \widetilde{E}\big(TX,\nabla_0^{TX},\nabla_1^{TX}\big) &:= \big(\widetilde{e}\big(TX,\nabla_s^{TX}\big), -\widetilde{e_b}\big(V,\nabla_s^{TX}\big)\big) \in \Omega^{m-1}\big(W,V,o(TX)\big)\,.\end{aligned} \tag{1.45}$$

If $V=\emptyset$, then $\widetilde{E}$ is the Chern–Simons form associated with the Euler class of TX, as defined in [BiZ1, (4.53)].

Let $\widetilde{e}(TY, \nabla_0^{TY}, \nabla_1^{TY})$ be the Chern–Simons class of smooth $m-2$-forms on V with values in the orientation line bundle $o(TY)$, which is defined modulo exact forms, and satisfies

$$d\widetilde{e}\big(TY, \nabla_0^{TY}, \nabla_1^{TY}\big) = e\big(TY, \nabla_1^{TY}\big) - e\big(TY, \nabla_0^{TY}\big)\,. \tag{1.46}$$

If $\dim X$ is odd, then we derive from (1.18) and (1.45):

$$\begin{aligned}\widetilde{e}\big(TX, \nabla_s^{TX}\big) &= 0\,,\\ 2\widetilde{e_b}\big(V, \nabla_s^{TX}\big) &= \widetilde{e}\big(TY, \nabla_0^{TY}, \nabla_1^{TY}\big) \text{ in } \Omega^{m-2}\big(V, o(TY)\big)/d\Omega(V, o(TY)\big)\,.\end{aligned} \tag{1.47}$$

Theorem 1.9. 1. *$E(TX,\nabla^{TX})$ is closed in the complex $(\Omega(W,V,o(TX)),d)$ and, modulo exact forms, it does not depend on the choice of g^{TX} and T^HW, i.e. the cohomology class $E(TX) = [E(TX,\nabla^{TX})] \in H^m(W,V,o(TX))$ does not depend on g^{TX} and T^HW.*

2. *For a path of metrics on TX and of horizontal sub-bundles of TW as above,*

$$d\widetilde{E}\big(TX, \nabla_0^{TX}, \nabla_1^{TX}\big) = E\big(TX, \nabla_1^{TX}\big) - E\big(TX, \nabla_0^{TX}\big)\,. \tag{1.48}$$

Modulo exact forms, $\widetilde{E}(TX, \nabla_0^{TX}, \nabla_1^{TX})$ does not depend on the choice of the paths g_s^{TX} and T_s^HW. Thus it defines the secondary relative Euler class of TX in the sense of Chern–Simons.

3. *We also have*

$$\int_0^1 ds\, i\big(\tfrac{\partial}{\partial s}\big)\, \mu\big(\nabla^{T\widetilde{X}}\big) = B\big(\nabla_1^{TX}\big) - B\big(\nabla_0^{TX}\big) - d^V \int_0^1 ds\, i\big(\tfrac{\partial}{\partial s}\big)\, B\big(\nabla^{T\widetilde{X}}\big)\,. \tag{1.49}$$

Proof. 1 is a simple application of (1.20a). From Theorem 1.4 we get the first part of 2; to get the second part of 2, we use a deformation argument as in subsection 1.4. Let $g_{s,1}^{TX}$ (resp. $T_{s,1}^HW$) be another family of metrics on TX (resp. sub-bundles of TW) such that $g_{i,1}^{TX} = g_i^{TX}$,

