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Continuity Properties of Integral Kernels Associated with Schrödinger Operators on Manifolds

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Abstract. For Schrödinger operators (including those with magnetic fields) with singular scalar potentials on manifolds of bounded geometry, we study continuity properties of some related integral kernels: the heat kernel, the Green function, and also kernels of some other functions of the operator. In particular, we show the joint continuity of the heat kernel and the continuity of the Green function outside the diagonal. The proof makes intensive use of the Lippmann–Schwinger equation.

1. Introduction

The analysis of Schrödinger operators occupies a central place in quantum mechanics. Suitably normalized, over the configuration space $\mathbb{R}^n$ these operators have the form

$$H_{A,U} = (-i\nabla - A)^2 + U\,, \tag{1.1}$$

where A is the magnetic vector potential and U is an electric potential. A huge literature is dedicated to the study of properties of $H_{A,U}$ in its dependence on A and U, see the recent reviews [48, 55, 56]. An essential feature of the quantum-mechanical operators in comparison to the differential operator theory is admitting singular potentials [19], although the operator itself preserves some properties like regularity of solutions [29].

In generalizing the Euclidean case it is natural to consider operators acting on curved spaces like Riemannian manifolds, where the operators take the form

$$H_{A,U} = -\Delta_A + U \tag{1.2}$$

with $-\Delta_A$ being the Bochner Laplacian. It is worthnoting that the study of the quantum-mechanical Hamiltonians on Riemannian manifolds goes back to Schrödinger [50] and is not only of mathematical interest. Besides the applications to quantum gravity and to other fields of quantum physics where geometrical methods play a crucial role, properties of the Schrödinger operators on curvilinear manifolds find extensive applications in contemporary nanophysics, see, e.g., [10, 25].

One of the important questions in the investigation of the Schrödinger operators is the continuity properties of related integral kernels, for example, of the Green function $G(x, y; z)$ or of the heat kernel. Many physically important quantities are expressed through the values of these kernels at some points or their restrictions onto submanifolds, and these values are meaningless or, strictly speaking, are not defined if the kernels are not continuous (as integral kernels are, generally speaking, only measurable). For example, the calculation of the so-called Wigner $\mathcal{R}$-matrix involves the values $G(a, b; z)$ of the Green function at certain fixed a and b; matrices of such form are widely used in the theory of zero-range potentials [1], the scattering theory [61] and the charge transport theory [62]. Other examples are provided by the heat kernel trace used in the quantum gravity [2, 59] or by the calculation of the density of states involving the trace of the renormalized Green function, i.e., its suitably renormalized restriction onto the diagonal [37]. We also note that the continuity of the Green function frequently implies *a priori* the continuity of the eigenfunctions. We remark that the differentiability of the kernels is also of interest in some problems [30].

For the Schrödinger operator (1.1) without magnetic vector potential ($A = 0$) acting over a Euclidean configuration space, the continuity of the naturally related integral kernels was proved by B. Simon [54] for (singular) potentials from the Kato classes. The continuity in the case of the presence of magnetic vector potentials was stated in [55] as an open problem; only several years ago Simon's results were extended in [7,8] to magnetic Schrödinger operators on domains in Euclidean space with vector and scalar potentials of Kato's type. In the both cases, the proof used certain probabilistic technique. A part of the results concerning bounds of for the kernels admits a generalization to elliptic operators with singular coefficients of a more general form, see, e.g., [14, 20, 31, 38, 39, 43, 63].

The theory of Schrödinger operators with singular potentials on manifolds is still far from complete. There are numerous works concerning the bounds for the heat kernels [3, 12, 21–24, 64], mapping properties [33, 34, 52, 57] or some particular questions of the spectral analysis [32, 44, 45, 51], but the attention has been mostly concentrated either on the free Laplace–Beltrami operator or on special potentials. Sufficiently wide conditions for the essential self-adjointness of the Schrödinger operators have been established only very recently [6, 53] (see also [40, 41] for further developments). In this paper we are interested in the regularity properties of the kernels rather than the bounds for them; these questions were not addressed in any of the previous works. Because of the above described situation in the study of Schrödinger operators on manifolds, our restrictions on vector and scalar potentials are slightly stronger than in the Euclidian case, and they are of a different nature: the Kato or Stummel classes used in [54] have some relationship to the probabilistic technique, while our conditions come mostly from the operator-theoretical methods and formulated in terms of L^p-spaces (see Subsection 2.6 below). Nevertheless, the class of potentials we consider is wide enough in order to include physically reasonable local singularities and to satisfy the requirement stated by B. Simon [54], as it includes all continuous functions and Coulomb-like local singularities.

As it was mentioned already, the study of Schrödinger operators in the Euclidean case involved some probabilistic tools like Brownian motion or the Feynman–Kac formula. Here we employ a completely different technique from operator theory. Our main tool is the Lippmann–Schwinger equation for self-adjoint operators A and B with common domain:

$$(A-\lambda)^{-1} - (B-\lambda)^{-1} = (B-\lambda)^{-1}(B-A)(A-\lambda)^{-1} .$$

If $(A-\lambda)^{-1}$ and $(B-\lambda)^{-1}$ are integral operators, so is the right-hand side, but its kernel tends to have better regularity properties then both of the kernels on the left. Such an observation being combined with arguments like elliptic regularity provides the continuity of the Green function, which can be transferred to other kernels (in particular, to the heat kernel), using a combination of operator methods from [13, 54].

We would like to emphasize that, in contrast to the probabilistic technique, our approach can be applied to higher order differential operators. Moreover, the higher the order of an elliptic operator, the easier it is to satisfy the conditions of the main Lemma 13, so that our methods can give new results also in the Euclidean case. Nevertheless, we restrict ourselves to Schrödinger operators on manifolds in this paper.

The paper is organized as follows. In Section 2, we collect some facts about Schrödinger operators on manifolds of bounded geometry and introduce the class of potentials A and U in (1.2) to deal with. Section 3 contains some important integral estimates. In Section 4, we derive some estimates for the resolvent norms as well as necessary bounds for the heat kernel. Section 5 is devoted to the proof of the main result, Theorem 21, which contains the continuity of integral kernels for various functions of the operator. In the last section, Section 6, we discuss briefly possible generalizations and perspectives.

2. Preliminaries

2.1. Geometry

By X we denote a complete connected Riemannian manifold with metric $g = (g_{ij})$. Throughout the paper we suppose that X is of bounded geometry, which means that the injectivity radius r_{inj} of X is strictly positive and all the covariant derivatives of arbitrary order of the Riemann curvature tensor are bounded. Examples are provided by homogeneous spaces with invariant metrics, compact Riemannian manifold and their covering manifolds, the leaves of a foliation of a compact Riemannian manifold with the induced metric; we refer to [47,52] for further examples and a more extensive discussion. We put $\nu := \dim X$; through the paper $d(x, y)$ denotes the geodesic distance between points $x, y \in X$, the open ball with center

$a \in X$ and radius r is denoted by $B(a,r)$, $D = \{(x,y) \in X \times X : x = y\}$ denotes the diagonal in $X \times X$. The integral of a function f on X with respect to the Riemann–Lebesgue measure on X is denoted by $\int_X f(x)\,dx$, and $V(a,r)$ denotes the Riemannian volume of $B(a,r)$. We also fix a number $r_0 = r_0(X)$, such that $0 < r_0 < r_{\text{inj}}$. The following properties of manifolds with bounded geometry will be used below (see, e.g., Sections A1.1 and 2.1 in [52], as well as [58] for proofs and additional bibliographical hints).

(V1). There is a constant $w_1 \geq 1$ such that for every $a, b \in X$ and $0 < r \leq r_0$

$$w_1^{-1} \leq \frac{V(a,r)}{V(b,r)} \leq w_1 \,.$$

(V2). There are constants $w_2 > 0$ and $\theta_X > 0$ such that for all $a \in X$ and $r > 0$

$$V(a,r) \leq w_2\, e^{\theta_X r} \,.$$

(V3). There is a constant $w_3 \geq 1$ such that in each ball $B(a, r_0)$ there holds $w_3^{-1} \leq \sqrt{\det[g_{ij}(x)]} \leq w_3$ with respect to the normal coordinates x in $B(a, r_0)$.

Put $V_s(r) := \sup_{x\in X} V(x,r)$, $V_i(r) := \inf_{x\in X} V(x,r)$. Then the properties (V1) and (V2) imply

(V4). $0 < V_i(r) \leq V_s(r) < \infty \quad \forall\, r > 0$,

(V5). $V_s(r) = O(r^\nu)$ as $r \to 0$.

Moreover, from the well-known Toponogov triangle comparison theorem (see, e.g., [4], p. 281) we have

(V6). If f_a denotes the inverse of the exponential map in $B(a, r_0)$, then there is a constant $w_4 \geq 1$ independent of a such that $w_4^{-1}\, d(x,y) \leq |f_a(x) - f_a(y)| \leq w_4\, d(x,y)$ for any $x, y \in B(a, r_0)$.

Lemma 1. *If $0 < r' \leq r''$, then there is a number $N \in \mathbb{N}$ such that each ball of radius r'' can be covered by at most N balls of radius r'. Moreover, $N \leq V_s((r'/2) + r'')/V_i(r'/2)$.*

Proof. Let a ball $B(x, r'')$ be given. Take a maximal system of points $x_1, \dots, x_n$ from $B(x, r'')$ such that the balls $B(x_j, r'/2)$ do not intersect each other. Then the balls $B(x_j, r')$ cover $B(x, r'')$. On the other hand, $V(x, (r'/2) + r'') \geq n V_i(r'/2)$, hence $n \leq V_s((r'/2) + r'')/V_i(r'/2)$. □

2.2. Spaces and kernels

Let f be a measurable function on X; if $f \in L^p(X)$, $1 \leq p \leq \infty$, then $\|f\|_p$ denotes the norm of f in $L^p(X)$, otherwise we write $\|f\|_p = \infty$. Let S be a bounded linear operator from $L^p(X)$ to $L^q(X)$ with norm $\|S\|_{p,q}$. Such an operator always has a kernel $K = K_S$ in the sense of distributions; if $K \in L^1_{\text{loc}}(X \times X)$, then K is called an *integral kernel* of S. The operator S with an integral kernel K_S is called an *integral* operator if for $f \in L^p(X)$ and for a.e. $x \in X$ we have $K_S(x,\cdot) f(\cdot) \in L^1(X)$ (see, e.g., [27]; note that we consider only everywhere defined integral operators according to the terminology of [27]). In virtue of the Closed Graph Theorem, we have for an integral operator S with the kernel K

$$Sf(x) = \int_X K(x,y)f(y)\,dx \quad \text{for a.e. } x\,.$$

Note that S having an integral kernel is not necessary an integral operator in the above sense: the simplest example is the Fourier transform in $L^2(\mathbb{R}^\nu)$. Another example related to the subject of the paper is the resolvent $R(\zeta)$ of the free Hamiltonian $-\Delta$ in $L^2(\mathbb{R}^\nu)$ for $\nu \geq 4$: $R(\zeta)$ is not an integral operator in $L^2(\mathbb{R}^\nu)$ but has an integral kernel (the Green function). The Gelfand–Dunford–Pettis Theorem gives a useful criterion for S to be an integral operator; before we state this theorem we agree on a following notation: If r, s is another pair of numbers with $1 \leq r, s \leq \infty$ we denote $\|S\|_{r,s} = \sup\{\|Sf\|_s : f \in L^p(X) \cap L^r(X), \|f\|_r \leq 1\}$ (the equality $\|S\|_{r,s} = \infty$ is not excluded. Evidently, this definition does not lead to contradiction in the case $p = r$, $q = s$. Now we state the Gelfand–Dunford–Pettis Theorem in the form given in [36, §3.3]:

Theorem 2. *Let S be a bounded operator from $L^p(X)$ to $L^\infty(X)$ and $p < \infty$. Then S is an integral operator and we have for its kernel K_S*

$$\|S\|_{p,\infty} = \sup \mathrm{ess}_{x\in X} \|K_S(x,\cdot)\|_{p'} \quad \textit{with} \quad p' = (1-p^{-1})^{-1}\,.$$

In particular, if S is a bounded operator from $L^p(X)$ to $L^q(X)$ with $p < \infty$ and for some $r < \infty$ the condition $\|S\|_{r,\infty} < \infty$ is satisfied, then S has an integral kernel K_S and $\sup \mathrm{ess}_{x\in X} \|K_S(x,\cdot)\|_{r'} < \infty$.

If K_1 and K_2 are two integral kernels of S, then $K_1(x,y) = K_2(x,y)$ a.e. in $X \times X$. If, in addition, K_1 and K_2 are *separately continuous* on $(X \times X) \setminus D$, then $K_1 = K_2$ everywhere on $(X \times X) \setminus D$.

An integral kernel $K(x,y)$ is called a *Carleman* kernel if

$$\int_X |K(x,y)|^2\,dy < \infty \quad \text{for a.e. } x \in X\,.$$

A bounded operator on $L^2(X)$ having a Carleman kernel is called also a *Carleman operator*. It is clear that any Carleman operator is an integral operator.

Remark. By the definition from [54], Carleman kernels K obey the following additional condition $\sup \mathrm{ess}_{x\in X} \|K(x,\cdot)\|_2 < \infty$. We use the definition of Carleman kernels from [35], which is wider then that from [54].

Fix $r > 0$ and for each real p, $p \geq 1$, introduce the space $L^p_{\mathrm{unif}}(X)$ (*uniformly local L^p-space*) by

$$L^p_{\mathrm{unif}}(X) = \left\{ f \in L^p_{\mathrm{loc}}(X) \,:\, \sup_{x\in X} \int_{B(x,r)} \left|f(x)\right|^p dx < \infty \right\}$$

with the norm

$$\|f\|_p^{(r)} = \left(\sup_{x\in X} \int_{B(x,r)} \left|f(x)\right|^p dx \right)^{1/p}.$$

According to Lemma 1, the definition of L^p_{unif} is independent of r and all the norms $\|\cdot\|^{(r)}_p$ with p fixed are mutually equivalent; we will denote $\|\cdot\|^{(r_0)}_p$ simply by $\|\cdot\|_{p,\text{unif}}$. It is clear that $L^p + L^\infty \subset L^p_{\text{unif}} \subset L^p_{\text{loc}}$ and $L^p_{\text{unif}} \subset L^q_{\text{unif}}$, if $p \geq q$.

