THE EQUIVARIANT INDEX THEOREM FOR TRANSVERSALLY ELLIPTIC OPERATORS AND THE BASIC INDEX THEOREM FOR RIEMANNIAN FOLIATIONS

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ABSTRACT. In this expository paper, we explain a formula for the multiplicities of the index of an equivariant transversally elliptic operator on a G-manifold. The formula is a sum of integrals over blowups of the strata of the group action and also involves eta invariants of associated elliptic operators. Among the applications is an index formula for basic Dirac operators on Riemannian foliations, a problem that was open for many years.

1. Introduction. In this note we announce two new results in index theory, namely an equivariant index theorem for transversally elliptic operators relative to a compact Lie group action and the basic index theorem for transversal Dirac operators in Riemannian foliations. The latter has been a well–known open problem in foliation theory for more than twenty years. Complete proofs of these results appear in [12] and [13].

Suppose that a compact Lie group G acts by isometries on a compact, connected Riemannian manifold M, and let $E = E^+ \oplus E^-$ be a graded, G-equivariant Hermitian vector bundle over M. We consider a first order G-equivariant differential operator $D = D^+ : \Gamma(M, E^+) \to \Gamma(M, E^-)$ that is transversally elliptic, and let D^- be the formal adjoint of D^+.

The group G acts on $\Gamma(M, E^\pm)$ by $(gs)(x) = g\cdot s(g^{-1}x)$, and the (possibly infinite-dimensional) subspaces $\ker(D^+)$ and $\ker(D^-)$ are G-invariant subspaces. Let $\rho : G \to U(V_\rho)$ be an irreducible unitary representation of G, and let $\chi_\rho = \operatorname{tr}(\rho)$ denote its character. Let $\Gamma(M, E^\pm)^\rho$ be the subspace of sections that is the direct sum of the irreducible G-representation subspaces of $\Gamma(M, E^\pm)$ that are unitarily equivalent to the representation ρ. It can be shown that the extended operators

$$\overline{D}_{\rho,s} : H^s\left(\Gamma\left(M, E^+\right)^\rho\right) \to H^{s-1}\left(\Gamma\left(M, E^-\right)^\rho\right)$$

are Fredholm and independent of s, so that each irreducible representation of G appears with finite multiplicity in $\ker D^\pm$. Let $a^\pm_\rho \in \mathbb{Z}_{\geq 0}$ be the multiplicity of ρ in $\ker(D^\pm)$.

The study of index theory for such transversally elliptic operators was initiated by M. Atiyah and I. Singer in the early 1970s ([2]). The virtual representation-valued index of D is given by

$$\operatorname{ind}^G(D) := \sum_\rho \left(a^+_\rho - a^-_\rho\right)[\rho],$$

where $[\rho]$ denotes the equivalence class of the irreducible representation ρ. The index multiplicity is

$$\operatorname{ind}^\rho(D) := a^+_\rho - a^-_\rho = \frac{1}{\dim V_\rho} \operatorname{ind}\left(D|_{\Gamma(M,E^+)^\rho \to \Gamma(M,E^-)^\rho}\right).$$

In particular, if ρ_0 is the trivial representation of G, then

$$\operatorname{ind}^{\rho_0}(D) = \operatorname{ind}\left(D|_{\Gamma(M,E^+)^G \to \Gamma(M,E^-)^G}\right),$$

where the superscript G implies restriction to G-invariant sections.

Atiyah's distributional index can be expanded in terms of the index multiplicities, which play the role of generalized Fourier coefficients

$$\operatorname{ind}_*(D)(\phi) = \sum_\rho \operatorname{ind}^\rho(D) \int_G \phi(g)\ \overline{\chi_\rho(g)}\, dg\ .$$

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From this formula, we see that the multiplicities determine the distributional index. Conversely, let $\alpha : G \to U(V_\alpha)$ be an irreducible unitary representation. Then

$$\operatorname{ind}_*(D)(\chi_\alpha) = \sum_\rho \operatorname{ind}^\rho(D) \int_G \chi_\alpha(g)\overline{\chi_\rho(g)}\,dg = \operatorname{ind}^\alpha D,$$

so that in principle complete knowledge of the distributional index is equivalent to knowing all of the multiplicities $\operatorname{ind}^\rho(D)$. Because the operator $D|_{\Gamma(M,E^+)^\rho \to \Gamma(M,E^-)^\rho}$ is Fredholm, all the indices $\operatorname{ind}^\rho(D)$ depend only on the stable homotopy class of the principal transverse symbol of D.