$T^H_{i,1}W = T^H_i W$ for $i = 0, 1$. Let $\overline{\pi} : \overline{W} = W \times \mathbb{R} \times \mathbb{R} \to \overline{S} = S \times \mathbb{R} \times \mathbb{R}$ be the fibration with fiber $\overline{X}$ induced by $\pi_W : W \to S$, and let $\overline{p}_W : \overline{W} \to W$ be the natural projection. Let $g^{TX}_{s,l}$ be a family of metrics on TX such that $g^{TX}_{s,0} = g^{TX}_s$ and $g^{TX}_{s,1} = g^{TX}_{s,1}$, $g^{TX}_{i,l} = g^{TX}_i$ for $i = 0, 1$, and in the same way, $T^H_{s,l}W$ is a family of paths connecting $T^H_s W$ and $T^H_{s,1}W$. Set $T^H\overline{W} = T^H_{s,l}W \oplus T\mathbb{R} \oplus T\mathbb{R}$. Then the construction above applies to the fibration $\overline{\pi}$. Let $\gamma_1 \in C^\infty(\overline{W}, \overline{p}^*_W \Lambda(T^*W) \otimes o(TX))$, $\gamma_2 \in C^\infty(V \times \mathbb{R} \times \mathbb{R}, \overline{p}^*_W \Lambda(T^*V) \otimes o(TX))$ be the coefficients of $ds\, dl$ in $e(T\overline{X}, \nabla^{T\overline{X}})$ and $e_b(V \times \mathbb{R} \times \mathbb{R}, \nabla^{T\overline{X}})$. Integrating the coefficient of $ds\, dl$ in the equations $d^{\overline{W}} e(T\overline{X}, \nabla^{T\overline{X}}) = 0$ and $d^{V\times\mathsf{R}\times\mathsf{R}} e_b(V \times \mathbb{R} \times \mathbb{R}, \nabla^{T\overline{X}}) = (j \times \mathrm{Id} \times \mathrm{Id})^* e(T\overline{X}, \nabla^{T\overline{X}})$ on $[0,1]^2$, we get

$$\begin{gathered}\widetilde{e}\big(TX, \nabla^{TX}_{s,1}\big) - \widetilde{e}\big(TX, \nabla^{TX}_{s,0}\big) = d^W \int_0^1 dl \int_0^1 ds\, \gamma_1\,, \\ \widetilde{e}_b\big(V, \nabla^{TX}_{s,0}\big) - \widetilde{e}_b\big(V, \nabla^{TX}_{s,1}\big) + d^V \int_0^1 dl \int_0^1 ds\, \gamma_2 = j^* \int_0^1 dl \int_0^1 ds\, \gamma_1\,.\end{gathered} \tag{1.50}$$

This gives the second part of 2.

From (1.20b), we get $d^{V\times\mathsf{R}} B(\nabla^{T\widetilde{X}}) = \mu(\nabla^{T\widetilde{X}})$. By comparing the coefficient of ds, we get $i(\frac{\partial}{\partial s})\mu(\nabla^{T\widetilde{X}}) = \frac{\partial}{\partial s} B(\nabla^{TX}_s) - d^V\big(i(\frac{\partial}{\partial s}) B(\nabla^{T\widetilde{X}})\big)$ which implies (1.49). □

Remark 1.10. Let us make explicit our sign conventions when integrating differential forms along the fiber for the fibration $W \to S$. If γ is a differential form on S and $(\sigma_1, \sigma_2) \in \Omega(W, V, o(TX))$ with compact support along the fibers, then

$$\begin{gathered}\int_X (\pi^*_W \gamma) \wedge \sigma_1 = \gamma\Big(\int_X \sigma_1\Big), \quad \int_Y (\pi^*_W \gamma) \wedge \sigma_2 = \gamma\Big(\int_Y \sigma_2\Big), \\ \int_{(X,Y)} (\sigma_1, \sigma_2) := \int_X \sigma_1 + (-1)^{\deg \sigma_2 - m} \int_Y \sigma_2\,.\end{gathered} \tag{1.51}$$

Then (1.51) is compatible with (0.2) in the following sense: if $(\sigma_1, \sigma_2) \in \Omega(W, V, o(TX))$ is closed (resp. exact), then $\int_{(X,Y)} (\sigma_1, \sigma_2)$ is a closed (resp. exact) form on S. In fact, let d^X be the exterior differentiation along the fiber X, then by [BGV, Prop. 10.1],

$$\begin{gathered}d^S \int_Y \sigma_2 = \int_Y d^V \sigma_2\,, \\ d^S \int_X \sigma_1 = \int_X \big(d^W \sigma_1 - d^X \sigma_1\big) = \int_X d^W \sigma_1 - (-1)^{\deg \sigma_1 - m + 1} \int_Y j^* \sigma_1\,.\end{gathered} \tag{1.52}$$

This implies that the form $\int_X e(TX,\nabla^{TX})\omega - (-1)^{\deg\omega}\int_Y e_b(V,\nabla^{TX})\omega$ is closed on S, for any closed form ω on W.