Lemma 3. *Let $f \in L^1_{\text{unif}}(X)$, $p \geq 1$, and $\omega > \theta_X$. Then for each $a \in X$, the function $g_a(x) = \exp(-\omega\, d(a,x)^p)f(x)$ belongs to $L^1(X)$ and $\|g_a\|_1 \leq c\,\|f\|_{1,\text{ unif}}$, where the constant c depend on ω only. Moreover,*

$$\int_{d(a,x)\geq r} \big|g_a(x)\big|\,dx \;\to\; 0 \quad \textit{as } r \to \infty$$

uniformly with respect to a and to f in the unit ball of $L^1_{\text{unif}}(X)$.

Proof. Let $n \in \mathbb{N}$ be arbitrary, then

$$\begin{aligned}\int_{B(a,n)} \big|g_a(x)\big|\,dx &= \sum_{k=1}^{n} \int_{B(a,k)\setminus B(a,k-1)} \big|g_a(x)\big|\,dx \\ &\leq \|f\|_{1,\text{ unif}} \sum_{k=1}^{n} N_k \exp\big(-\omega(k-1)\big)\,,\end{aligned}$$

where N_k is the minimal number of balls of radius r_0 covering the ball $B(a,k)$. Using Lemma 1 and the estimate (V2), we get $N_k \leq c''\exp(\theta_X\,k)$, where c'' is independent of n. Passing to the limit $n\to\infty$, we get the estimate $\|g_a\|_1 \leq c\,\|f\|_{1,\text{ unif}}$.

Represent now ω in the form $\omega = \omega' + \omega''$, where $\omega' > \theta_X$, $\omega'' > 0$. Then

$$\begin{aligned}\int\limits_{d(a,x)\geq r} \big|g_a(x)\big|\,dx &\leq \exp(-\omega'' r^p)\int\limits_X \exp\big(-\omega' d(a,x)^p\big)\big|f(x)\big|\,dx \\ &\leq c'\exp\big(-\omega'' r^p\big)\|f\|_{1,\text{unif}}\,.\end{aligned}$$

□

2.3. Self-adjoint operators

Let S be a self-adjoint operator in $L^2(X)$, not necessarily bounded. We denote by $\operatorname{spec}(S)$ the spectrum of S and by $\operatorname{res}(S)$ the resolvent set $\mathbb{C}\setminus\operatorname{spec}(S)$. For $\zeta \in \operatorname{res}(S)$ we denote by $R_S(\zeta)$ (or simply by $R(\zeta)$) the resolvent of S: $R_S(\zeta) = (S-\zeta)^{-1}$. The kernel of $R(\zeta)$ in the sense of distributions is called the *Green function* of S and will be denoted by $G_S(x,y;\zeta)$. For $\kappa > 0$ and $\zeta \in \operatorname{res}(S)$, $\operatorname{Re}\zeta < \inf\operatorname{spec}(S)$, we will consider the power $R^\kappa_S(\zeta)$ of $R(\zeta)$ defined by

$$R^\kappa_S(\zeta) = \frac{1}{\Gamma(\kappa)}\int_0^\infty e^{-t(S-\zeta)}\,t^{\kappa-1}\,dt\,, \tag{2.1}$$

where the integral is taken in the space of bounded operators in $L^2(X)$ (it converges absolutely there). It is clear that for an integer κ, (2.1) gives the usual power of $R(\zeta)$. The (distributional) kernel for $R^\kappa_S(\zeta)$ will be denoted by $G^{(\kappa)}_S(x,y;\zeta)$. Note, that instead of S we will use for resolvents, propagators and their kernels other subscripts identifying the operator S and will omit these subscripts, if confusion is excluded.

For numbers p and q with $1 \le p \le q \le \infty$ we will consider the following condition on the operator S assuming S is semibounded below:

(S_{pq}). there exist constants $B_{p,q} > \max(-\inf \operatorname{spec}(S), 0)$ and $C_{p,q} > 0$ such that for every $t > 0$

$$\|e^{-tS}\|_{p,q} \le C_{p,q}\, t^{-\gamma} \exp(B_{p,q}\, t)\,, \quad \text{where} \quad \gamma = \frac{1}{2}\nu(p^{-1} - q^{-1})\,.$$

The proof of the following Theorem 4 is contained in the proofs of the Theorems B.2.1–B.2.3 in [54]. We include the proof for sake of completeness.

Theorem 4. *Let S be a self-adjoint semibounded below operator in $L^2(X)$ obeying the condition* (S_{pq}) *above for some p and q with $1 \le p \le q \le \infty$ and let $\kappa > 0$ with $p^{-1} - q^{-1} < 2\kappa/\nu$ be given. Then the following assertions are true:*

(1) $\|R^\kappa(\zeta)\|_{p,q} < \infty$, *if one of the following conditions is satisfied:*
 (a) $\operatorname{Re}\zeta < -B_{p,\infty}$;
 (b) *κ is an integer and $p \le 2 \le q$.*
 Moreover, $\|R^\kappa(\zeta)\|_{p,q} \to 0$ *as* $\operatorname{Re}\zeta \to -\infty$.

(2) *Assume additionally that $p \le 2 \le q$. Take a Borel function f on* $\operatorname{spec}(S)$ *satisfying for $\xi \in \operatorname{spec}(S)$ the estimate $\big|f(\xi)\big| \le b\big(|\xi|+1\big)^{-\kappa}$ with some $b > 0$. Then $\|f(S)\|_{p,q} < Cb$ where $C > 0$ is independent of b. Suppose $q = \infty$, then $f(S)$ is an integral operator, if $\kappa > \nu/2p$. Moreover, in the last case $f(S)$ is a Carleman operator if $p = 2$ and has has an integral kernel $F(x,y)$ bounded by the constant Cb if $p = 1$.*

(3) *Suppose $q = \infty$ and take $\zeta \in \operatorname{res}(S)$. Then $R^\kappa(\zeta)$ has an integral kernel if one of the following conditions is satisfied:* (a) $\operatorname{Re}\zeta < -B_{p,\infty}$; (b) *$\kappa$ is an integer and $p \le 2$. Moreover, in both cases, $R^\kappa(\zeta)$ is a Carleman operator if $p = 2$, and the integral kernel of $R^\kappa(\zeta)$ is bounded, if $p = 1$.*

Proof. According to (2.1),

$$\|R^\kappa(\zeta)\|_{p,q} \le \frac{1}{\Gamma(\kappa)} \int_0^\infty \|e^{-tS}\|_{p,q} e^{t \operatorname{Re}\zeta}\, t^{\kappa-1}\, dt\,,$$

if $\operatorname{Re}\zeta < -B_{p,q}$. Therefore, in this case (1a) is proven and $\|R^\kappa(\zeta)\|_{p,q} \to 0$ as $\operatorname{Re}\zeta \to -\infty$. Before completing the proof of the item (1) we prove the item (2).

Fix E, $E < B_{p,q}$, and let $g(\xi) = (\xi - E)^\kappa f(\xi)$. Represent κ as the sum $\kappa = \kappa' + \kappa''$ such that $p^{-1} - 2^{-1} < 2\kappa'/\nu$, $2^{-1} - q^{-1} < 2\kappa''/\nu$, then by (1a) we have $b_1 := \|R^{\kappa'}(E)\|_{p,2} < \infty$ and $b_2 := \|R^{\kappa''}(E)\|_{2,q} < \infty$. Moreover, $|g(\xi)| \le b_3 < \infty$ for all $\xi \in \operatorname{spec}(S)$. Since $f(S) = R^{\kappa''}(E) g(S) R^{\kappa'}(E)$, we get $\|f(S)\|_{p,q} \le bC$ with $b = b_1 b_2 b_3$. The last statements of the item (2) follow immediately from Theorem 4. The subitem (1b) follows easily from (2); the item (3) is the consequence of Theorem 2 and items (1), (2). $\square$

Corollary 5. *Let the operator S satisfy the conditions of Theorem* 4. *Then the following assertions are true:*

(1) *Let $0 \le p^{-1} - q^{-1} < 2/\nu$ and numbers r, s be taken such that $1 \le r, s \le \infty$, $r^{-1} = p^{-1} - s^{-1}$. If $W \in L^s(X)$, then $\|R(E)W\|_{r,q} < \infty$ for $E < 0$ with sufficiently large $|E|$; moreover $\|R(E)W\|_{r,q} \to 0$ as $E \to -\infty$.*

(2) *Let in addition $q = \infty$. Then for any $W \in L^p(X)$ we have*

(2a) *$\|R(E)W\|_{\infty,\infty} < \infty$ for $E < 0$ with sufficiently large $|E|$ and $\|R(E)W\|_{\infty,\infty} \to 0$ as $E \to -\infty$;*

(2b) *$\|\,|W|^{1/2}R(E)|W|^{1/2}\|_{2,2} < \infty$ for $E < 0$ with sufficiently large $|E|$ and $\|\,|W|^{1/2}R(E)|W|^{1/2}\|_{2,2} \to 0$ as $E \to -\infty$.*

Proof. (1) Since W is a continuous mapping from L^r to L^p, the proof follows from the item (1) of the theorem.
(2a) This item is a particular case of (1).
(2b) It follows from (2a) by duality $\||W|\,R(E)\,\|_{1,1} \to 0$ as $E \to -\infty$, therefore the item (2b) follows from the Stein interpolation theorem (see the approach (2a) to the proof of formula (A26) in [54]). □

Remark. We emphasize that the item (3) of Theorem 4 can be considerably refined for functions of Schrödinger operators in the Euclidian spaces, see, e.g., [5, 20].

For our purpose, a class of Carleman operators S in $L^2(X)$ is important; this class consists of operators with integral kernels K having the following continuity conditions:

(C1). for every $f \in L^2(X)$ the function $g_f(x) = \displaystyle\int_X K(x, y)f(y)\,dy$ is continuous;

(C2). the function $X \ni x \mapsto \displaystyle\int_X |K(x, y)|^2\,dy$ is continuous.

Remark. In virtue of (C1) the image of an operator S with the corresponding kernel K consists of continuous functions. Moreover, S is a continuous mapping from $L^2(X)$ to the space $C(X)$ endowed with the topology of uniform convergence on compact sets. Note that the inclusion $S\big(L^2(X)\big) \subset C(X)$ alone does not imply the continuity of the functions in (C1), these need only be continuous after a modification on a set of measure zero.

Proposition 6. *If a kernel K fulfills the conditions* (C1) *and* (C2)*, then the mapping $F : x \mapsto K(x, \cdot)$ from X to the Hilbert space $L^2(X)$ is continuous.*

Proof. The condition (C1) shows that F is continuous with respect to the weak topology of $L^2(X)$, and (C2) implies that $x \mapsto \|F(x)\|$ is continuous. □

Using the proofs of Lemmas B.7.8 and B.7.9 from [54], we obtain easy the following theorem:

Theorem 7. (1) *Let Q, S, and T be bounded operators in $L^2(X)$ such that S and T have Carleman kernels with the properties* (C1) *and* (C2) *above. Then S^*QT is a Carleman operator with a continuous kernel in $X \times X$.*

(2) *Let S be a self-adjoint operator in $L^2(X)$ and f be a Borel function on* $\operatorname{spec}(S)$ *such that for all* $\xi \in \operatorname{spec}(S)$ *there holds* $\left|f(\xi)\right| \le b\left(|\xi|+1\right)^{-2\kappa}$ *with* $b > 0$, $\kappa > 0$. *If for some* $\zeta \in \operatorname{res}(S)$ *the operator* $R_S^\kappa(\zeta)$ *has a Carleman kernel with properties* (C1) *and* (C2), *then* $f(S)$ *is a Carleman operator and its kernel* $F(x,y)$ *is continuous in* $X \times X$. *Moreover, if* $\|R_S^\kappa(\zeta)\|_{2,\infty} \le c$, *then* $|F(x,y)| \le b\,c^2$ *for all* $x, y \in X$.

2.4. Schrödinger operators and related kernels

We denote by H_0 the Laplace–Beltrami operator on X, $H_0 = -\Delta$ (the Schrödinger operator of a free charged particle on X). The corresponding resolvent, the Green function and the integral kernel of the Schrödinger semigroup (heat kernel) e^{-tH_0} are denoted by $R_0(\zeta)$, $G_0(x,y\,;\zeta)$, and $P_0(x,y\,;t)$, respectively. Let $A = \sum_{j=1}^{\nu} A_j\,dx^j$ be a 1-form on X, for simplicity we suppose here $A_j \in C^\infty(X)$. The functions A_j can be considered as the components of the vector potential of a *magnetic field* on X. On the other hand, A defines a connection ∇_A in the trivial line bundle $X \times \mathbb{C}$, $\nabla_A u = du + iuA$; by $-\Delta_A = \nabla_A^* \nabla_A$ we denote the corresponding Bochner–Laplacian. The operator $H_A = -\Delta_A$ is essentially self-adjoint on $C_0^\infty(X)$. In addition, we consider a scalar potential U of an electric field on X, which is a real-valued measurable function, $U \in L^2_{\mathrm{loc}}(X)$; if $H_A + U$ is essentially self-adjoint on $C_0^\infty(X)$, then its closure (the *magnetic Schrödinger operator*) is denoted by $H_{A,U}$. The corresponding resolvent, Green function, and the heat kernel will be denoted by $R_{A,U}$, $G_{A,U}$, and $P_{A,U}$, respectively.