Consider now the heat kernel expression for the index multiplicities. The usual McKean-Singer argument shows that, in particular, for every $t > 0$, the index $\operatorname{ind}^\rho(D)$ may be expressed as the following iterated integral:

$$\operatorname{ind}^\rho(D) = \int_{x\in M}\int_{g\in G} \operatorname{str}\, g\cdot K\left(t, g^{-1}x, x\right)\ \overline{\chi_\rho(g)}\ dg\ |dx|$$
$$= \int_{x\in M}\int_{g\in G} \left(\operatorname{tr}\, g\cdot K^+\left(t, g^{-1}x, x\right) - \operatorname{tr}\, g\cdot K^-\left(t, g^{-1}x, x\right)\right)\ \overline{\chi_\rho(g)}\ dg\ |dx| \tag{1}$$

where $K^\pm(t,\cdot,\cdot) \in \Gamma\left(M\times M, E^\mp \boxtimes (E^\pm)^*\right)$ is the kernel for $e^{-t\left(D^\mp D^\pm + C - \lambda_\rho\right)}$ on $\Gamma(M, E^\pm)$, letting $|dx|$ denote the Riemannian density over M.

A priori, the integral above is singular near sets of the form

$$\bigcup_{G_x\in[H]} x\times G_x \subset M\times G,$$

where the isotropy subgroup G_x is the subgroup of G that fixes $x \in M$, and $[H]$ is a conjugacy class of isotropy subgroups.

Over the last twenty years numerous papers have appeared that express $\operatorname{ind}_*(D)$ and

$$\int_M \left(\operatorname{tr}\, g\cdot K^+\left(t, g^{-1}x, x\right) - \operatorname{tr}\, g\cdot K^-\left(t, g^{-1}x, x\right)\right)\ |dx|$$

in terms of topological and geometric quantities, as in the Atiyah-Segal-Singer index theorem for elliptic operators [4] or the Berline-Vergne Theorem for transversally elliptic operators [6],[7]. However, until now there has been very little known about the problem of expressing $\operatorname{ind}^\rho(D)$ in terms of topological or geometric quantities which are determined at the different strata of the G-manifold M. The special case when all of the isotropy groups are the same dimension was solved by M. Atiyah in [2], and this result was utilized by T. Kawasaki to prove the Orbifold Index Theorem (see [26]). Our analysis is new in that the integral over the group in (1) is performed first, before integration over the manifold, and thus the invariants in our index theorem are very different from those seen in other equivariant index formulas.

Our main theorem (Theorem 11) expresses $\operatorname{ind}^\rho(D)$ as a sum of integrals over the different strata of the action of G on M, and it involves the eta invariant of associated equivariant elliptic operators on spheres normal to the strata. The result is the following.

Equivariant Index Theorem. *The equivariant index* $\operatorname{ind}^\rho(D)$ *is given by the formula*

$$\begin{aligned}\operatorname{ind}^\rho(D) &= \int_{G\diagdown \widetilde{M_0}} A_0^\rho(x)\ \widetilde{|dx|}\ + \sum_{j=1}^r \beta\left(\Sigma_{\alpha_j}\right)\ ,\\ \beta\left(\Sigma_{\alpha_j}\right) &= \frac{1}{2\dim V_\rho}\sum_{b\in B}\frac{1}{n_b \operatorname{rank} W^b}\left(-\eta\left(D_j^{S+,b}\right) + h\left(D_j^{S+,b}\right)\right)\int_{G\diagdown\widetilde{\Sigma_{\alpha_j}}} A_{j,b}^\rho(x)\ \widetilde{|dx|}\ .\end{aligned}$$

The notation will be explained later; e.g. the integrands $A_0^\rho(x)$ and $A_{j,b}^\rho(x)$ are the familar Atiyah-Singer integrands corresponding to local heat kernel supertraces of induced elliptic operators over closed manifolds. Even in the case when the operator D is elliptic, this result was not known previously. Further, the formula above gives a method for computing eta invariants of Dirac-type operators on quotients of spheres by compact group actions; these have been computed previously only in some special cases. We emphasize that every part of the formula is explicitly computable from local information provided by the operator and manifold. Even the eta invariant of the operator $D_j^{S+,b}$ on a sphere is calculated directly from the principal symbol of the operator D at one point of a singular stratum. Examples show that all of the terms in the formula above are nontrivial.