2 Secondary Classes and Mathai–Quillen Form

In this section, we construct the secondary relative Euler class for manifolds with boundary introduced in section 1, using the Mathai–Quillen form.

This section is organized as follows. In subsection 2.1, we recall the Mathai–Quillen construction of the transgressed Euler forms for Euclidean vector bundles in the formalism of Berezin integrals [BiZ1, §3b)-d)]. In subsection 2.2, we construct the secondary relative Euler class.

2.1 A transgressed Euler class. Let W be a smooth manifold with boundary V. Let $\pi_E : E \to W$ be a real vector bundle of dimension m. Let h^E be a Euclidean metric on E, let ∇^E be a Euclidean connection on (E,h^E) with curvature R^E; then R^E is a smooth section of $\Lambda^2(T^*W)\otimes \mathrm{End}(E)$. Also, $\pi_E^*\nabla^E$ is a Euclidean connection on $\pi_E^*(E,h^E)$ with curvature $\pi_E^*R^E$, a smooth section of $\Lambda^2(T^*E)\otimes \mathrm{End}(\pi_E^*E)$. As in (1.2) and (1.13), we identify R^E with $\dot{R}^E \in C^\infty(W,\Lambda^2(T^*W)\,\widehat{\otimes}\,\widehat{\Lambda^2 E^*})$.

Let $\{h_i\}_{i=1}^m$ be an orthonormal basis of E and let $\{h^i\}_{i=1}^m$ be the corresponding dual basis of E^*. The connection ∇^E defines a horizontal subspace T^HE of TE such that $TE = T^HE\oplus E$. Let P^E be the projection $TE\to E$ and let $P^{E*} : E^*\to T^*E$ be the transpose of P^E. Then P^E is a section of $T^*E\otimes E$ and $P^E_Z = (\pi_E^*\nabla^E)Z$. Denote by $E\ni Z\to Z^b\in E^*$ the isomorphism induced by the metric h^E, then P^E will be identified with the section of $T^*E\,\widehat{\otimes}\,\widehat{E^*}$ given by $\dot{P}^E = \sum_{i=1}^m (P^{E*}h^i)\otimes \widehat{h^i}$.

Definition 2.1. For $t>0$, let A_t be the section of $\Lambda(T^*E)\,\widehat{\otimes}\,\pi_E^*\widehat{\Lambda E^*}$ on E given by

$$A_t(Z) = \tfrac{1}{2}\pi_E^*\dot{R}^E + \sqrt{t}\dot{P}^E_Z + t|Z|^2 \quad \text{for all } Z\in E\,. \tag{2.1}$$

We will apply the formalism of Berezin integrals, summarized in subsection 1.1, to TE: If ω is a smooth section of $\Lambda(T^*E)\,\widehat{\otimes}\,\pi_E^*\widehat{\Lambda E^*}$, $\int^B\omega$ is a smooth section of $\Lambda(T^*E)\,\widehat{\otimes}\,\pi_E^*o(E)$, i.e. a smooth differential form on E with values in $\pi_E^*o(E)$. Set $e(E,\nabla^E) := \mathrm{Pf}[R^E/2\pi]$. Then $e(E,\nabla^E)$ is a smooth closed section of $\Lambda^{\dim E}(T^*W)\,\widehat{\otimes}\,o(E)$. If $V=\emptyset$, $e(E,\nabla^E)$ is a Chern–Weil representative of the rational Euler class of E. Of course, if $\dim E$ is odd, then $e(E,\nabla^E)=0$.