For real valued functions U on X we denote as usual $U_+ := \max(U,0)$, $U_- := \max(-U,0) \equiv U_+ - U$. The following result of M. Shubin plays one of the crucial part below (see [53, Theorem 1.1]):

Theorem 8. *Let U be a real-valued function on X such that $U_+ \in L^2_{\mathrm{loc}}(X)$ and $U_- \in L^p_{\mathrm{loc}}(X)$ with $p = \nu/2$ if $\nu \ge 5$, $p > 2$ if $\nu = 4$, and $p = 2$ if $\nu \le 3$. If $H_A + U$ is semi-bounded below on $C_0^\infty(X)$, then $H_A + U$ is essentially self-adjoint on $C_0^\infty(X)$.*

The properties of $P_0(x,y\,;t)$ we need below are presented in the following theorem, see [21, Formula 3.14]:

Theorem 9. *The function $P_0(x,y\,;t)$ is of class C^∞ on $X \times X \times (0,\infty)$ and*

$$0 \le P_0(x,y\,;t) \le \frac{C_P}{\min(t^{\nu/2},1)}\left(1 + \frac{d(x,y)^2}{t}\right)^{\frac{\nu}{2}+1} \exp\left(-\frac{d(x,y)^2}{4t} - \lambda t\right), \quad (2.2)$$

where $C_P > 0$, $\lambda = \inf \operatorname{spec}(H_0)$. Moreover,

$$\sup_{x,t} \int_X P_0(x,y\,;t)\,dy \le 1\,. \quad (2.3)$$

2.5. Kato's inequality

We recall that an everywhere defined linear operator $S : L^p(X) \to L^q(X)$ is said to be *positive in the sense of the point-wise order* or *positivity preserving*, if $Sf(x) \geq 0$ a.e for every $f \in L^p(X)$ with $f(x) \geq 0$ a.e.; such an operator is bounded [60]. A positive operator S *dominates* an everywhere defined linear operator $T : L^p(X) \to L^q(X)$ in the sense of the point-wise order, if for all $f \in L^p(X)$ we have $|Tf(x)| \leq S|f|(x)$ a.e. If T is dominated by a positive operator, then T is bounded and there is a positive operator $|T| : L^p(X) \to L^q(X)$ with the following properties:

(1) T is dominated by $|T|$,
(2) if S is another positive operator which dominates T, then $S - |T|$ is positive preserving (in symbols: $|T| \leq S$ *in the sense of the point-wise order*); it is clear that in this case $\|T\|_{p,q} \leq \|S\|_{p,q}$.

Moreover, every bounded linear operator from $L^p(X)$ to $L^\infty(X)$ is dominated by a positive operator S and we have for integral kernels $|K_T(x,y)| \leq K_S(x,y)$ a.e. [49, 60].

The main tool to extend results obtained for a Schrödinger operator without magnetic fields to that with a nontrivial magnetic field is the following theorem which combines [6, Theorem 5.7] and [28, Theorem 2.15].

Theorem 10. *Let U satisfy the condition of Theorem 8 and let $H_A + U$ be semi-bounded below on $C_0^\infty(X)$. Then the following assertions are true.*

(1) *Semigroup dominations: For every t, $t > 0$, we have $|e^{-tH_{A,U}}| \leq e^{-tH_{0,U}}$ in the sense of the point-wise order in $L^2(X)$; hence, $|P_{A,U}(x,y;t)| \leq P_{0,U}(x,y;t)$ for a.e. $x, y \in X$.*
(2) *Resolvent dominations: For every E, $E < 0$, with sufficiently large $|E|$, we have $|R_{A,U}(E)| \leq R_{0,U}(E)$ in the sense of the point-wise order in $L^2(X)$; hence, $|G_{A,U}(x,y;E)| \leq G_{0,U}(x,y;E)$ for a.e. $x, y \in X$.*

2.6. Admissible potentials, convenient kernels

The main results of the paper require some properties of considered potentials and kernels. We call a potential U *admissible*, if $U_+ \in L^{p_0}_{\mathrm{loc}}(X)$ and $U_- \in \sum_{i=1}^n L^{p_i}(X)$, where $2 \leq p_i \leq \infty$ if $\nu \leq 3$, $\nu/2 < p_i \leq \infty$ if $\nu \geq 4$ $(0 \leq i \leq n)$ (we stress that p_i depend on U). The class of all admissible potentials will be denoted by $\mathcal{P}(X)$. It is clear that $\mathcal{P}(X)$ is a saturated cone in the space of all measurable real valued functions $L^0_{\mathbb{R}}(X)$ on X, i.e., if $U_1, U_2 \in \mathcal{P}(X)$, then

- $\lambda_1, \lambda_2 \in \mathbb{R}_+$ implies $\lambda_1 U_1 + \lambda_2 U_2 \in \mathcal{P}(X)$;
- $V \in L^0_{\mathbb{R}}(X)$, and $U_1 \leq V \leq U_2$ implies $V \in \mathcal{P}(X)$.

We show in Section 4 that $H_A + U$ is essentially self-adjoint and semi-bounded below on $C_0^\infty(X)$ if $U \in \mathcal{P}(X)$.

To use the Lippmann–Schwinger equation we need some restriction on the integral kernels which control the behavior of the kernels near the diagonal and on the infinity. The norm estimates of the Green functions from Theorem 4 show a usefulness of the following classes of kernels. Let $0 \leq \alpha < \nu$, $1 \leq p \leq \infty$. We denote by $\mathcal{K}(\alpha, p)$ the class of all measurable functions K *everywhere defined* on $(X \times X) \setminus D$ if $\alpha > 0$ and on $X \times X$ if $\alpha = 0$, and obeying the conditions

(L1). for a constant $c = c(K) > 0$ there holds $|K(x,y)| \le c\max(1, d(x,y)^{-\alpha})$ for *all* $(x,y) \in (X \times X) \setminus D$ if $\alpha > 0$ and for *all* x, $y \in X$ otherwise;

(L2). for every $r > 0$ (or, which is the same, for all sufficiently small $r > 0$) there holds

$$\lfloor K \rfloor_{p,r} := \max\Big(\sup \mathrm{ess}_{x\in X} \left\|\chi_{X\setminus B(x,r)}\, K(x,\cdot)\right\|_p , \\ \sup \mathrm{ess}_{y\in X} \left\|\chi_{X\setminus B(y,r)}\, K(\cdot,y)\right\|_p \Big) < \infty ,$$

where χ_A denotes the characteristic function of $A \subset X$.

By $\mathcal{K}_0(\alpha,p)$ we will denote the subclass of $\mathcal{K}(\alpha,p)$ consisting of all functions K from $\mathcal{K}(\alpha,p)$ obeying the condition

(L3). $\lim\limits_{r\to\infty} \lfloor K \rfloor_{p,r} = 0$.

Below we list the simplest properties of the classes $\mathcal{K}(\alpha,p)$ and $\mathcal{K}_0(\alpha,p)$ which are needed below.

(K1). If $\alpha > 0$, then the condition (L1) is equivalent to each of the following ones:

(L1a). for a constants $r > 0$ and $c > 0$ there holds: $|K(x,y)| \le c\, d(x,y)^{-\alpha}$ if $0 < d(x,y) < r$ and $|K(x,y)| \le c$ if $d(x,y) \ge r$;

(L1b). for every $r > 0$ there is a constants $c > 0$ such that: $|K(x,y)| \le c\, d(x,y)^{-\alpha}$ if $0 < d(x,y) < r$ and $|K(x,y)| \le c$ if $d(x,y) \ge r$.

For $\alpha = 0$ the condition $0 < d(x,y) < r$ must be replaced by $d(x,y) < r$.

(K2). $\mathcal{K}(\alpha_1,p) \subset \mathcal{K}(\alpha_2,p)$ and $\mathcal{K}_0(\alpha_1,p) \subset \mathcal{K}_0(\alpha_2,p)$, if $0 \le \alpha_1 \le \alpha_2 < \nu$ and $1 \le p \le \infty$.

(K3). If K satisfies (L1), then $K \in \mathcal{K}(\alpha,\infty)$. In particular, $\mathcal{K}(\alpha,p) \subset \mathcal{K}(\alpha,\infty)$ for all $p \ge 1$ and α, $0 \le \alpha < \nu$. Therefore $\mathcal{K}(\alpha,p) \subset \mathcal{K}(\alpha,q)$ if $p \le q \le \infty$, and $\mathcal{K}_0(\alpha,p) \subset \mathcal{K}_0(\alpha,q)$ if $p \le q < \infty$.

(K4). All the classes $\mathcal{L} = \mathcal{K}(\alpha,p)$ or $\mathcal{L} = \mathcal{K}_0(\alpha,p)$ are order ideals in the space of measurable functions $L^0(X\times X)$, i.e., $\mathcal{L}$ is a linear subspace in $L^0(X \times X)$ with the property: If $K \in \mathcal{L}$, $L \in L^0(X\times X)$ and $|L(x,y)| \le |K(x,y)|$ for all (x,y), then $L \in \mathcal{L}$.

From Lemma 12 below (Section 3) we get obviously the following property

(K5). $\mathcal{K}(\alpha,p) \subset L^1_{\mathrm{loc}}(X\times X)$ for all p and α with $1 \le p \le \infty$, $0 \le \alpha < \nu$.

The next lemma delivers an important class of functions belonging to $\mathcal{K}_0(\alpha,p)$.

Lemma 11. *Let $P(x,y;t)$ be a measurable function on $X \times X \times (0,\infty)$ such that for some constants c_j, $c_j > 0$, $j = 1,2,3$, we have the estimate*

$$\left|P(x,y;t)\right| \le c_1 t^{-\nu/2} \exp\left(c_2 t - c_3 \frac{d(x,y)^2}{t}\right). \tag{2.4}$$

Let $\alpha = 0$ if $\nu = 1$, α be an arbitrary number from the interval $(0,\nu)$ if $\nu = 2$, and $\alpha = \nu - 2$ if $\nu \ge 3$. Let $\gamma \ge 0$; for any $\zeta \in \mathbb{C}$, $\mathrm{Re}\,\zeta < -c_2$, define the following *kernel* $K_\gamma(x,y;\zeta)$ *by the Laplace transform*

$$K_\gamma(x,y;\zeta) = \int_0^\infty t^\gamma\, P(x,y;t)\, e^{t\zeta}\, dt\,.$$

Then K_γ *belongs to all the classes* $\mathcal{K}_0(\alpha,p)$ *with* $1 \le p \le \infty$ *for all* ζ *with* $\operatorname{Re}\zeta$ *sufficiently close to* $-\infty$.

Proof. Since the kernel $t^\gamma P(x,y;t)$ admits the estimate of the type (2.4), it sufficient to consider the case $\gamma = 0$ only. It is well known that for any fixed c, $c > 0$, the function $K_0(x,E)$ defined for $x > 0$, $E < 0$ as the integral

$$K_0(x,E) := \int_0^\infty t^{-\nu/2} \exp(Et - cx^2 t^{-1})\, dt\,,$$

has the following asymptotic properties:

(1) for fixed $E < 0$ there holds $K_0(x,E) = O(h(x))$ as $x \to 0$, where

$$h(x) = \begin{cases} x^{-\nu+2}\,, & \text{if } \nu > 2\,,\\ |\log x|\,, & \text{if } \nu = 2\,,\\ 1\,, & \text{if } \nu = 1\,. \end{cases}$$

(2) For every $\delta > 0$ there exist $c'_\delta, c''_\delta > 0$ such that for $|x| \ge \delta$ we have $K_0(x,E) \le c'_\delta \exp(-c''_\delta x)$; here c'_δ is independent of E with $E \le -1$ whereas $c''_\delta \to +\infty$ as $E \to -\infty$.

Now the property (L1) for K follows from (1) and (L3) from (2) (if $p < \infty$ we use additionally Lemma 3). □

3. Auxiliary results concerning convenient kernels

Lemma 12. (1) *Let* $\alpha \in \mathbb{R}$, $\alpha < \nu$, *and* $a \in X$. *Then for every* $x \in X$ *and* $r > 0$ *there holds*

$$J_1(x) := \int_{B(a,r)} d(x,y)^{-\alpha}\, dy < \infty\,. \tag{3.1}$$

Moreover, there exists a constant $\tilde c_\alpha$ *depending only on* α, *such that if* $r \le r_0/3$, *then*

$$J_1(x) \le \begin{cases} \tilde c_\alpha r^{\nu-\alpha}\,, & \text{if } \alpha > 0\,,\\ \tilde c_\alpha\, r^\nu \big(r + d(a,x)\big)^{-\alpha} & \text{otherwise}\,. \end{cases}$$

(2) *Let* $0 < \alpha_1$, $\alpha_2 < \nu$ *and* $\beta = \alpha_1 + \alpha_2 - \nu$. *Then there is a constant* $c > 0$ *such that for any* $a \in X$ *and any* r, $0 < r < r_0$, *we have for* $x, y \in B(a,r)$, $x \ne y$:

$$J_2(x,y) := \int\limits_{B(a,r)} d(x,z)^{-\alpha_1} d(y,z)^{-\alpha_2}\, dz \le \begin{cases} c\, d(x,y)^{-\beta}\,, & \text{if } \beta > 0\,,\\ c\,\big(|\log d(x,y)| + 1\big)\,, & \text{if } \beta = 0\,,\\ c\,, & \text{otherwise}\,. \end{cases}$$

Proof. (1) According to (V5) we can choose $c' > 0$ in such a way that $V_s(r) \le c' r^\nu$ if $r \le r_0$.

Let $\alpha \le 0$, then $d(x,y) \le r + d(a,x)$ for $y \in B(a,r)$, therefore

$$\int_{B(a,r)} \big(d(x,y)\big)^{-\alpha}\, dy \le \big(r + d(a,x)\big)^{-\alpha} V_s(r)\,,$$

in particular, for $r \le r_0$

$$J_1(x) \le c' r^\nu \big(r + d(a,x)\big)^{-\alpha}\,. \tag{3.2}$$

Let now $\alpha > 0$. Suppose firstly $d(a,x) \ge 2r$. Then for $y \in B(a,r)$ we have $d(x,y) \ge d(a,x) - d(a,y) \ge r$. Therefore

$$\int_{B(a,r)} \big(d(x,y)\big)^{-\alpha}\, dy \;\le\; V_s(r) r^{-\alpha}\,.$$

In particular, for $r \le r_0$

$$J_1(x) \le\; c'\, r^{\nu-\alpha}\,, \tag{3.3}$$

Now suppose $d(a,x) < 2r$. Then for $y \in B(a,r)$ we have $d(x,y) \le d(a,x) + d(a,y) < 3r$. Therefore

$$\int_{B(a,r)} \big(d(x,y)\big)^{-\alpha}\, dy \le \int_{B(x,3r)} \big(d(x,y)\big)^{-\alpha}\, dy = \int_0^{3r} \rho^{-\alpha}\, dV(x,\rho)\,,$$

where the integral in the right-hand side is Stieltjes with respect to the increasing function $\rho \to V(x,\rho)$. Using the integration by part we get:

$$\int_{B(x,3r)} \big(d(x,y)\big)^{-\alpha}\, dy = V(x,3r)(3r)^{-\alpha} + \alpha \int_0^{3r} \frac{V(x,\rho)}{\rho^{\alpha+1}}\, d\rho < \infty$$

since $V(x,\rho) \le c'' \rho^\nu$ by (V5). In particular, if $r \le r_0/3$, then

$$J_1(x) \le \frac{3^{\nu-\alpha}\nu c'}{\nu - \alpha}\, r^{\nu-\alpha}\,. \tag{3.4}$$

Now the result follows from (3.2), (3.3), and (3.4).