The de Rham operator provides an important example illustrating the computability of the formula, yielding a new theorem expressing the equivariant Euler characteristic in terms of ordinary Euler characteristics of the strata of the group action (Theorem 12).

One of the primary motivations for obtaining an explicit formula for $\mathrm{ind}^{\rho}(D)$ was to use it to produce a basic index theorem for Riemannian foliations, thereby solving a problem that has been open since the 1980s. In fact the basic index theorem is a consequence of the invariant index theorem corresponding to the trivial representation ρ_0. This theorem is stated below, with more details in Section 6.

Basic Index Theorem. *The basic index is given by the formula*

$$\begin{aligned} \mathrm{ind}_b\left(D_b^E\right) &= \int_{\widetilde{M_0}\diagup\overline{\mathcal{F}}} A_{0,b}(x)\ \widetilde{|dx|} + \sum_{j=1}^{r}\beta(M_j) \\ \beta(M_j) &= \frac{1}{2}\sum_{\tau}\frac{1}{n_\tau \mathrm{rank}\ W^\tau}\left(-\eta\left(D_j^{S+,\tau}\right) + h\left(D_j^{S+,\tau}\right)\right)\int_{\widetilde{M_j}\diagup\overline{\mathcal{F}}} A_{j,b}^{\tau}(x)\ \widetilde{|dx|}, \end{aligned}$$

where the sum is over all components of singular strata and over all canonical isotropy bundles W^τ, only a finite number of which yield nonzero terms $A_{j,b}^{\tau}$.

Several techniques in this paper are new and have not been previously explored. First, the fact that $\mathrm{ind}^{\rho}(D)$ is invariant under G-equivariant homotopies is used in a very specific way, and we keep track of the effects of these homotopies so that the formula for the index reflects data coming from the original operator and manifold. In Section 4 we describe a process of blowing up, cutting, and reassembling the G-manifold into what is called the desingularization, which also involves modifying the operator and vector bundles near the singular strata as well. The result is a G-manifold that has less intricate structure and for which the heat kernels are easier to evaluate. The key idea is to relate the local asymptotics of the equivariant heat kernel of the original manifold to the desingularized manifold; at this stage the eta invariant appears through a direct calculation on the normal bundle to the singular stratum. More precisely, we compute the local contribution of the supertrace of a general constant coefficient equivariant heat operator in the neighborhood of a singular point of an orthogonal group action on a sphere. It is here that the equivariant index is related to a boundary value problem, which explains the presence of eta invariants in the main theorem.

Another new idea in this paper is the decomposition of equivariant vector bundles over G-manifolds with one orbit type. A crucial step in the proof required the construction of a subbundle of an equivariant bundle over a G-invariant part of a stratum that is the minimal G-bundle decomposition that consists of direct sums of isotypical components of the bundle. We call this decomposition the *fine decomposition* and define it in Section 2. A more detailed account of this method will appear in [20].

The relevant properties of the supertrace of the equivariant heat kernel are discussed in Section 3. We apply the heat kernel analysis, representation theory, and fine decomposition to produce a heat kernel splitting formula. This process leads to a reduction formula for the equivariant heat supertrace, from which the Equivariant Index Theorem 11 follows. Examples show that all the terms in the index formula are in general nontrivial.

We note that a recent paper of Gorokhovsky and Lott addresses this transverse index question on Riemannian foliations. Using a different technique, they are able to prove a formula for the basic index of a basic Dirac operator that is distinct from our formula, in the case where all the infinitesimal holonomy groups of the foliation are connected tori and if Molino's commuting sheaf is abelian and has trivial holonomy (see [18]).

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2. The refined isotypical decomposition. Let X be a smooth Riemannian manifold on which a compact Lie group G acts by isometries with single orbit type $[H]$ (see Section 4). Let X^H be the fixed point set of

H, and for $\alpha \in \pi_0\left(X^H\right)$, let X_α^H denote the corresponding connected component of X^H. Let $N = N(H)$ be the normalizer.