Definition 2.2. We define two differential forms on E with values in $\pi_E^* o(E)$ by

$$a_t(Z) = \int^B \exp\big(- A_t(Z)\big)\,, \quad b_t(Z) = \int^B \frac{\widehat{Z}^{b}}{2\sqrt{t}} \wedge \exp\big(- A_t(Z)\big)\,. \tag{2.2}$$

Let $\int_E$ denote the fiber integral of forms on E taking values in $\pi_E^* o(E)$. Now we state an important result of Mathai and Quillen [MQ, Thm. 6.4] (also cf. [BiZ1, Thms. 3.4, 3.5]).

Theorem 2.3. *The forms a_t are closed of degree m. For $t > 0$, the forms a_t represent the Thom class of E, such that $\int_E a_t = 1$. Moreover, for $t > 0$ we have the relations*

$$\begin{aligned} a_0 &= \pi_E^* e(E, \nabla^E)\,, \\ b_t &= -\tfrac{1}{2t} i(Z) a_t\,, \qquad \tfrac{\partial a_t}{\partial t} = -db_t\,. \end{aligned} \tag{2.3}$$

Then

$$\psi(E, \nabla^E) := \lim_{\varepsilon \to 0} \int_\varepsilon^{1/\varepsilon} b_t dt \tag{2.4}$$

exists as an $(m-1)$-form on $E \setminus \{0\}$ with values in $\pi_E^* o(E)$.

Definition 2.4. The form $\psi(E, \nabla^E)$ is called the Mathai–Quillen form.

In fact, $\psi(E, \nabla^E)$ is defined on E as a current (cf. [BiZ1, Thm. 3.7]) and

$$\begin{aligned} d\psi(E, \nabla^E) &= \pi_E^* e(E, \nabla^E) - \delta_W\,, \\ \psi(E, \nabla^E)(\lambda Z) &= (\mathrm{sign}\lambda)^m \psi(E, \nabla^E)(Z), \ \text{for}\ \lambda \in \mathbb{R} \setminus \{0\}\,. \end{aligned} \tag{2.5}$$

Remark 2.5. If $E = TW$, then the restriction of $\psi(E, \nabla^E)$ to the sphere bundle $S(E)$ of E coincides with the transgressed Euler class defined by Chern (cf. [Che2], [Z, §3.5]).

The following result was essentially obtained in [BiZ1, (6.20)].

Proposition 2.6. *Let $j_S : S(E) \to E$ be the natural injection. Let $E^\perp \subset \pi_E^* E$ be the relative tangent bundle of the fibration $\pi_S : S(E) \to W$. We identify $o(E^\perp)$ to $\pi_S^* o(E)$ as in (1.14), in viewing $S(E)$ as the boundary of the unit ball $D(E)$ on each fiber. Let ∇^{E} be the connection on $E^\perp$ induced by $\pi_E^* \nabla^E$ by taking the orthogonal projection. If $\dim E$ is odd, then on $S(E)$,*

$$j_S^* \psi(E, \nabla^E) = \tfrac{1}{2} e(E^\perp, \nabla^{E}\ )\,. \tag{2.6}$$

Proof. Let $L \subset \pi_E^* E$ be the tautological line bundle on $S(E)$, then $E^\perp$ is the orthogonal complement of L in $(\pi_S^* E, h^{\pi_S E})$. Let P^L, P^{E} be the orthogonal projections from $(\pi_S^* E, h^{\pi_S E})$ onto $L, E^\perp$, then $\nabla^{E} = P^{E}\ \pi_S^* \nabla^E$. Set $\nabla^L = P^L \pi_S^* \nabla^E$, $A = \pi_S^* \nabla^E - (\nabla^{E} \oplus \nabla^L)$, then A is a one-form on $S(E)$