(2) Since property (2) is local, it follows from (V3) and (V6) that we can restrict ourselves by the proof for the case $X = \mathbb{R}^\nu$, $y = 0$. Then (2) follows from the inequality $|x - a| < r$ and the following assertion:

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 < \nu$, then for any $a \in \mathbb{R}^\nu$, $r > 0$, and $x \in B(a,r)$, $x \ne 0$, there holds

$$I(x) := \int\limits_{B(a,r)} \frac{dz}{|x-z|^{\alpha_1}|z|^{\alpha_2}}$$
$$\le \begin{cases} \dfrac{c'}{|x|^{\alpha_1+\alpha_2-\nu}} + \dfrac{c''}{(|a|+r)^{\alpha_1+\alpha_2-\nu}}\,, & \textit{if } \alpha_1+\alpha_2 \ne \nu\,, \\ c' \log \dfrac{|a|+r}{|x|} + c''\,, & \textit{otherwise}\,, \end{cases} \tag{3.5}$$

where the constants c' and c'' are positive and depend only on α_1 and α_2.

We start the proof of this assertion with the change of variables $z = |x|u$ in the integral (3.5); the result is

$$I(x) = |x|^{\nu-\alpha_1-\alpha_2} \int\limits_{B\left(\frac{a}{|x|},\frac{r}{|x|}\right)} \left|e_x - u\right|^{-\alpha_1} \left|u\right|^{-\alpha_2} du\,,$$

where $e_x = x/|x|$. Let $B = B\big(0, \frac{|a|+r}{|x|}\big)$, then

$$I(x) \le |x|^{\nu-\alpha_1-\alpha_2} \int_B \left|e_x - u\right|^{-\alpha_1} \left|u\right|^{-\alpha_2} du\,.$$

Denote $B' = B(0,2)$ and

$$c := \int_{B'} \left|e_x - u\right|^{-\alpha_1} \left|u\right|^{-\alpha_2} du$$

(c is independent of x in virtue of the rotational symmetry consideration). Since $|u| - 1 \le |e_x - u| \le 1 + |u|$, we have $2^{-1}|u| \le |e_x - u| \le 2|u|$ if $|u| \ge 2$; hence, $|e_x - u|^{-\alpha_1} \le 2^{|\alpha_1|}|u|^{-\alpha_1}$ for such values of u. Therefore,

$$\begin{aligned} I(x) &\le |x|^{\nu-\alpha_1-\alpha_2}\left(c + 2^{|\alpha_1|}\int_{B\setminus B'} \left|u\right|^{-\alpha_1-\alpha_2} du\right) \\ &= |x|^{\nu-\alpha_1-\alpha_2}\left(c + 2^{|\alpha_1|} s_\nu \int_2^{\frac{|a|+r}{|x|}} \rho^{\nu-1-\alpha_1-\alpha_2}\, d\rho\right), \end{aligned}$$

where s_ν is the area of the unit sphere in $\mathbb{R}^\nu$. Calculating the integral, we get the result. □

The following lemma plays the main part in the article. Below we denote as usual $p' = \frac{p}{p-1}$ for $1 \le p \le \infty$.

Lemma 13. *Take $K_j \in \mathcal{K}(\alpha_j, q_j)$, $j = 1, 2$, and let $W = \sum_{k=1}^n W_k$, where $W_k \in L^{p_k}(X)$. Denote $p_{\min} = \min\limits_{1\le k\le n} p_k$, $p_{\max} = \max\limits_{1\le k\le n} p_k$ and suppose that the following conditions are satisfied:*

(a) $\dfrac{1}{p_{\max}} + \dfrac{1}{q_1} + \dfrac{1}{q_2} = 1$;

(b) $p_{\min} \ge 1$, *if* $\alpha_1 = \alpha_2 = 0$, *and* $p_{\min} > \nu/(\nu - \max(\alpha_1, \alpha_2))$ *otherwise.*

Then for the function $F(x, y, z) = K_1(x, z)W(z)K_2(z, y)$, the following assertions are true.

(1) $F(x, y, \cdot) \in L^1(X)$ *for* $x \ne y$; *therefore the function* J,

$$J(x, y) = \int_X F(x, y, z)\, dz\,,$$

is well-defined on $(X \times X) \setminus D$.

(2) *Denote* $\alpha := \max\big(0,\, \alpha_1+\alpha_2-\frac{\nu}{p'_1},\dots,\, \alpha_1+\alpha_2-\frac{\nu}{p'_n}\big)$. *Then* $J\in\mathcal{K}(\alpha,\infty)$ *if* $p'_k(\alpha_1+\alpha_2)\neq\nu$ *for all* k, $k=1\dots n$. *Otherwise* $J\in\mathcal{K}(\alpha,\infty)$ *if* $\alpha>0$, *and* $J\in\mathcal{K}(\beta,\infty)$ *with arbitrary* $\beta>0$, *if* $\alpha\le 0$ (*we assume here* $\infty\cdot 0=0$).

(3) *Let* $p_{\max}<\infty$ *or at least one of the functions* K_j $(j=1,2)$ *belong to* $\mathcal{K}_0(\alpha_j,q_j)$ *with* α_j *and* q_j *obeying the conditions* (a) *and* (b). *Then the function* J *has the continuity properties listed below:*

(3a) *if* $K_1(\cdot,z)$ *is continuous in* $X\setminus\{z\}$ *for a.e.* $z\in X$, *then* $J(\cdot,y)$ *is continuous in* $X\setminus\{y\}$ for all $y\in X$;

(3b) *if* $K_2(z,\cdot)$ *is continuous in* $X\setminus\{z\}$ *for a.e.* $z\in X$, *then* $J(x,\cdot)$ *is continuous in* $X\setminus\{x\}$ for all $x\in X$;

(3c) *if* $K_1(\cdot,z)$ *and* $K_2(z,\cdot)$ *are continuous in* $X\setminus\{z\}$ *for a.e.* $z\in X$, *then* J *is continuous in* $(X\times X)\setminus D$.

(4) *Let* $\alpha_1+\alpha_2<\nu$. *If* $\alpha_1+\alpha_2\neq 0$, *assume additionally that* $W_k\in L^q_{\mathrm{loc}}(X)$ *for some* $q>\nu/(\nu-\alpha_1-\alpha_2)$ *and all* k, $k=1,\dots,n$. *Then* $F(x,y,\cdot)\in L^1(X)$ *for all* $x,y\in X$, *so that* J *is well-defined on* $X\times X$. *Moreover, if* $p_{\max}<\infty$ *or at least one of the conditions* $K_j\in\mathcal{K}_0(\alpha_j,q_j)$ $(j=1,2)$ *is satisfied, then the following continuity properties take place:*

(4a) *if* $K_1(\cdot,z)$ *is continuous in* $X\setminus\{z\}$ *for a.e.* $z\in X$, *then* $J(\cdot,y)$ *is continuous in* X for all $y\in X$;

(4b) *If* $K_2(z,\cdot)$ *is continuous in* $X\setminus\{z\}$ *for a.e.* $z\in X$, *then* $J(x,\cdot)$ *is continuous in* X for all $x\in X$;

(4c) *If* $K_1(\cdot,z)$ *and* $K_2(z,\cdot)$ *are continuous in* $X\setminus\{z\}$ *for a.e.* $z\in X$, *then then* J *is continuous in* $X\times X$.

Proof. First of all we conclude from the property (K3) (see Section 2.6) that for every k, $k=1,\dots,n$, there are $q_j^{(k)}$ such that $K_j\in\mathcal{K}_j(\alpha_j,\,q_j^{(k)})$ and the following properties are satisfied:

$$(\mathrm{a}_k)\quad \frac{1}{p_k}+\frac{1}{q_1^{(k)}}+\frac{1}{q_2^{(k)}}=1\,.$$

Moreover, it is clear that for all k

(b_k) $p_k\ge 1$, if $\alpha_1=\alpha_2=0$, and $p_k>\nu/(\nu-\max(\alpha_1,\alpha_2))$ otherwise.

Therefore, to prove the properties (1), (3) and (4) we can suppose $n=1$ since J is additive with respect to W. This is true for the property (2) as well, it sufficient to take into consideration (K2) from Section 2.6. Hence, further we consider the case $n=1$ only.

(1) Fix $x,y\in X$ such that $x\neq y$ and take η, $0<\eta<d(x,y)/2$. In the ball $B(x,\eta)$, we estimate $|F(x,y,z)|\le c d(x,z)^{-\alpha_1}|W(z)|$; therefore if $\alpha_1=0$, the inclusion $F(x,y,\cdot)\in L^1(B(x,\eta))$ is obvious. If $\alpha_1>0$, the inequality $p>\nu/(\nu-\max(\alpha_1,\alpha_2))$ implies $\alpha_1 p'<\nu$, hence $F(x,y,\cdot)\in L^1(B(x,\eta))$ in virtue of the Hölder inequality and the item (1) of Lemma 12. Similarly $F(x,y,\cdot)\in L^1(B(y,\eta))$. For the set $Z\equiv Z(x,y,\eta)=X\setminus(B(x,\eta)\cup B(y,\eta))$ we have $F(x,y,\cdot)\in L^1(Z(x,y,\eta))$ by the (L2) from the definition of the classes $\mathcal{K}$ and by Hölder again. Thus, $F(x,y,\cdot)\in L^1(X)$.

(2) Take r, $0 < r < r_0/2$. Then for $d(x,y) \geq 2r$ we have by Hölder and (L2):

$$\begin{aligned}\big|J(x,y)\big| &\leq \int_{X\setminus B(x,r)} \big|F(x,y,z)\big|\,dz + \int_{X\setminus B(y,r)} \big|F(x,y,z)\big|\,dz \\ &\leq 2\|W\|_p \lfloor K_1\rfloor_{q_1,r} \lfloor K_2\rfloor_{q_2,r}\,.\end{aligned}$$

Let now $0 < d(x,y) < 2r$. Take a ball $B(a,r)$ with $x, y \in B(a,r)$. For $p > 1$ we have as above, using additionally (L1),

$$\begin{aligned}|J(x,y)| \leq c &\int_{B(a,2r)} d(x,z)^{-\alpha_1} d(y,z)^{-\alpha_2} |W(z)|\,dz \\ &+ \int\limits_{X\setminus B(x,r)} |F(x,y,z)|\,dz + \int\limits_{X\setminus B(y,r)} |F(x,y,z)|\,dz \\ &\leq c\,\|W\|_p \left(\int\limits_{B(a,2r)} d(x,z)^{-\alpha_1 p'} d(y,z)^{-\alpha_2 p'}\,dz \right)^{1/p'} \\ &+ 2\|W\|_p \lfloor K_1\rfloor_{q_1,r} \lfloor K_2\rfloor_{q_2,r}\,,\end{aligned}$$

with a constant $c > 0$. Using now Lemma 12(2), we see that $J \in \mathcal{K}(\alpha,\infty)$ with required α. If $p = 1$, then with necessity $\alpha_1 = \alpha_2 = 1$ and the proof is obvious.

(3) Fix points $x_0, y_0 \in X$, $x_0 \neq y_0$ and take a number η such that $0 < \eta < d(x_0,y_0)/3$. Further fix $\epsilon > 0$ and show that η can be chosen in such a way that

- $|J(x,y_0) - J(x_0,y_0)| < \epsilon$ for $x \in B(x_0,\eta/2)$ in the case (3a);
- $|J(x_0,y) - J(x_0,y_0)| < \epsilon$ for $y \in B(y_0,\eta/2)$ in the case (3b);
- $|J(x,y) - J(x_0,y_0)| < \epsilon$ for $x \in B(x_0,\eta/2)$, $y \in B(y_0,\eta/2)$ in the case (3c).

For this purpose we take a number R, $R > 2d(x_0,y_0)$, then for every points $x \in B(x_0,\eta/2)$, $y \in B(y_0,\eta/2)$, the following estimate takes place

$$\begin{aligned}\big|J(x,y) - J(x_0,y_0)\big| \leq &\int\limits_{B(x_0,\eta)} \big|F(x,y,z)\big|\,dz + \int\limits_{B(x_0,\eta)} \big|F(x_0,y_0,z)\big|\,dz \\ &+ \int\limits_{B(y_0,\eta)} \big|F(x,y,z)\big|\,dz + \int\limits_{B(y_0,\eta)} \big|F(x_0,y_0,z)\big|\,dz \\ &+ \Bigg| \int\limits_{Z(x_0,y_0,\eta)\cap B(x_0,R)} \big[F(x,y,z) - F(x_0,y_0,z)\big]\,dz \Bigg| \\ &+ \int\limits_{X\setminus B(x_0,R)} \big|F(x,y,z)\big|\,dz + \int\limits_{X\setminus B(x_0,R)} |F(x_0,y_0,z)|\,dz\,,\end{aligned} \tag{3.6}$$

where as before $Z(x_0,y_0,\eta) = X \setminus \big(B(x_0,\eta) \cup B(y_0,\eta)\big)$. For $z \in B(x_0,\eta)$ we have $|F(x,y,z)| \leq c\,d(x,z)^{-\alpha_1}\,|W(z)|$, where c does not depend on x, y, and z. Since

$p'\alpha_1 < \nu$ for $1/p' + 1/p = 1$ we have, by Lemma 12(1) and the Hölder inequality,

$$\int_{B(x_0,\eta)} \bigl|F(x,y,z)\bigr|\,dz \le c'\eta^{(\nu/p')-\alpha_1} \tag{3.7}$$

where c' is independent of x and y. Similarly,

$$\int_{B(y_0,\eta)} \bigl|F(x,y,z)\bigr|\,dz \le c''\eta^{(\nu/p')-\alpha_2} \tag{3.8}$$

with c'' independent of x and y again. We stress that (3.7) and (3.8) are valid *for all* $x \in B(x_0, \eta/2)$, $y \in B(y_0, \eta/2)$, in particular, for $x = x_0$, $y = y_0$. Now we chose η such that $2c'\eta^{(\nu/p')-\alpha_1} + 2c''\eta^{(\nu/p')-\alpha_2} < \epsilon/3$. The sum of the last two terms in (3.6) are estimated from above with the help of the Hölder inequality by

$$2\,\lfloor K_1\rfloor_{q_1,R-d}\lfloor K_2\rfloor_{q_2,R}\,\|\chi_R W\|_p\,,$$

where $d = d(x_0, y_0)$ and χ_R is the characteristic function of the set $X \setminus B(x_0, R)$. Therefore we can assume by appropriate choice of R this sum is $< \epsilon/3$. Denoting $M(\eta, R) := Z(x_0, y_0, \eta) \cap B(x_0, R)$, it remains to prove for the obtained η and R that the following functions are continuous:

- $B(x_0, \eta/2) \ni x \mapsto \int_{M(\eta,R)} F(x, y_0, z)\,dz$ in the case (3a);
- $B(y_0, \eta/2) \ni y \mapsto \int_{M(\eta,R)} F(x_0, y, z)\,dz$ in the case (3b);
- $B(x_0, \eta/2) \times B(y_0, \eta/2) \ni (x, y) \mapsto \int_{M(\eta,R)} F(x, y, z)\,dz$ in the case (3c).