Definition 1. We denote $X_\alpha = GX_\alpha^H$, and X_α is called a **component of X relative to G**.

Remark 2. The space X_α is not necessarily connected, but it is the inverse image of a connected component of $G\diagdown X = N\diagdown X^H$ under the projection $X \to G\diagdown X$. Also, note that $X_\alpha = X_\beta$ if there exists $n \in N$ such that $nX_\alpha^H = X_\beta^H$. If X is a closed manifold, then there are a finite number of components of X relative to G.

We now introduce a decomposition of a G-bundle $E \to X$ over a G-space with single orbit type $[H]$. Let E_α be the restriction $E|_{X_\alpha^H}$. For $\sigma : H \to U(W_\sigma)$ an irreducible unitary representation, let $\sigma^n : H \to U(W_\sigma)$ be the irreducible representation defined by

$$\sigma^n(h) = \sigma\left(n^{-1}hn\right).$$

We let $\left.E_\alpha^{[\sigma]}\right|_{X_\alpha^H}$ denote the $[\sigma]$-isotypical component of E over X_α^H, meaning that each $E_{\alpha,x}^{[\sigma]}$ is the subspace of E_x that is a direct sum of irreducible H-representation subspaces of type $[\sigma]$. We define

$$E_{\alpha,[\sigma],x}^N = \operatorname{span}\left\{E_{\alpha,x}^{[\sigma^n]} : n \in N \text{ and } nX_\alpha^H = X_\alpha^H\right\}.$$

The N-orbits yield an N-bundle $E_{\alpha,[\sigma]}^N$ over $NX_\alpha^H \subseteq X^H$, and a similar bundle may be formed over each distinct NX_β^H, with $\beta \in \pi_0\left(X^H\right)$. Further, observe that since each bundle $E_{\alpha,[\sigma]}^N$ is an N-bundle over NX_α^H, it defines a unique G bundle $E_{\alpha,[\sigma]}^G$.

Definition 3. The G-bundle $E_{\alpha,[\sigma]}^G$ over the submanifold X_α is called a **fine component** or the **fine component of $E \to X$ associated to** $(\alpha, [\sigma])$.

If $G\diagdown X$ is not connected, one must construct the fine components separately over each X_α. If E has finite rank, then E may be decomposed as a direct sum of distinct fine components over each X_α. In any case, $E_{\alpha,[\sigma]}^N$ is a finite direct sum of isotypical components over each X_α^H.

Definition 4. The direct sum decomposition of $E|_{X_\alpha}$ into subbundles E^b that are fine components $E_{\alpha,[\sigma]}^G$ for some $[\sigma]$, written

$$E|_{X_\alpha} = \bigoplus_b E^b ,$$

is called the **refined isotypical decomposition** (or **fine decomposition**) of $E|_{X_\alpha}$.

In the case where $G\diagdown X$ is connected, the group $\pi_0(N/H)$ acts transitively on the connected components $\pi_0\left(X^H\right)$, and thus $X_\alpha = X$. We comment that if $[\sigma, W_\sigma]$ is an irreducible H-representation present in E_x with $x \in X_\alpha^H$, then $E_x^{[\sigma]}$ is a subspace of a distinct E_x^b for some b. The subspace E_x^b also contains $E_x^{[\sigma^n]}$ for every n such that $nX_\alpha^H = X_\alpha^H$.

Remark 5. Observe that by construction, for $x \in X_\alpha^H$ the multiplicity and dimension of each $[\sigma]$ present in a specific E_x^b is independent of $[\sigma]$. Thus, $E_x^{[\sigma^n]}$ and $E_x^{[\sigma]}$ have the same multiplicity and dimension if $nX_\alpha^H = X_\alpha^H$.

Remark 6. The advantage of this decomposition over the isotypical decomposition is that each E^b is a G-bundle defined over all of X_α, and the isotypical decomposition may only be defined over X_α^H.

Definition 7. Now, let E be a G-equivariant vector bundle over X, and let E^b be a fine component as in Definition 3 corresponding to a specific component $X_\alpha = GX_\alpha^H$ of X relative to G. Suppose that another G-bundle W over X_α has finite rank and has the property that the equivalence classes of G_y-representations present in $E_y^b, y \in X_\alpha$ exactly coincide with the equivalence classes of G_y-representations present in W_y, and that W has a single component in the fine decomposition. Then we say that W is **adapted** to E^b.