taking values in the skew-adjoint endomorphisms of $\pi_S^* E$ which exchange $E^\perp$ and L. Let σ_L be the tautological section of $\pi_S^* E$ such that $-\sigma_L$ is the inward pointing unit vector of $S(E)$ in $D(E)$. Then $j_S^* P^E = \pi_S^* \nabla^E \sigma_L = A\sigma_L \in T^*S(E) \otimes E^\perp$. Set $\dot{S}_{\sigma_L} = \frac{1}{2}\sum_j g^j \wedge (\widehat{A(g_j)\sigma_L})^b \in T^*S(E) \widehat{\otimes} \widehat{E^{\perp *}}$, with $\{g_j\}$ an orthonormal basis of $TS(E)$ and $\{g^j\}$ its dual basis. As in (1.16), we have

$$\dot{R}^E \quad = \pi_S^* \dot{R}^E - 2\dot{S}_{\sigma_L}^2 + a \wedge \widehat{(\sigma_L)}^b. \tag{2.7}$$

Now by (1.14), (2.2), (2.4) and (2.7), if $\dim E$ is odd,

$$\begin{aligned} j_S^* \psi(E, \nabla^E) &= \int_0^\infty \frac{dt}{2\sqrt{t}} \int^B \widehat{(\sigma_L)}^b \exp\left(-\frac{1}{2}\pi_S^* \dot{R}^E - 2\sqrt{t}\dot{S}_{\sigma_L} - t\right) \\ &= \int^B \widehat{(\sigma_L)}^b \exp\left(-\frac{1}{2}\pi_S^* \dot{R}^E\right) \sqrt{\pi} \sum_{k=0}^\infty \frac{(-\dot{S}_{\sigma_L})^k}{2\Gamma\left(\frac{k}{2}+1\right)} \\ &= \frac{\sqrt{\pi}}{2} \int^B \widehat{(\sigma_L)}^b \exp\left(-\frac{1}{2}\dot{R}^E \quad\right) = \frac{1}{2} e(E^\perp, \nabla^E \quad). \end{aligned} \tag{2.8}$$

□

2.2 Secondary relative Euler class. In this subsection, we assume that there exists a non-vanishing section, σ, of E on V; we assume $|\sigma|_{h^E} = 1$. Then it follows from (2.5), that on V,

$$d\sigma^* \psi(E, \nabla^E) = j^* e(E, \nabla^E). \tag{2.9}$$

Let $(\Omega(W, V, o(E)), d)$ be the algebraic mapping cone of j^*, with j^* as in subsection 1.5, but with $o(TX)$ replaced by $o(E)$. Thus we will consider

$$E_\sigma(E, \nabla^E) := \big(e(E, \nabla^E), \sigma^* \psi(E, \nabla^E)\big) \in H^m\big(W, V, o(E)\big), \tag{2.10}$$

as the Chern–Weil representative associated to the *relative Euler class* of (E, ∇^E).

Let $(h_0^E, \nabla_0^E), (h_1^E, \nabla_1^E)$ be two pairs of metrics and Euclidean connections on E. Let $(h_s^E)_{s\in\mathbb{R}}$ be a smooth family of metrics on E connecting h_0^E, h_1^E. Let $p_W : W \times \mathbb{R} \to W$ be the natural projection such that $(\widetilde{E} = p_W^* E,\ h^{\widetilde{E}}_{|W\times\{s\}} = h_s^E)$ is a Euclidean vector bundle on $W \times \mathbb{R}$. The section σ induces naturally a normalized section of $(\widetilde{E}, h^{\widetilde{E}})|_{V\times\mathbb{R}}$ which we denote by $\widetilde{\sigma}$. Choose a Euclidean connection $\nabla^{\widetilde{E}}$ on $(\widetilde{E}, h^{\widetilde{E}})$ such that $\nabla^{\widetilde{E}}_{|W\times\{0\}} = \nabla_0^E$, $\nabla^{\widetilde{E}}_{|W\times\{1\}} = \nabla_1^E$ and denote by ∇_s^E its restriction to $W \times \{s\}$. We set (cf. (1.45))