For this purpose we note that for $(x, y) \in B(x_0, \eta/2) \times B(y_0, \eta/2)$ and $z \in Z(x_0, y_0, \eta) \cap B(x_0, R)$ we have the estimate $\bigl|F(x, y, z)\bigr| \le \text{const}\,\bigl|W(z)\bigr|$ and $W \in L^1(B(x_0, R))$. Therefore the required continuity properties follow from the Lebesgue majorization theorem and conditions (3a)–(3c).

(4) Fix η with $0 < \eta < r_0$. To prove $F(x, y, \cdot) \in L^1(X)$ we need to consider only the case $x = y =: x_0$. But in this case we have with a constant $c > 0$ the estimates $|F(x_0, x_0, z)| \le c\,d(x_0, z)^{-\alpha_1-\alpha_2}\,|W(z)|$ for all $z \in B(x_0, \eta)$, $z \ne x_0$. Therefore, the inclusion $F(x_0, x_0, \cdot) \in L^1(B(x_0, \eta))$ is a consequence of Lemma 12(1), the inequality $(\alpha_1 + \alpha_2)q' < \nu$ and the Hölder inequality. The inclusion $F(x_0, x_0, \cdot) \in L^1(X \setminus B(x_0, \eta))$ follows from (L2) and the Hölder again.

For proving the properties (4a)–(4c) we proceed as in the proof of the item (3) and use the notations of this proof. Now we must consider only the case $x_0 = y_0$;

in this case we estimate

$$\begin{aligned}|J(x,y)-J(x_0,x_0)| \leq &\int\limits_{B(x_0,\eta)} \big|F(x,y,z)\big|\,dz + \int\limits_{B(x_0,\eta)} \big|F(x_0,x_0,z)\big|\,dz \\ &+ \Big| \int\limits_{B(x_0,R)\setminus B(x_0,\eta)} \Big(F(x,y,z)-F(x_0,x_0,z)\Big)\,dz \Big| \\ &+ \int\limits_{X\setminus B(x_0,R)} \big|F(x,y,z)\big|\,dz + \int\limits_{X\setminus B(x_0,R)} \big|F(x_0,x_0,z)\big|\,dz\,.\end{aligned}$$

The sum of the first two terms has the upper bound of the form $c'\eta^{(\nu/q')-\alpha_1-\alpha_2}$ where the exponent is strictly positive; the sum of the last two terms is estimated by $2\,\lfloor K_1\rfloor_{q_1,R-d}\lfloor K_2\rfloor_{q_2,R}\,\|\chi_R W\|_p$ as before, and to use the Lebesgue majorization theorem again we have the estimate $\big|F(x,y,z)\big| \leq \text{const}\,\big|W(z)\big|$ for $z\in B(x_0,R)\setminus B(x_0,\eta)$. ☐

The suppositions of Lemma 13 are essential. Indeed, there holds the following

Proposition 14. *There is a positive symmetric kernel $K\in C^\infty(\mathbb{R}^2)$ such that* (1) *K is Carleman, moreover, for any $a\in\mathbb{R}$ the functions $K(a,\cdot)$ and $K(\cdot,a)$ belong to $C_0^\infty(\mathbb{R})$; (2) K defines a bounded operator S in $L^2(\mathbb{R})$; (3) for some $f\in L^2(\mathbb{R})$ the function $g(x)=\int_{\mathbb{R}} K(x,y)f(y)\,dy$ is not equal a.e. to any continuous function on $\mathbb{R}$.*

Proof. To obtain a kernel K with the required properties we use a construction from [16]. Fix a function $\phi\in C^\infty(\mathbb{R})$ such that $\phi(x)=0$ if $x\leq 0$, $\phi(x)=1$ if $x\geq 1$, $\phi'(x)>0$ if $0<x<1$, and set $\phi_1(x):=\phi(x+1)\phi(2-x)$, $\phi_2(x)=\phi(x-2)$, $\psi(x)=\phi(2x)\phi(2-2x)$. Define the kernel $K(x,y):=M(x,y)+M(y,x)$ with $M(x,y):=\phi_1(x)\,\phi_2(y)\,L(x,y)$ and

$$L(x,y):=\begin{cases}0\,, & \text{for } x\leq 0 \text{ or } y\leq 0\,,\\ x^{-1}\psi(y-x^{-1})\,, & \text{for } x,y>0\,,\end{cases}$$

Let us prove (1). First we note that $\operatorname{supp} L\subset\{(x,y)\in\mathbb{R}^2 : x,y>0,\, xy>1\}$ and that the restriction of L to the set $\{(x,y)\in\mathbb{R}^2 : x>0, y>0\}$ is a C^∞-function, therefore $M\in C^\infty(\mathbb{R}^2)$. Denote $U_x=\operatorname{supp}\psi(\cdot-x^{-1})$, $V_x=\operatorname{supp}\psi(y-\cdot^{-1})$. It is easy to see that $U_x\subset\{y\in\mathbb{R} : x,y>0, ,x^{-1}<y<1+x^{-1}\}$, $V_y\subset\{x\in\mathbb{R} : x,y>0, ,y^{-1}<x<(y-1)^{-1}\}$, and (1) is proven.

Let us prove (2); actually we prove that M defines a bounded operator in $L^2(\mathbb{R})$. Denote $f(y)=\min(1,|y|^{-1})$; due to the Schur Theorem [27, Theorem 5.2] it is sufficient to prove that

$$\int_{\mathbb{R}} M(x,y)f(y)\,dy\leq C_1\,,\qquad \int_{\mathbb{R}} M(x,y)\,dx\leq C_2\,f(y)$$

with some constants $C_1, C_2 > 0$. We have

$$\int_{\mathbb{R}} M(x,y)f(y)\,dy \le x^{-1}\int_{y\ge 2,\ y\in U_x} \psi(y-x^{-1})f(y)\,dy\,.$$

If $y \in U_x$, then $x^{-1} \le y$, therefore $f(y) \le x$. Hence

$$\int_{\mathbb{R}} M(x,y)f(y)\,dy \le \int_{\mathbb{R}} \psi(y-x^{-1})\,dy = \int_{\mathbb{R}} \psi(x)\,dx < \infty\,.$$

On the other hand, if $y \le 2$, then $\int_{\mathbb{R}} M(x,y)\,dx = 0$. Suppose $y > 2$, then

$$\int_{\mathbb{R}} M(x,y)\,dx \le \int_{x>0,\ x\in V_y} x^{-1}\psi(y-x^{-1})\,dx\,.$$

If $x > 0$ and $x \in V_y$, then $y^{-1} \le x \le (y-1)^{-1}$, hence

$$\int_{\mathbb{R}} M(x,y)\,dx \le \int_{y^{-1}}^{(y-1)^{-1}} x^{-1}\,dx = \ln\big(1+(y-1)^{-1}\big) \le (y-1)^{-1} \le 2y^{-1}\,.$$

As a result, we have $\int_{\mathbb{R}} M(x,y)\,dx \le 2f(y)$ and the item (2) is proven. To prove (3), we first note that $f \in L^2(\mathbb{R})$ and then show that the function $g(x) = \int_{\mathbb{R}} K(x,y)f(y)\,dy$, (where f is defined above) is piecewise continuous in a neighborhood of the point $x = 0$ and has a jump at this point. Since $M(\cdot,x) = 0$ if $x < 2$, it is sufficient to prove that the function $h(x) = \int_{\mathbb{R}} M(x,y)f(y)\,dy$, is piecewise continuous and has a jump at the point $x = 0$. It is clear that $h(x) = 0$ if $x < 0$ or $x > 2$. Let $0 < x < 1/3$, then

$$h(x) \ge \int_3^\infty M(x,y)f(y)\,dy = x^{-1}\int_{y\ge 3,\, y\in U_x} \psi(y-x^{-1})f(y)\,dy\,.$$

If $y > 3$ and $y \in U_x$, then $y \le 1 + x^{-1}$, therefore, for the same values of y, $f(y) = y^{-1} \ge x(x+1)^{-1}$. Hence, we have for $0 < x < 1/3$

$$h(x) \ge (x+1)^{-1}\int_{y\ge 3,\, y\in U_x} \psi(y-x^{-1})\,dy = (x+1)^{-1}\int_{\mathbb{R}} \psi(y)\,dy \ge \frac{3}{4}\int_{\mathbb{R}} \psi(y)\,dy\,,$$

and the item (3) is proven. □

Take the kernel K and the function f from Proposition 14, then setting $K_1 = K$, $K_2 = 1$, $W = f$ in Lemma 13 we get a discontinuous function J, which demonstrates the importance of the assumptions in Lemma 13.

4. Norm estimates for the kernels

We start with an auxiliary result.

Lemma 15. *Let $V \in \mathcal{P}(X)$ be semi-bounded below: $V \ge -C_V$, where $C_V \ge 0$, then:*

(1) *$H_A + V$ is semi-bounded below and essentially self-adjoint on $C_0^\infty(X)$. For every $t > 0$ we have $|e^{-tH_{A,V}}| \le e^{C_V t}e^{-tH_0}$ in the sense of point-wise order.*

(2) *Let* $1 \le p \le q \le \infty$*. Then* $\|e^{-tH_{A,V}}\|_{p,q} \le C_{p,q}\, t^{-\gamma} \exp(B_{p,q} t)$*, where* $\gamma = \frac{1}{2}\nu(p^{-1} - q^{-1})$ *and* $B_{p,q}$*,* $C_{p,q} \ge 0$ *(i.e.,* $H_{A,V}$ *obeys the condition* (S_{pq}) *from Subsection* 2.3 for all p, q *with* $1 \le p \le q \le \infty$*).*

(3) *for* $t > 0$*,* $e^{-tH_{A,V}}$ *is an integral operator and* $|P_{A,V}(x, y; t)| \le e^{C_V t} P_0(x, y; t)$ *for a.e.* x, y*.*

(4) *for* $\kappa > 0$ *and* $E < 0$ *with sufficiently large* $|E|$*, the operator* $R^{\kappa}_{A,V}(E)$ *has an integral kernel* $G^{(\kappa)}_{A,V}(x, y; E)$ *obeying the condition* $|G^{(\kappa)}_{A,V}(x, y; E)| \le G^{(\kappa)}_0(x, y; E + C_V)$*. In particular, at least for* $\kappa \ge 1$ *we have* $G^{(\kappa)}_{A,V}(E) \in \mathcal{K}_0(\alpha, p)$ *for all* p*,* $1 \le p \le \infty$*, where* $\alpha = 0$ *if* $\nu = 1$*,* α *be an arbitrary number from the interval* $(0, \nu)$ *if* $\nu = 2$ *and* $\alpha = \nu - 2$ *if* $\nu \ge 3$*.*

Proof. (1) It is clear that the operator $H_A + V$ is semi-bounded below, therefore it is essentially self-adjoint on $C^\infty_0(X)$ by Theorem 8. In particular, $H_{0,V}$ is essentially self-adjoint on $\mathcal{D}(H_0) \cap \mathcal{D}(V)$. Hence, we can use the Trotter product formula and for $f \in L^2(X)$ we get

$$\exp(-tH_{0,V})f = \lim_{n\to\infty} \big(\exp(-tH_0/n)\exp(-tV/n)\big)^n f \tag{4.1}$$

with respect to the L^2-norm. Equation (4.1) shows that $0 \le e^{-tH_{0,V}} f \le e^{C_V t} e^{-tH_0} f$, if $f \ge 0$; in virtue of Theorem 10, the item (1) is proven.

(2) Inequality (2.3) means that $\sup\{\|e^{-tH_0}\|_{\infty,\infty} : t \ge 0\} \le 1$. On the other hand, we obtain from (2.2)

$$\sup_{x,y} P_0(x, y; t) \le \frac{\tilde{C}_P}{\min(t^{\nu/2}, 1)} \tag{4.2}$$

with $\tilde{C}_P \ge C_P$. This means that $\|e^{-tH_0}\|_{1,\infty} \le \tilde{C}_P \max(t^{-\nu/2}, 1)$. Using the Stein interpolation theorem (Theorem IX.21 from [46]) we finish the proof of the item (2).

(3) Theorem 2 and item (2) imply the first statement; the estimate follows from the estimate in (1).

(4) The existence of integral kernels is a consequence of the item (2) and Theorem 4. To get the estimates on the kernels we use the transformation (2.1) for the kernels from the item (3). The last assertion is an immediate consequence of Lemma 11. □

Remark. We stress again that the kernels $P_{A,V}$ and $G^{(\kappa)}_{A,V}$ are defined not uniquely but only modulo a negligible function. Moreover, $R^{\kappa}_{A,V}(\zeta)$ can be not an integral operator for every $\zeta \in \mathrm{res}(H_{A,V})$; i.e., this is the case, if $\kappa = 1$, $\nu \ge 4$.