Lemma 8. *In the definition above, if another G-bundle W over X_α has finite rank and has the property that the equivalence classes of G_y-representations present in $E_y^b, y \in X_\alpha$ exactly coincide with the equivalence classes of G_y-representations present in W_y, then it follows that W has a single component in the fine* *decomposition and hence is adapted to E^b. Thus, the last phrase in the corresponding sentence in the above definition is superfluous.*

Proof. Suppose that we choose an equivalence class $[\sigma]$ of H-representations present in W_x, $x \in X_\alpha^H$. Let $[\sigma']$ be any other equivalence class; then, by hypothesis, there exists $n \in N$ such that $nX_\alpha^H = X_\alpha^H$ and $[\sigma'] = [\sigma^n]$. Then, observe that $nW_x^{[\sigma]} = W_{nx}^{[\sigma^n]} = W_x^{[\sigma^n]}$, with the last equality coming from the rigidity of irreducible H-representations. Thus, W is contained in a single fine component, and so it must have a single component in the fine decomposition. ☐

In what follows, we show that there are naturally defined finite-dimensional vector bundles that are adapted to any fine components. We enumerate the irreducible representations $\left\{[\rho_j, V_{\rho_j}]\right\}_{j=1,2,\ldots}$ of G. Let $[\sigma, W_\sigma]$ be any irreducible H-representation. Let $G \times_H W_\sigma$ be the corresponding homogeneous vector bundle over the homogeneous space G/H. Then the L^2-sections of this vector bundle decompose into irreducible G-representations. In particular, let $\left[\rho_{j_0}, V_{\rho_{j_0}}\right]$ be the equivalence class of irreducible representations that is present in $L^2\left(G/H, G \times_H W_\sigma\right)$ and that has the lowest index j_0. Then Frobenius reciprocity implies

$$0 \neq \operatorname{Hom}_G\left(V_{\rho_{j_0}}, L^2\left(G/H, G \times_H W_\sigma\right)\right) \cong \operatorname{Hom}_H\left(V_{\operatorname{Res}\left(\rho_{j_0}\right)}, W_\sigma\right),$$

so that the restriction of ρ_{j_0} to H contains the H-representation $[\sigma]$. Now, for a component X_α^H of X^H, with $X_\alpha = GX_\alpha^H$ its component in X relative to G, the trivial bundle

$$X_\alpha \times V_{\rho_{j_0}}$$

is a G-bundle (with diagonal action) that contains a nontrivial fine component $W_{\alpha,[\sigma]}$ containing $X_\alpha^H \times \left(V_{\rho_{j_0}}\right)^{[\sigma]}$.

Definition 9. We call $W_{\alpha,[\sigma]} \to X_\alpha$ the **canonical isotropy G-bundle associated to** $(\alpha, [\sigma]) \in \pi_0\left(X^H\right) \times \widehat{H}$. Observe that $W_{\alpha,[\sigma]}$ depends only on the enumeration of irreducible representations of G, the irreducible H-representation $[\sigma]$ and the component X_α^H.

Lemma 10. *Given any G-bundle $E \to X$ and any fine component E^b of E over some $X_\alpha = GX_\alpha^H$, there exists a canonical isotropy G-bundle $W_{\alpha,[\sigma]}$ adapted to $E^b \to X_\alpha$.*

3. Equivariant heat kernel and equivariant index. We now review some properties of the equivariant index and equivariant heat kernel that are known to experts in the field (see [2], [10], [11], [5]). With notation as in the introduction, let $E = E^+ \oplus E^-$ be a graded, G-equivariant vector bundle over M. We consider a first order G-equivariant differential operator $D^+ : \Gamma\left(M, E^+\right) \to \Gamma\left(M, E^-\right)$ which is transversally elliptic, and let D^- be the formal adjoint of D^+. The restriction $D^{\pm,\rho} = D^\pm|_{\Gamma(M,E)^\rho}$ behaves in a similar way to an elliptic operator. Let $\{X_1, ..., X_r\}$ be an orthonormal basis of the Lie algebra of G. Let $\mathcal{L}_{X_j}$ denote the induced Lie derivative with respect to X_j on sections of E, and let $C = \sum_j \mathcal{L}_{X_j}^* \mathcal{L}_{X_j}$ be the Casimir operator on sections of E. Let $\lambda_\rho \geq 0$ be the eigenvalue of C associated to the representation type $[\rho]$. The following argument can be seen in some form in [2]. Given a section $\alpha \in \Gamma\left(M, E^+\right)^\rho$, we have