$$\begin{aligned} \widetilde{e}(E, \nabla_s^E) &:= \int_0^1 ds\, i\left(\tfrac{\partial}{\partial s}\right) e(\widetilde{E}, \nabla^{\widetilde{E}}) \quad \in \Omega^{m-1}\big(W, o(E)\big), \\ \widetilde{e_{b,\sigma}}(V, \nabla_s^E) &:= \int_0^1 ds\, i\left(\tfrac{\partial}{\partial s}\right) \widetilde{\sigma}^* \psi(\widetilde{E}, \nabla^{\widetilde{E}}) \quad \in \Omega^{m-2}\big(V, o(E)\big), \\ \widetilde{E}_\sigma(E, \nabla_0^E, \nabla_1^E) &:= \big(\widetilde{e}(E, \nabla_s^E), -\widetilde{e_{b,\sigma}}(V, \nabla_s^E)\big) \quad \in \Omega^{m-1}\big(W, V, o(E)\big). \end{aligned} \tag{2.11}$$

PROPOSITION 2.7. *The cohomology class $E_\sigma(E,\nabla^E)$ in $H^m(W,V,o(E))$ does not depend on the choice of (∇^E,h^E). Moreover, for a path (∇^E_s,h^E_s) as above, we have*

$$d\widetilde{E}_\sigma(E,\nabla^E_0,\nabla^E_1)=E_\sigma(E,\nabla^E_1)-E_\sigma(E,\nabla^E_0)\,. \tag{2.12}$$

Modulo exact forms, $\widetilde{E}_\sigma(E,\nabla^E_0,\nabla^E_1)$ does not depend on the choice of $(\nabla^E_s,h^E_s)_{s\in[0,1]}$. Thus $\widetilde{E}_\sigma$ defines the secondary relative Euler class of E in the sense of Chern–Simons.

Proof. Since $e(\widetilde{E},\nabla^{\widetilde{E}})$ is closed on $W\times\mathbb{R}$, we have

$$d^W\widetilde{e}(E,\nabla^E_s)=e(E,\nabla^E_1)-e(E,\nabla^E_0)\,. \tag{2.13}$$

By (2.9), we have $d^{V\times\mathsf{R}}\widetilde{\sigma}^*\psi(\widetilde{E},\nabla^{\widetilde{E}})=(j\times\mathrm{Id})^*e(\widetilde{E},\nabla^{\widetilde{E}})$, hence by comparing the coefficient of ds we get

$$\frac{\partial}{\partial s}\widetilde{\sigma}^*\psi\big(E,\nabla^E_s\big)-d^V\Big[i\left(\tfrac{\partial}{\partial s}\right)\widetilde{\sigma}^*\psi\big(\widetilde{E},\nabla^{\widetilde{E}}\big)\Big]=(j\times\mathrm{Id})^*\Big[i\left(\tfrac{\partial}{\partial s}\right)e\big(\widetilde{E},\nabla^{\widetilde{E}}\big)\Big]\,. \tag{2.14}$$

Integrating both sides over $s\in[0,1]$, we get

$$j^*\widetilde{e}\big(E,\nabla^E_s\big)+d^V\widetilde{e}_{b,\sigma}\big(E,\nabla^E_s\big)=\sigma^*\psi\big(E,\nabla^E_1\big)-\sigma^*\psi\big(E,\nabla^E_0\big)\,, \tag{2.15}$$

which together with (2.13) implies (2.12). To get the last part of Proposition 2.7, we use (2.9) and a deformation argument as in the proof of Theorem 1.9. □

REMARK 2.8. The classes $E_\sigma,\widetilde{E}_\sigma$ depend on the section σ, in fact $E_{-\sigma}=-E_\sigma$, if $\dim E$ is odd. But in the situation of subsection 1.2, there exists a natural choice of the unit section σ, normally e_{n}; thus Proposition 2.7 is applicable.