Define kernels $K_\nu(x, y)$,

$$K_\nu(x, y) = \begin{cases} d(x, y)^{2-\nu}, & \text{if } \nu \ne 2, \\ |\log d(x, y)|, & \text{if } \nu = 2, \end{cases} \tag{4.3}$$

and for each function f from $L^1_{\rm loc}(X)$ and each $r > 0$ define the quantities (*"Kato norms"*)

$$\|f\|_{\rm K}^{(r)} := \sup_{x\in X} \int_{d(x,y)\le r} K_\nu(x,y)\,|f(y)|\,dy\,. \tag{4.4}$$

If $\|f\|_{\rm K}^{(r)} < \infty$ for some $r > 0$, then this holds for any $r > 0$.

Lemma 16. *Let $f \in L^p_{\rm unif}(X)$ where $p = 1$ if $\nu = 1$ and $p > \nu/2$ otherwise. Then*

$$\lim_{r\downarrow 0} \|f\|_{\rm K}^{(r)} = 0$$

uniformly in the unit ball $\|f\|_{p,{\rm unif}} \le 1$.

Proof. This is an immediate consequence of Lemma 12(1). □

Remark. Lemma 16 means that $L^p_{\rm unif}(X)$ is a subspace of the corresponding "Kato class", which can be defined on the manifold X in the same way as in the case of the Euclidean space $\mathbb{R}^\nu$ [13, 18].

Below we need the following lemmas.

Lemma 17. *Let $F(\rho,t)$ be a measurable function on $(0,\infty)\times(0,\infty)$ which obeys for each ρ and t the condition*

$$0 \le F(\rho\,,t) \le \frac{1}{\min(t^{\nu/2},1)} \exp\left(-\frac{\rho^2}{a^2t}\right)\,,$$

where $a > 0$ is fixed. For $0 \le t \le 1$ denote

$$Q(\rho,t) = \int_0^t F(\rho,\,s)\,ds\,.$$

Then with some constant $c_\nu > 0$ we have:

$$Q(\rho,\,t) \le c_\nu \frac{a^{\nu-2}}{\rho^{\nu-2}} \exp\left(-\frac{\rho^2}{2a^2t}\right) \qquad \textit{for } \nu \ge 3\,, \tag{4.5}$$

$$Q(\rho,\,t) \le \begin{cases} \left|\log(\rho^2/a^2t)\right| + 1\,, & \textit{if } \rho^2 < a^2t\,,\\ c_\nu \exp\left(-\dfrac{\rho^2}{2a^2t}\right), & \textit{if } \rho^2 \ge a^2t\,, \end{cases} \qquad \textit{for } \nu = 2\,, \tag{4.6}$$

and

$$Q(\rho,\,t) \le \begin{cases} 2\sqrt{t}\,, & \textit{if } \rho^2 < a^2t\,,\\ c_\nu \dfrac{\rho}{a} \exp\left(-\dfrac{\rho^2}{2a^2t}\right), & \textit{if } \rho^2 \ge a^2t\,, \end{cases} \qquad \textit{for } \nu = 1\,. \tag{4.7}$$

Proof. By the change of variable we obtain

$$\begin{aligned} Q(\rho,\, t) &\le \frac{a^{\nu-2}}{\rho^{\nu-2}} \int\limits_{\rho^2/a^2 t}^{+\infty} s^{\nu/2-2}\, e^{-s}\, ds \\ &\le \frac{a^{\nu-2}}{\rho^{\nu-2}} \exp\left(-\frac{\rho^2}{2ta^2}\right) \int\limits_{\rho^2/a^2 t}^{+\infty} s^{\nu/2-2}\, e^{-s/2}\, ds\,. \end{aligned} \tag{4.8}$$

Denoting

$$c_\nu := \int_0^{+\infty} s^{\nu/2-2}\, e^{-s/2}\, ds\,,$$

we get (4.5). Let $\nu \le 2$; in the case $\rho^2 < a^2 t$ we represent

$$\begin{aligned} \int_{\rho^2/a^2 t}^{+\infty} s^{\nu/2-2}\, e^{-s}\, ds &= \int_{\rho^2/a^2 t}^{1} s^{\nu/2-2}\, e^{-s}\, ds \,+ \int_{1}^{+\infty} s^{\nu/2-2}\, e^{-s}\, ds \\ &\le \int_{\rho^2/a^2 t}^{1} s^{\nu/2-2}\, ds \,+ e^{-1}\,, \end{aligned}$$

and from the first inequality in (4.8) obtain immediately (4.6) and (4.7) for the considered case. If $\rho^2 \ge a^2 t$ we denote

$$c_\nu := \int_1^{+\infty} s^{\nu/2-2}\, e^{-s/2}\, ds\,,$$

and finish the proof of (4.6) and (4.7). □

Lemma 18. *Let $W \in L^p_{\text{unif}}(X)$ where $p = 1$ if $\nu = 1$ and $p > \nu/2$ otherwise, and let $P(x, y; t) = F(d(x, y),\, t)$ where F is from Lemma 17. Then for all sufficiently small $t > 0$ there holds*

$$\sup_x \int_X \int_0^t P(x, y\,; s)\, |W(y)|\, ds\, dy < \infty\,.$$

Moreover,

$$\lim_{t\downarrow 0} \sup_x \int_X \int_0^t P(x, y\,; s)\, |W(y)|\, ds\, dy = 0$$

uniformly with respect to W in the unit ball of $L^p_{\text{unif}}(X)$.

Proof. We can suppose $0 < t < 1$ and $a\sqrt{t} \le r_0$. Using the notation of Lemma 17 we have

$$
\begin{aligned}
\sup_x \int_X \int_0^t & P(x,y\,;s)\,|W(y)|\,dsdy \\
&\le \sup_x \int\limits_{d(x,y)\le a\sqrt{t}} Q(d(x,y),t)\,|W(y)|\,dy \\
&\quad + \sup_x \int\limits_{a\sqrt{t}<d(x,y)\le a\sqrt[4]{t}} Q(d(x,y),t)\,|W(y)|\,dy \\
&\quad + \sup_x \int\limits_{d(x,y)>a\sqrt[4]{t}} Q(d(x,y),t)\,|W(y)|\,dy \;=: F_1(t)+F_2(t)+F_3(t)\,.
\end{aligned}
$$

Consider the function $F_1(t)$. From Lemma 17 $F_1(t) \le 2\sqrt{t}\|W\|_{1,\mathrm{unif}}$ if $\nu = 1$. For $\nu \ge 3$ we obtain $F_1(t) \le \mathrm{const}\,\|W\|_{\mathrm{K}}^{(r)}$ where $r = a\sqrt{t}$. Since $d(x,y) \le a\sqrt{t}$ implies $|\log a^2 t| \le |\log d(x,y)^2|$, in the case $\nu = 2$ the inequality (4.6) implies $F_1(t) \le \mathrm{const}\,\|W\|_{\mathrm{K}}^{(r)}$ from Lemma 17 with $r = a\sqrt{t}$ again. Hence, $F_1(t) \to 0$ as $t \to 0$ uniformly in the unit ball of $L^p_{\mathrm{unif}}(X)$ due to Lemma 16.

In the region $d(x,y) > a\sqrt{t}$ we have according to Lemma 17 (in the case $\nu = 2$ we consider sufficiently small t) $Q(d(x,y),t) \le \mathrm{const} K_\nu(x,y)$ with kernels K_ν from (4.3). Hence, $F_2(t) \le \mathrm{const}\|W\|_{\mathrm{K}}^{(r)}$ with $r = a\sqrt[4]{t}$, and by Lemma 16 $F_2(t) \to 0$ uniformly in the unit ball of $L^p_{\mathrm{unif}}(X)$ as $t \to 0$.

Finally, consider $F_3(t)$. Choose now t_0, $t_0 > 0$, such that $(5t_0a^2)^{-1} > \theta_X$. According to Lemma 17 we have for $t < t_0$ in the region $d(x,y) > a\sqrt[4]{t}$:

$$
\begin{aligned}
Q\big(d(x,y),t\big) &\le \mathrm{const}\, d(x,y)^{2-\nu} \exp\left(-\frac{d(x,y)^2}{4t_0a^2}\right) \exp\left(-\frac{1}{4\sqrt{t}}\right) \\
&\le \mathrm{const}\, d(x,y)^{1-\nu} \exp\left(-\frac{d(x,y)^2}{5t_0a^2}\right) \exp\left(-\frac{1}{4\sqrt{t}}\right) \\
&\le \mathrm{const}\, t^{(1-\nu)/4} \exp\left(-\frac{d(x,y)^2}{5t_0a^2}\right) \exp\left(-\frac{1}{4\sqrt{t}}\right).
\end{aligned}
\tag{4.9}
$$

In virtue of Lemma 3, for each $x \in X$ the function

$$
g_x(y) = \exp\left(-\frac{d(x,y)^2}{5t_0a^2}\right)|W(y)|
$$

belongs to $L^1(X)$ and $\|g_x\|_1 \le c'\|W\|_{1,\,\mathrm{unif}}$ where c' is independent of t and x. Therefore, we have from (4.9)

$$
F_3(t) \le \mathrm{const}\, t^{(1-\nu)/4} \exp\left(-\frac{1}{4\sqrt{t}}\right)\|W\|_{1,\mathrm{unif}}\,,
$$

thus $F_3(t) \to 0$ as $t \to 0$ uniformly in the unit ball of L^1_{unif} and hence, of L^p_{unif}. $\square$

The following theorem is the main result of the section.

Theorem 19. *Let $U \in \mathcal{P}(X)$, then the following assertions are true*

(1) *$H_A + U$ is essentially self-adjoint and semi-bounded below on $C_0^\infty(X)$.*

(2) *Let $1 \le p \le q \le \infty$. Then $\|e^{-tH_{A,U}}\|_{p,q} \le C_{p,q}\, t^{-\gamma} \exp(B_{p,q}t)$, where $\gamma = \frac{1}{2}\nu(p^{-1} - q^{-1})$ and $B_{p,q}$, $C_{p,q} \ge 0$ (i.e., $H_{A,U}$ obeys the condition* (S_{pq}) *from Subsection* 2.3 for all p, q *with* $1 \le p \le q \le \infty$*).*

(3) *There are $C > 0$ and $a > 0$ such that for any compact sets $K_1, K_2 \subset X$ with $d := \operatorname{dist}(K_1, K_2) > 0$ we have for all t, $0 < t < 1$,*

$$\|\chi_1\, e^{-tH_{A,U}}\, \chi_2\|_{1,\infty} \le Ct^{-\nu/2} e^{-d^2/a^2 t}\,,$$

where χ_j is the characteristic function of K_j, $j = 1, 2$.

(4) *For $\zeta \in \operatorname{res}(H_{A,U})$ with $\operatorname{Re}\zeta < 0$ and sufficiently large $|\operatorname{Re}\zeta|$, the kernel $G^{(\kappa)}_{A,U}(\zeta) := G^{(\kappa)}_{A,U}(\cdot,\cdot;\zeta)$ of $R^{\kappa}_{A,U}(\zeta)$ exists for each $\kappa > 0$ and $G^{(\kappa)}_{A,U}(\zeta) \in \mathcal{K}(\alpha, q)$ where q, $1 \le q \le \infty$, is arbitrary, and $\alpha = \nu - 2\kappa$ for $\kappa < \nu/2$, $0 < \alpha < \nu$ is arbitrary for $\kappa = \nu/2$, and $\alpha = 0$ for $\kappa > \nu/2$.*

Proof. We can represent U in the form $U = V - W$, where $V \in \mathcal{P}(X)$ and is semi-bounded below, and $W = \sum_{j=1}^n W_j$ where $W_j \ge 0$, $W_j \in L^{p_j}(X)$ with $2 \le p_j < \infty$ if $\nu \le 3$ and $\nu/2 < p_j < \infty$ otherwise.

(1) Since

$$\begin{aligned}
\|R^{1/2}_{A,V}(E) W R^{1/2}_{A,V}(E)\|_{2,2} &\le \sum_{j=1}^n \|R^{1/2}_{A,V}(E) W_j R^{1/2}_{A,V}(E)\|_{2,2} \\
&= \sum_{j=1}^n \|\, W_j^{1/2} R^{1/2}_{A,V}(E)\|^2_{2,2} \\
&= \sum_{j=1}^n \|\, W_j^{1/2} R_{A,V}(E) W_j^{1/2}\|_{2,2}\,,
\end{aligned}$$

we have according to Corollary 5(2b) that $\|R^{1/2}_{A,V}(E) W R^{1/2}_{A,V}(E)\|_{2,2} \to 0$ as $E \to -\infty$. Therefore, W is form-bounded with respect to $H_{A,V}$ and the item (1) follows from Theorem 8.

(2) As shown in the proof of the inequality (B11) in [54], it is sufficient to prove the following relations:

(R1). there is $T > 0$ such that $\sup_{0 \le t \le T} \|e^{-tH_{A,U}}\|_{\infty,\infty} < \infty$;

(R2). there are $\tilde{B} > 0$ and $\tilde{C} > 0$ such that $\|e^{-tH_{A,U}}\|_{2,\infty} \le \tilde{C}\, t^{\nu/4} \exp(\tilde{B}\, t)$ for all $t > 0$.

Taking into account Theorem 10, we have to prove (R1) and (R2) for the case $A = 0$ only. For this purpose we use the ideas of the proofs of Theorem B.1.1 from [54] and Theorem 2.1 from [13]. Let us start with (R1). First of all, from Lemmas 15 and 18 we see that

$$\lim_{t \downarrow 0} \Big\| \int_0^t e^{-sH_{0,V}} W\, ds \Big\|_{\infty,\infty} = 0$$

uniformly in W from the unit ball of $L^p(X)$. Let $W^{(n)}(x)=\sum_{j=1}^{n} \min(W_j(x), n)$ and $H_n = H_{0,V} - W^{(n)}$. Since $0 \le W^{(n)} \le W$ for all n, we can find constants $T > 0$ and η, $0 < \eta < 1$, such that

$$\Big\| \int_0^T e^{-sH_{0,V}} W^{(n)}\, ds \Big\|_{\infty,\infty} \le \eta$$

for all n. Fix now t, $0 < t < T$. Using the Dyson–Phillips expansion we show that $\|e^{-tH_n}\|_{\infty,\infty} \le (1-\eta)^{-1}$, see the proof of Theorem 2.1 from [13]. On the other hand, $H_n := H_{0,V} - W^{(n)}$ tends to $H_{0,U}$ in the strong resolvent sense [46, Theorem VIII.25]. Let $\phi \in L^2(X)$, $\|\phi\|_\infty \le 1$, then $\|e^{-tH_n}\phi - e^{-tH_{0,U}}\phi\|_2 \to 0$ and $\|e^{-tH_n}\phi\|_\infty \le (1-\eta)^{-1}$ for all n. We can extract a subsequence $(e^{-tH_{n_k}}\phi)_{k\ge 1}$ which tends to $e^{-tH_{0,U}}\phi$ a.e., hence $\|e^{-tH_{0,U}}\phi\|_\infty \le (1-\eta)^{-1}$ and the statement (R1) is proven.