$$D^- D^+ \alpha = \left(D^- D^+ + C - \lambda_\rho\right) \alpha.$$

Then $D^- D^+ + C - \lambda_\rho$ is self-adjoint and elliptic and has finite dimensional eigenspaces consisting of smooth sections. The $[\rho]$-part $K^{[\rho]}$ of the heat kernel of $e^{-tD^- D^+}$ is the same as the $[\rho]$-part of the heat kernel $K\left(t, \cdot, \cdot\right)$ of $e^{-t\left(D^- D^+ + C - \lambda_\rho\right)}$. One can show that

$$\operatorname{ind}^\rho(D) = \int_{x \in M} \int_{g \in G} \operatorname{str}\, g \cdot K\left(t, g^{-1}x, x\right)\, \overline{\chi_\rho(g)}\, dg\, \left|dx\right|, \tag{2}$$

where $\chi_\rho(g)$ Since the heat kernel K changes smoothly with respect to G-equivariant deformations of the metric and of the operator D and the right hand side is an integer, we see that $\operatorname{ind}^\rho(D)$ is stable under such homotopies of the operator D^+ through G-equivariant transversally elliptic operators. This implies that the indices $\operatorname{ind}^G(D)$ and $\operatorname{ind}_g(D)$ mentioned in the introduction depend only on the G-equivariant homotopy class of the principal transverse symbol of D^+. Since the integral above is independent of t, we may compute its asymptotics as $t \to 0^+$ to determine $\operatorname{ind}^\rho(D)$.

One important idea is that the asymptotics of $K\left(t,g^{-1}x,x\right)$ as $t\to 0$ are completely determined by the operator's local expression along the minimal geodesic connecting $g^{-1}x$ and x, if $g^{-1}x$ and x are sufficiently close together. If the distance between $g^{-1}x$ and x is bounded away from zero, there is a constant $c>0$ such that $K\left(t,g^{-1}x,x\right)=\mathcal{O}\left(e^{-c/t}\right)$ as $t\to 0$. For these reasons, it is clear that the asymptotics of the supertrace $\operatorname{ind}^{\rho}(D)$ are locally determined over spaces of orbits in M, meaning that the contribution to the index is a sum of t^0-asymptotics of integrals of $\int_U\int_G \operatorname{str}\left(g\cdot K\left(t,g^{-1}w,w\right)\right)\overline{\chi_\rho(g)}\,dg\,|dw|$ over a finite collection of saturated sets $U\subset M$ that are unions of orbits that intersect a neighborhood of a point of M. However, the integral of the t^0 asymptotic coefficient of $\int_G \operatorname{str}\left(g\cdot K\left(t,g^{-1}w,w\right)\right)\overline{\chi_\rho(g)}\,dg$ over all of M is not the index, and the integral of the t^0 asymptotic coefficient of $\int_M \operatorname{str}\left(g\cdot K\left(t,g^{-1}w,w\right)\right)\,\mathrm{dvol}_M$ over all of G is not the index. Thus, the integrals over M and G may not be separated when computing the local index contributions. In particular, the singular strata of the group action may not be ignored.

4. Desingularizing along a singular stratum. In the first part of this section, we will describe some standard results from the theory of Lie group actions (see [9], [25]). Such G-manifolds are stratified spaces, and the stratification can be described explicitly. As above, G is a compact Lie group acting on a smooth, connected, closed manifold M. We assume that the action is effective, meaning that no $g\in G$ fixes all of M. (Otherwise, replace G with $G\diagup\{g\in G: gx=x \text{ for all } x\in M\}$.) Choose a Riemannian metric for which G acts by isometries; average the pullbacks of any fixed Riemannian metric over the group of diffeomorphisms to obtain such a metric.