Set $\dot{S}_\sigma=\frac{1}{2}\sum_j f^j\wedge(\widehat{\nabla^E_{f_j}\sigma})^b\in T^*W\,\widehat{\otimes}\,\widehat{E^*}$ with $\{f_j\}$ a basis of TW and $\{f^j\}$ its dual basis. From Definition 2.1, for $t\geq 0$, $\sigma^*A_t=\frac{1}{2}\dot{R}^E+2\sqrt{t}\dot{S}_\sigma+t$. By (2.4), as in (2.8),

$$\sigma^*\psi(E,\nabla^E)=\sqrt{\pi}\int^B\widehat{\sigma}^b\exp\left(-\frac{1}{2}j^*\dot{R}^E\right)\sum_{k=0}^{\infty}\frac{(-\dot{S}_\sigma)^k}{2\Gamma\left(\frac{k}{2}+1\right)}\,. \tag{2.16}$$

Now if we take $\sigma=e_{\mathsf{n}}$, then by (1.15), $\dot{S}=\dot{S}_\sigma$. By (1.14), (1.17), and (2.16), we thus get the following result.

PROPOSITION 2.9. *With the notation of section 1 we have*

$$e_{\mathsf{n}}^*\psi\big(TX,\nabla^{TX}\big)=e_b(V,\nabla^{TX})\,. \tag{2.17}$$

3 Local Gauss–Bonnet–Chern Theorem

In this section, we establish a local version of the Gauss–Bonnet–Chern theorem for manifolds with boundary using heat-kernel methods. We do not assume that the metric has product structure near the boundary.

This section is organized as follows. In subsection 3.1, we state the Gauss–Bonnet–Chern theorem for manifolds with boundary and in subsections 3.2–3.8 we prove a local version. In subsection 3.2, using [BiZ1, §4h)] and a key estimate established in [RS, Lemma 5.12] (also cf. [Gr, (2.4.22)]), we reduce the problem to a problem near the boundary Y of X, and we explain that we can localize the problem. In subsection 3.3, we recall the Lichnerowicz formula from [BiZ1, Thm. 4.13]. In subsection 3.4, we reformulate our boundary condition and in subsection 3.5, we construct a trivialization and rescale the coordinate $Z \in T_{y_0}X$; we also use the Bismut–Zhang rescaling on Clifford variables along the boundary Y. Moreover we calculate the limits of the operators and the boundary conditions. In subsection 3.6, we establish uniform estimates on the heat kernels of the rescaled operators with suitable boundary conditions. In subsection 3.7, we obtain the explicit fundamental solution of our model problem. In subsection 3.8, we use the result of subsection 3.7 to prove Theorem 3.2, the local Gauss–Bonnet–Chern theorem.

We use the notation in subsection 1.3.

3.1 The Gauss–Bonnet–Chern theorem. Let X be a compact smooth manifold with boundary Y and dimension $\dim X = m$, and let F be a flat complex vector bundle over X with flat connection ∇^F.

Denote by $\Omega(X,F) := \oplus_{p=0}^m \Omega^p(X,F) := \oplus_{p=0}^m C^\infty(X, \Lambda^p(T^*X) \otimes F)$ the space of smooth differential forms on X with values in F. The bundle $\Lambda(T^*X)$ is $\mathbb{Z}$-graded, and so it possesses a natural $\mathbb{Z}_2$-grading. The flat connection ∇^F extends naturally to a differential, d^F, on $\Omega(X,F)$. The cohomology of the complex $(\Omega(X,F), d^F)$ is called the (absolute) de-Rham cohomology of X with coefficients in F, and is denoted by $H^\bullet(X,F)$. Define the Euler characteristic of F and X, respectively, by

$$\chi(X,F) := \sum_{p=0}^m (-1)^p \dim_{\mathbb{C}} H^p(X,F), \quad \chi(X) := \chi(X,\mathbb{C}). \tag{3.1}$$

Consider next an arbitrary Riemannian metric g^{TX} on X and an arbitrary Hermitian metric h^F on F, and denote by $\langle \cdot , \cdot \rangle_{\Lambda(T^*X)\otimes F}$ the induced Hermitian metric on $\Lambda(T^*X) \otimes F$, and let dv_X be the Riemannian volume element on (TX, g^{TX}), then we can view dv_X as a section of $\Lambda^m(T^*X) \otimes o(TX)$. We define a Hermitian product on $\Omega(X,F)$ by

$$\langle \sigma, \sigma' \rangle := \int_X \langle \sigma, \sigma' \rangle_{\Lambda(T^*X)\otimes F} dv_X \,, \tag{3.2}$$

for $\sigma, \sigma' \in \Omega(X,F)$; the Hilbert space obtained by completion is denoted by $L^2(X, \Lambda(T^*X) \otimes F)$. We consider d^F as an unbounded operator in