To proceed further we need the following "Schwarz inequality"

$$\big|(e^{-tH_{0,U}} f)(x)\big|^2 \le \big|e^{-t(H_{0,V}-2W)}1(x)\big|\, \big|e^{-tH_{0,V}}|f|^2(x)\big| \quad \text{for a.e. } x \in X\,, \tag{4.10}$$

where $f \in L^2(X)$. Replaced W by cut-off functions $W^{(n)}$ defined above we can repeat the proof of Lemma 6.4 from [15] (see also proof of Theorem 2.1 from [13]) to derive (4.10) with $W^{(n)}$ instead of W, and then extend this inequality to W by the limiting considerations above. The property (R1) implies that for all $t > 0$ we have $\|e^{-t(H_{0,V}-2W)}\|_{\infty,\infty} \le C_1 e^{tB_1}$ with some B_1, $C_1 > 0$ (see the mentioned prof from [13]), whereas Lemma 15(2) and inequality (4.2) imply for $f \in L^2(X)$, $\|f\| \le 1$,

$$\big\| e^{-tH_{0,V}} |f|^2 \big\|_\infty \le \frac{C_2}{\min(t^{\nu/2}, 1)} e^{B_2 t}\,,$$

with some B_2, $C_2 > 0$. Using (4.10) we finish checking the property (R2) and, therefore, the proof of the item (2).

(3) To prove this item it is sufficient to follow the proof of Proposition B.4.2 from [54].

(4) The existence of the integral kernels follows from Theorem 4(3). Arguing further as in the proof of Lemma B.7.6 in [54], we can show that for each $d > 0$ there is a constant $c_d > 0$ such that for $\zeta \in \operatorname{res}(H_{A,U})$, where $\operatorname{Re}\zeta < 0$ and $|\operatorname{Re}\zeta|$ is sufficiently large we have $|G_{A,U}(x,y;\zeta)| \le c_d$ for $d(x,y) \ge d$. Moreover, if $\nu \ne 2\kappa$, then $|G_{A,U}(x,y;\zeta)| \le c_d\, d(x,y)^\alpha$ for $d(x,y) \le d$ with α given in the item (4) of the theorem. In the case $\kappa = \nu/2$ it is sufficient to replace the inequality in the item (3) by $\|\chi_1\, e^{-tH_{A,U}}\, \chi_2\|_{1,\infty} \le Ct^{-(\nu+\epsilon)/2} e^{-d^2/b^2 t}$ with $\epsilon > 0$, $b > a$, and repeat the arguments of the proof of Theorem B.4.3 from [54]. Thus, we show that $G_{A,U}(\zeta) \in \mathcal{K}(\alpha, \infty)$ for noted ζ and α.

According to Theorems 2 and 4, we have $\lfloor G_{A,U}^{(\kappa)} \rfloor_{p',r} < \infty$ for every $r < r_0$ if $p^{-1} < 2\kappa/\nu$ if $p^{-1} < 2\kappa/\nu$. This condition is satisfies, if $p = \infty$, hence, if $p' = 1$. Therefore, $G_{A,U}(\zeta) \in \mathcal{K}(\alpha, 1)$ for ζ and α as above. Thus, by the property (K3) of the classes $\mathcal{K}$ (see Section 2.6) the theorem is proved. $\square$

5. Continuity of the kernels

Before stating the main result (Theorem 21 below) we prove the following lemma.

Lemma 20. *Let f be a real-valued function from $L^p_{\mathrm{loc}}(X)$, where $1 \le p < \infty$, and $f \ge c$ with a constant $c \in \mathbb{R}$. Then there exists a real-valued function g from $C^\infty(X)$ such that $g \ge c$ and $f - g \in L^q(X)$ for all $1 \le q \le p$.*

Proof. Fix $a \in X$ and for integers n, $n \ge 1$, denote $Y_n = B(a,n) \setminus \overline{B(a,n-1)}$. Fix a real sequence α_n, $\alpha_n > 0$, such that $\sum \alpha_n \le 1$ and denote by f_n the restriction of f to the set Y_n. Since the measure of Y_n is finite, for every n we can find a real-valued function g_n, $g_n \in C_0^\infty(X)$, such that $g_n \ge c$, $\operatorname{supp}(g_n) \subset Y_n$, and $\max(\|f_n - g_n\|_p^p, \|f_n - g_n\|_1) \le \alpha_n$. Since the family (Y_n) is locally finite, the point-wise sum $g = \sum g_n$ exists and $g \in C^\infty(X)$. It is clear that $g \ge c$ and $\max(\|f-g\|_p, \|f-g\|_1) \le 1$, i.e., $f - g \in L^p(X) \cap L^1(X)$; hence, $f - g \in L^q(X)$ for all $1 \le q \le p$. □

Now we are in position to prove the main result of the paper.

Theorem 21. *Let a potential U, $U \in \mathcal{P}(X)$, be given.*

(1) *For $t > 0$ the operator $e^{-tH_{A,U}}$ has an integral kernel $P_{A,U}(x,y\,;\,t)$ which is jointly continuous in $X \times X \times (0,\infty)$.*

(2) *For any* bounded *Borel set $S \subset \mathbb{R}$, the corresponding spectral projection for $H_{A,U}$ has a continuous in $X \times X$ integral kernel.*

(3) *Let $\kappa > 0$ and $\zeta \in \operatorname{res}(H_{A,U})$. Then the Green function $G^{(\kappa)}_{A,U}(\cdot,\cdot;\zeta)$ is continuous in $(X \times X) \setminus D$ if one of the following conditions is valid:*
   (a) $\operatorname{Re}\zeta < 0$ *and* $|\operatorname{Re}\zeta|$ *is sufficiently large,*
   (b) κ *is an integer.*

   Moreover, if $\kappa > \nu/4$, then under these conditions $G^{(\kappa)}_{A,U}(\cdot,\cdot;\zeta)$ is a Carleman kernel with the properties (C1) *and* (C2) *from Subsection* 2.3*; in particular, the image of $R^\kappa(\zeta)$ consists of continuous functions.*

(4) *If f is a Borel function on $\operatorname{spec}(H_{A,U})$ obeying the condition $|f(\xi)| \le b(|\xi|+1)^{-\kappa}$ with some $b > 0$ and $\kappa > \nu/2$, then the operator $f(H_{A,U})$ has an integral kernel $F(x,y)$ which is continuous on $X \times X$. Moreover $\sup\{|F(x,y)| : x, y \in X\} \le C\,b < \infty$ where C depends only on κ.*

(5) *If $\kappa > \nu/2$, then for all $\zeta \in \operatorname{res}(H_{A,U})$ the kernel $G^{(\kappa)}_{A,U}(\cdot,\cdot;\zeta)$ is a bounded continuous function on the whole space $X \times X$.*

(6) *Each eigenfunction of $H_{A,U}$ is bounded and continuous.*

(7) *Let k be an integer, $k \ge 1$. Then the map $\zeta \mapsto G^{(k)}_{A,U}(x,y\,;\,\zeta)$ is holomorphic in $\operatorname{res}(H_{A,U})$ for all $x, y \in X$ if $k > \nu/2$, and for $x \ne y$ otherwise. Moreover $\partial G^{(k)}_{A,U}(x,y\,;\,\zeta)/\partial\zeta = kG^{(k+1)}_{A,U}(x,y\,;\,\zeta)$ for (x,y) above.*

Proof. Using Lemma 20 we represent U in the form $U = V + W$, where V and W have the properties

$$V \in C^{\infty}(X) \text{ and is semi-bounded below}; \tag{5.1}$$

$$\begin{gathered} W = \sum\nolimits_{j=0}^{n} W_j\,, \quad W_j \in L^{p_j}(X)\,, \\ 2 \le p_j < \infty \text{ if } \nu \le 3 \text{ and } \nu/2 < p_j < \infty \text{ otherwise}, \ 0 \le j \le n\,. \end{gathered} \tag{5.2}$$

Let κ be any strictly positive number; denote by α_κ the number $\nu - 2\kappa$ if $\kappa < \nu/2$, an arbitrary number from the interval $(0, \nu)$ if $\kappa = \nu/2$, and 0 if $\kappa > \nu/2$. Then we have by Theorem 19(4) and the properties (K2) and (K3) from Section 2.6

for $E < 0$ with sufficiently large $|E|$ the kernels $G^{(\kappa)}_{A,V}(\cdot,\cdot;E)$ and $G^{(\kappa)}_{A,U}(\cdot,\cdot;E)$ exist and belong to all the classes $\mathcal{K}(\beta, q)$ with $1 \le q \le \infty$, $\alpha_\kappa \le \beta < \nu$. (5.3)

Moreover, by Lemma 15(4),

for $E < 0$ with sufficiently large $|E|$ and for $\kappa \ge 1$ we have $G^{(\kappa)}_{A,V}(\cdot,\cdot;E) \in \mathcal{K}_0(\beta, q)$ for every q, $1 \le q < \infty$ and β, $\alpha_\kappa \le \beta < \nu$. (5.4)

Further, by virtue of (5.1), we have the following continuity properties:

$G_{A,V}(x, y; \zeta)$ can be chosen from $C^\infty\big((X \times X)\setminus D\big)$ if $\nu \ge 2$ and from $C^\infty(X \times X)$ if $\nu = 1$. (5.5)

The first statement in (5.5) follows from the standard elliptic regularity considerations [52]; the second one can be found in [42].

Now we show that for $E < 0$ with sufficiently large $|E|$ and for every integer k, $k \ge 1$ there holds

$$R^k_{A,U}(E) = R^{k-1}_{A,U}(E) R_{A,V}(E) - R^k_{A,U}(E) W R_{A,V}(E)\,, \tag{5.6}$$

$$R^k_{A,U}(E) = R_{A,V}(E) R^{k-1}_{A,U}(E) - R_{A,V}(E) W R^k_{A,U}(E)\,. \tag{5.7}$$

Passing on to adjoint operators we derive (5.7) from (5.6), therefore, we consider (5.6) only. Obviously, it is sufficient to prove (5.6) for the case $k = 1$. Using item (2b) from Corollary 5 and Theorem 19 we get $\|\,|W|^{1/2} R^{1/2}_{A,V}(E)\|_{2,2} = \|R^{1/2}_{A,V}(E)\,|W|\,R^{1/2}_{A,V}(E)\|^{1/2}_{2,2} < \infty$ and similarly $\|R^{1/2}_{A,U}(E)\,|W|^{1/2}\,\|_{2,2} < \infty$. Denote, as usual,

$$\operatorname{sign} W(x) = \begin{cases} \dfrac{W(x)}{|W(x)|}\,, & \text{if } W(x) \ne 0\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Then $\|\operatorname{sign} W\,|W|^{1/2} R^{1/2}_{A,V}(E)\|_{2,2} < \infty$, therefore $\|R^{1/2}_{A,U}(E) W R^{1/2}_{A,V}(E)\|_{2,2} < \infty$, hence $\|R_{A,U}(E) W R_{A,V}(E)\|_{2,2} < \infty$. It remains to prove that both of the sides of the equation

$$R_{A,U}(E) = R_{A,V}(E) - R_{A,U}(E) W R_{A,V}(E)\,. \tag{5.8}$$

coincide on a dense subset in $L^2(X)$. Consider functions $f = (H_{A,V} - E)\phi$ where ϕ runs over $C_0^\infty(X)$; these functions form a dense subset since $H_{A,V}$ is essentially self-adjoint on $C_0^\infty(X)$. Further, $\phi \in \mathcal{D}(H_{A,U})$ and $W\phi \in L^2(X)$, therefore $R_{A,U} f = R_{A,U}((H_{A,U} - E)\phi - W\phi) = \phi - R_{A,U} W\phi$. Since $\phi = R_{A,V} f$, we get the result. Worth noting that (5.8) is nothing else than the Lippmann–Schwinger equation for the potential W.

Using (5.2) and (5.3) we get with the help of Lemma 13 that for $x \neq y$ and $k \geq 1$

$$G^{(k)}_{A,U}(x,\cdot;E)W(\cdot)\, G_{A,V}(\cdot,y;E) \in L^1(X\times X)\,, \tag{5.9}$$

$$G_{A,V}(x,\cdot;E)W(\cdot)\, G^{(k)}_{A,U}(\cdot,y;E) \in L^1(X\times X)$$

for all $E < 0$ with sufficiently large $|E|$. Similarly, using Lemma 13 with $W \equiv 1$, we get for $k \geq 2$ and, for the same E,

$$G^{(k-1)}_{A,U}(x,\cdot;E)G_{A,V}(\cdot,y;E)\,, \quad G_{A,V}(x,\cdot;E)G^{(k-1)}_{A,U}(\cdot,y;E) \in L^1(X\times X)\,.$$

Therefore, the following functions are well defined:

$$J^{(k)}_1(x,y;E) := \int_X G^{(k)}_{A,U}(x,z;E)W(z)G_{A,V}(z,y;E)\,dz\,,$$

$$J^{(k)}_2(x,y;E) := \int_X G_{A,V}(x,z;E)W(z)G^{(k)}_{A,U}(z,y;E)\,dz\,, \tag{5.10}$$

for $k \geq 1$, and

$$L^{(k)}_1(x,y;E) := \int_X G^{(k-1)}_{A,U}(x,z;E)G_{A,V}(z,y;E)\,dz\,,$$

$$L^{(k)}_2(x,y;E) := \int_X G_{A,V}(x,z;E)G^{(k-1)}_{A,U}(z,y;E)\,dz\,, \tag{5.11}$$

for $k \geq 2$. Moreover, the integrals in (5.10) and (5.11) converge absolutely. Denote $L^{(1)}_j(x,y;E) := G_{A,V}(x,y;E)$ for $j = 1,2$. We show that for all $k \geq 1$ the functions $L^{(k)}_1(E) - J^{(k)}_1(E)$ and $L^{(k)}_2(E) - J^{(k)}_2(E)$ are the integral kernels of $R^{(k)}_{A,U}(E)$, i.e.,

$$G^{(k)}_{A,U}(x,y;E) = L^{(k)}_j(x,y;E) - J^{(k)}_j(x,y;E)\,, \tag{5.12}$$

for a.e. $(x,y) \in L^1(X\times X)$ $(j = 1,2)$. By the item (2) of Lemma 13 and the property (K5) from Section 2.6 all the kernels $J^{(k)}_j$ and $L^{(k)}_j$ belong to $L^1_{\mathrm{loc}}(X\times X)$. According to (5.6) and (5.7) it remains to show that for $\phi,\psi \in C_0^\infty(X)$, $k \geq 1$ and $j = 1,2$

$$\langle \psi \,|\, R^k_{A,U}(E)\phi\rangle = \int_{X\times X} L^{(k)}_j(x,y)\overline{\psi(x)}\phi(y)\,dxdy - \int_X J^{(k)}_j(x,y)\overline{\psi(x)}\phi(y)\,dxdy\,.$$

Firstly, we show that

$$\big\langle \psi \,\big|\, R^k_{A,U}(E)WR_{A,V}(E)\phi\big\rangle = \int_X J^{(k)}_1(x,y)\overline{\psi(x)}\phi(y)\,dxdy\,. \tag{5.13}$$

It was shown by the proof of equation (5.8) that $\| \, |W|^{1/2} R^{1/2}_{A,V}(E)\|_{2,2} < \infty$ and $\| \, |W|^{1/2} R^{1/2}_{A,U}(E)\|_{2,2} < \infty$, therefore $\| \operatorname{sign} W \, |W|^{1/2} R_{A,V}(E)\|_{2,2} < \infty$ and $\| \, |W|^{1/2} R^{k}_{A,U}(E)\|_{2,2} < \infty$. This means that the functions

$$f_1(z) := \operatorname{sign} W(z) \, |W|^{1/2}(z) \int_X G_{A,V}(z,x;E)\, \phi(x)\, dx$$

and

$$f_2(z) := |W|^{1/2}(z) \int_X G^{(k)}_{A,U}(z,y;E)\, \psi(y)\, dy$$

are from $L^2(X)$. It is clear that $\langle \psi \,|\, R^k_{A,U}(E) W R_{A,V}(E)\phi\rangle = \langle f_2 \,|\, f_1\rangle$. In virtue of (5.9) and absolute convergence of integrals (5.10), we can change order of integration in the integral expression for $\langle f_2 \,|\, f_1\rangle$ and obtain (5.13). Similarly we prove that

$$\big\langle \psi \,\big|\, R^{k-1}_{A,U}(E) R_{A,V}(E)\phi \big\rangle = \int_X L^{(k)}_1(x,y)\overline{\psi(x)}\phi(y)\, dxdy\,.$$

Hence, (5.12) is proved for $j=1$. The case $j=2$ is reduced to $j=1$ by the simple consideration:

$$\big\langle \psi \,\big|\, R^{k}_{A,U}(E)\phi \big\rangle = \big\langle R^{k-1}_{A,U}(E) R_{A,V}(E)\psi \,\big|\, \phi \big\rangle - \big\langle R^{k}_{A,U}(E) W R_{A,V}(E)\psi \,\big|\, \phi \big\rangle.$$

From (5.2) and (5.3) it is easy to see that conditions (a) and (b) of Lemma 13 are satisfied with $K_1 = G^{(k)}_{A,U}(E)$ $(k \geq 1)$ and $K_2 = G_{A,V}(E)$ with W given in (5.2). The same is true for $W \equiv 1$ in Lemma 13. Moreover, in virtue of (5.4) the functions K_1, K_2, and W satisfy the additional conditions from the item (3) of the lemma and if $k > \nu/2$ they satisfy the additional conditions of the item (4). Therefore, by (5.5) and Lemma 13(3b) (or item (4b), if $k > \nu/2$), for all $x \in X$, the function $G^{(k)}_{A,U}(x,\cdot;E)$ is continuous in $X \setminus \{x\}$ (respectively, in X) for all $k \geq 1$. Now taking $K_1 = G_{A,V}(E)$, $K_2 = G^{(k)}_{A,U}(E)$ and using the item (3c) of Lemma 13, we get the first statement of the following assertion concerning the properties of the kernels $G^{(k)}_{A,U}(E)$:

> $G^{(k)}_{A,U}(E)$ is continuous in $(X \times X) \setminus D$ for all $k \geq 1$, and in $X \times X$ for all $k > \nu/2$. Moreover, if $k > \nu/2$, then $G^{(k)}_{A,U}(E)$ is a Carleman kernel obeying the conditions (C1) and (C2) from Subsection 2.3 (5.14)

To prove the second statement we note firstly that $G^{(k)}_{A,U}(E)$ is a Carleman kernel for $k > \nu/2$ by Theorem 4. Further,

$$\int_X \big| G^{(k)}_{A,U}(x,y\,;E)\big|^2\, dy = G^{(2k)}_{A,U}(x,x\,;E)$$

and $G^{(2k)}_{A,U}(x,x\,;E)$ is continuous in x. Hence, the property (C2) is valid. The property (C1) we get from Lemma 13 if we set $K_1 = G^{(k)}_{A,U}(E)$, $W = f$, $K_2 \equiv 1$ and take into consideration (5.4) and (5.14).

Since $\|R^{\kappa}_{A,U}(\zeta)\|_{2,\infty} < \infty$ by Theorems 4(1) and 19 we can apply the item (2) of Theorem 7 and obtain the following assertion:

$$\text{There is an integer } n > 1 \text{ such that for any Borel function } f \text{ defined on the spectrum of } H_{A,U} \text{ and having the property } \big|f(\xi)\big| \le b\big(|\xi|+1\big)^{-n} \text{ with } b>0, \text{ the operator } f(H_{A,U}) \text{ has a Carleman continuous kernel } F(x,y) \text{ with property } \big|F(x,y)\big| \le bc^2, \text{ where } c \text{ is independent of } f. \tag{5.15}$$

The assertion (5.15) allows us to deduce all the items (1)–(7) of the theorem step by step.

(1) Consider for $t > 0$ the function $f(\xi) = e^{-t\xi}$. Since $H_{A,U}$ is semi-bounded, there exists $b > 0$ such that $|f(\xi)| \le b(1 + |\xi|)^{-n}$ on the spectrum of $H_{A,U}$. Therefore by (5.15), the operator $\exp(-tH_{A,U})$ has a kernel $P_{A,U}(x, y\,; t)$ which is jointly continuous in $x, y \in X$ at any fixed t, $t > 0$. Fix now any $t_0 > 0$, then in a neighborhood of t_0 we have the estimate $|e^{-t\xi} - e^{-t_0\xi}| \le b(t)(1 + |\xi|)^{-n}$, where $b(t) \to 0$ if $t \to t_0$. By (5.15), $|P_{A,U}(x, y\,; t) - P_{A,U}(x, y\,; t_0)| \le c^2 b(t)$. Now using the continuity of $P_{A,U}(x, y\,; t)$ with respect to (x, y) we complete the proof of (1).

(2) This is an immediate consequence of (5.15).

(3) Using the item (2) of Theorem 19 with $p = 1$ and $q = \infty$ for $t \ge 1$ and item (3) of this theorem for $0 < t < 1$ we see that the integral in the expression

$$\frac{1}{\Gamma(\kappa)} \int_0^\infty P_{A,U}(x, y\,; t) e^{t\zeta}\, t^{\kappa-1}\, dt\,, \tag{5.16}$$

converges absolutely and locally uniformly in $(X \times X) \setminus D$ if $\operatorname{Re} \zeta < B_{1,\infty}$. As a result, expression (5.16) defines a continuous function on $(X \times X) \setminus D$; denote this function by $M(x, y; \zeta)$. Now applying item (2) of Theorem 19 with $p = q = \infty$ and properties of dominated operators (see Section 2.5) we show that for every pair of compactly supported functions ϕ, ψ from $L^\infty(X)$

$$\int\limits_X \int\limits_X \big|P_{A,U}(x, y; t)\phi(x)\psi(y)\big|\, dxdy \, < \infty$$

for every $t > 0$. Therefore, using equation (2.1) and the Fubini theorem we obtain

$$\big\langle \phi \,|\, R^{(\kappa)}_{A,U}\, \psi \big\rangle = \int\limits_X \int\limits_X F(x, y; \zeta)\overline{\phi(x)}\psi(y)\, dxdy\,,$$

where the integral converges absolutely. Hence, $M(\cdot\,, \cdot\,; \zeta) \in L^1_{\mathrm{loc}}(X \times X)$ and by the uniqueness property of kernels (Section 2.2), $G^{(\kappa)}_{A,U}(x, y; \zeta) = M(x, y; \zeta)$. Thus, in the case (a) the proof is completed.

To prove the item in the case (b), we take $n > \nu/2$ from (5.15), fix $E_0 < 0$ such that the kernels $G^{(1)}_{A,U}(E_0)$, $\ldots$, $G^{(n)}_{A,U}(E_0)$ are continuous in $(X \times X) \setminus D$, and consider an arbitrary $\zeta \in \operatorname{res}(H_{A,U})$. Let $f_\zeta(\xi) = (\xi - E_0)^{-n}(\xi - \zeta)^{-1}$ for $\xi \in \operatorname{spec}(H_{A,U})$. Then $|f_\zeta(\xi)| \le b(\zeta)(|\xi| + 1)^{-n}$ for all $\xi \in \operatorname{spec}(H_{A,U})$ where

$b(\zeta) > 0$ is locally bounded in ζ from $\mathrm{res}(H_{A,U})$. Using the identity

$$\frac{1}{\xi-\zeta} = \frac{1}{\xi-E_0} + \frac{\zeta-E_0}{(\xi-E_0)^2} + \ldots + \frac{(\zeta-E_0)^{n-1}}{(\xi-E_0)^n} + \frac{(\zeta-E_0)^n}{(\xi-E_0)^n(\xi-\zeta)} \tag{5.17}$$

we get for the kernels

$$\begin{aligned} G^{(1)}_{A,U}(\zeta) = {} & G^{(1)}_{A,U}(E_0) + (\zeta-E_0)G^{(2)}_{A,U}(E_0) \\ & + \cdots + (\zeta-E_0)^{n-1}G^{(n)}_{A,U}(E_0) + (\zeta-E_0)^n F_\zeta\,, \end{aligned}$$

where F_ζ is the kernel for $f_\zeta(H_{A,U})$. Therefore for $\kappa = 1$, the part (b) follows from the part (a) and the assertion (5.15). To get these items for any positive integer κ it is sufficient to consider the κ-th power of both the sides of (5.17), and represent the right-hand side of the obtained expression as the sum of products of terms in the right-hand side of (5.17). In this sum, the addends containing the non-zero powers of $\frac{(\zeta-E_0)^n}{(\xi-E_0)^n(\xi-\zeta)}$ have an estimate from above by $\mathrm{const}(|\xi|+1)^{-n}$ on $\mathrm{spec}(H_{A,U})$ and therefore have continuous kernels by the assertion (5.15). According to the part (a), the kernels of the remaining addends are continuous outside the diagonal D, and we get the result.

If $\kappa > \nu/4$, the kernel $G^{(\kappa)}(E)$ is Carleman in virtue of Theorems 4(2) and 19. The arguments used by the proof of (5.14) show that these kernels obey the required properties (C1) and (C2).

(4) Taking into consideration the items (1) and (3) we can prove the item (4) by the arguments used in the proof of the statement (5.15).

(5) This is an immediate consequence of the item (4).

(6) The continuity of the eigenfunctions of $H_{A,U}$ follows from the last statement in the item (3). Since $\|e^{-tH_{A,U}}\|_{2,\infty} < \infty$ (see Theorem 19), any eigenfunction of $H_{A,U}$ is bounded.

(7) To get the derivative $\partial G_{A,U}(x,y\,;\,\zeta)/\partial\zeta$ at a point $\zeta_0 \in \mathrm{res}(H_{A,U})$, we use the expansion (5.17) with E_0 replaced by ζ_0, and ξ replaced by $H_{A,U}$. Due to the item (4), for sufficiently large n the last term in the right-hand side of (5.17) will have an integral kernel which is uniformly bounded as ζ is in some small neighborhood of ζ_0. This proves the requested equality for $k = 1$. For $k > 1$ one should consider the k-th powers in the both sides of (5.17) and use the same arguments. $\square$

6. Concluding remarks

It would be interesting to understand whether the estimates obtained admit a generalization to the potentials from the Kato class on the manifold, see (4.4). In this connection it would be also useful to know whether the above definition of the Kato class is sufficient for these purposes or one needs more restrictive conditions for the non-flat case. This question is still open.

At the same time, we emphasize that the approach presented here works not only to prove the continuity properties, but also allows a more detailed analysis of the Green function. Let us mention one of possible applications. In some problems connected with the renormalization technique the asymptotic behavior near the diagonal D is important. Some corresponding estimates in the Euclidian space were proved in [54], in particular, in $L^2(\mathbb{R}^3)$ the Green function G_V of $-\Delta + V$ with V from the Kato class was shown to satisfy the estimate

$$\frac{C_1}{|x-y|} \leq \big|G_V(x,y;\zeta)\big| \leq \frac{C_2}{|x-y|}$$

for small $|x-y|$ with some $C_1, C_2 > 0$. Related properties for singular magnetic potentials are discussed, e.g., in [26]. In [9] we represented the Green function in lower dimensions ($\nu \leq 3$) in the form $G_{A,U}(x,y;\zeta) = F_{A,U}(x,y) + G^{\mathrm{ren}}_{A,U}(x,y;\zeta)$, where the second term on the right hand side is continuous in the whole space $X \times X$, and described the dependence of the singularity $F_{A,U}$ on the magnetic and electric potentials. It came out that this singularity may differ from the standard one (fundamental solution for the Laplace operator) if the electric potential becomes singular.

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