$L^2(X,\Lambda(T^*X)\otimes F)$ with domain $\Omega_0(X,F):=\{\sigma\in\Omega(X,F);\ \operatorname{supp}\sigma\cap Y=\emptyset\}$. The adjoint operator d^{F*} is also defined on $\Omega_0(X,F)$. Let F^* be the dual of F. Then ∇^F defines a connection ∇^F on F^* by the identification $F\to F^*$ induced by the metric h^F. We denote by $d^{F\otimes o(TX)}$ the differential on $\Omega(X,F^*\otimes o(TX))$ induced by the flat connection ∇^F on F^*. Let $*^F$ be the Hodge operator on $*^F:\Lambda(T^*X)\otimes F\to\Lambda(T^*X)\otimes F^*\otimes o(TX)$ defined by $(\sigma\wedge *^F\sigma')_F:=\langle\sigma,\sigma'\rangle_{\Lambda(T\ X)\otimes F}\,dv_X$. Then on $\Omega^p(X,F)$,

$$d^{F*}=(-1)^p(*^F)^{-1}d^{F\otimes o(TX)}*^F, \tag{3.3}$$

and so is

$$D:=d^F+d^{F*}. \tag{3.4}$$

Then $D^2=d^Fd^{F*}+d^{F*}d^F:\Omega^p(X,F)\to\Omega^p(X,F)$ is the Hodge Laplacian associated to the pair of metrics g^{TX} and h^F .

Next we need to define self-adjoint extensions of D by elliptic boundary conditions. To do so, we use the metric on X to identify the normal bundle N to Y in X with the orthogonal complement of TY in $TX|_Y$. Denote then by e_{n} the inward pointing unit normal vector field along Y. Then we put with $i(\cdot)$ interior multiplication

$$\begin{aligned}\Omega^p_a(X,F)&:=\big\{\sigma\in\Omega^p(X,F)\,;\ i(e_{\mathsf{n}})\sigma=i(e_{\mathsf{n}})(d^F\sigma)=0\text{ on }Y\big\}\,;\\ D_a&:=D|\Omega_a(X,F):=D|\oplus_{p=0}^m\Omega^p_a(X,F)\,;\\ H^p_a(X,F)&:=\operatorname{Ker}D_a\cap\Omega^p_a(X,F)\,.\end{aligned} \tag{3.5}$$

Note that D_a is essentially self-adjoint.

Theorem 3.1. *We have a canonical isomorphism*

$$H^p_a(X,F)\simeq H^p(X,F)\,. \tag{3.6}$$

Proof. If h^F is flat, (3.6) was proved in [RS, Prop. 4.2], (cf. also [BrüL, Thm. 4.2], [G, Thm. 2.7.3], [Mü1, p. 239]), but the same proof works in the general case. □

Recall that ∇^{TX} is the Levi–Civita connection on (TX,g^{TX}) and that the forms $e(TX,\nabla^{TX})$, $e(TY,\nabla^{TY})$ and $e_b(Y,\nabla^{TX})$ are defined in subsection 1.3. In the case $Y=\emptyset$, the Gauss–Bonnet–Chern theorem [Che2] gives

$$\chi(X,F)=\operatorname{rk}(F)\int_X e(TX,\nabla^{TX})\,.$$

If $Y\neq\emptyset$, then $e_b(Y,\nabla^{TX})$ is an $o(TY)$-valued $m-1$-form on Y. If m is odd, then

$$e(TX,\nabla^{TX})=0\,,\quad e_b(Y,\nabla^{TX})=\tfrac12 e(TY,\nabla^{TY})\,. \tag{3.7}$$

Thus we obtain a reformulation of the Gauss–Bonnet–Chern theorem [Ch1,2], [G, §2.7] in terms of Berezin integrals. This result is essentially