For $x\in M$, the isotropy or stabilizer subgroup $G_x<G$ is defined to be $\{g\in G: gx=x\}$. The orbit $\mathcal{O}_x$ of a point x is defined to be $\{gx: g\in G\}$. Since $G_{xg}=gG_xg^{-1}$, the conjugacy class of the isotropy subgroup of a point is fixed along an orbit.

On any such G-manifold, the conjugacy class of the isotropy subgroups along an orbit is called the **orbit type**. On any such G-manifold, there are a finite number of orbit types, and there is a partial order on the set of orbit types. Given subgroups H and K of G, we say that $[H]\leq[K]$ if H is conjugate to a subgroup of K, and we say $[H]<[K]$ if $[H]\leq[K]$ and $[H]\neq[K]$. We may enumerate the conjugacy classes of isotropy subgroups as $[G_0],...,[G_r]$ such that $[G_i]\leq[G_j]$ implies that $i\leq j$. It is well-known that the union of the principal orbits (those with type $[G_0]$) form an open dense subset M_0 of the manifold M, and the other orbits are called **singular**. As a consequence, every isotropy subgroup H satisfies $[G_0]\leq[H]$. Let M_j denote the set of points of M of orbit type $[G_j]$ for each j; the set M_j is called the **stratum** corresponding to $[G_j]$. If $[G_j]\leq[G_k]$, it follows that the closure of M_j contains the closure of M_k. A stratum M_j is called a **minimal stratum** if there does not exist a stratum M_k such that $[G_j]<[G_k]$ (equivalently, such that $\overline{M_k}\subsetneq\overline{M_j}$). It is known that each stratum is a G-invariant submanifold of M, and in fact a minimal stratum is a closed (but not necessarily connected) submanifold. Also, for each j, the submanifold $M_{\geq j}:=\bigcup\limits_{[G_k]\geq[G_j]}M_k$ is a closed, G-invariant submanifold. To prove the main theorem of this paper, we decompose the G-manifold using tubular neighborhoods of the minimal strata.

We will now construct a new G-manifold N that has a single stratum (of type $[G_0]$) and that is a branched cover of M, branched over the singular strata. A distinguished fundamental domain of M_0 in N is called the **desingularization of** M and is denoted $\widetilde{M}$. We also refer to [1] for their recent related explanation of this desingularization (which they call *resolution*). To simplify this discussion, we will assume that the codimension of each singular stratum is at least two.

A sequence of modifications is used to construct N and $\widetilde{M}\subset N$. Let M_j be a minimal stratum. Let $T_\varepsilon(M_j)$ denote a tubular neighborhood of radius ε around M_j, with ε chosen sufficiently small so that all orbits in $T_\varepsilon(M_j)\setminus M_j$ are of type $[G_k]$, where $[G_k]<[G_j]$. Let

$$N^1=(M\setminus T_\varepsilon(M_j))\cup_{\partial T_\varepsilon(M_j)}(M\setminus T_\varepsilon(M_j))$$

be the manifold constructed by gluing two copies of $(M\setminus T_\varepsilon(M_j))$ smoothly along the boundary. Since the $T_\varepsilon(M_j)$ is saturated (a union of G-orbits), the G-action lifts to N^1. Note that the strata of the G-action on N^1 correspond to strata in $M\setminus T_\varepsilon(M_j)$. If $M_k\cap(M\setminus T_\varepsilon(M_j))$ is nontrivial, then the stratum corresponding to isotropy type $[G_k]$ on N^1 is

$$N_k^1=(M_k\cap(M\setminus T_\varepsilon(M_j)))\cup_{(M_k\cap\partial T_\varepsilon(M_j))}(M_k\cap(M\setminus T_\varepsilon(M_j))).$$

Thus, N^1 is a G-manifold with one fewer stratum than M, and $M \setminus M_j$ is diffeomorphic to one copy of $(M \setminus T_\varepsilon(M_j))$, denoted $\widetilde{M}^1$ in N^1. In fact, N^1 is a branched double cover of M, branched over M_j. If N^1 has one orbit type, then we set $N = N^1$ and $\widetilde{M} = \widetilde{M}^1$. If N^1 has more than one orbit type, we repeat the process with the G-manifold N^1 to produce a new G-manifold N^2 with two fewer orbit types than M and that is a 4-fold branched cover of M. Again, $\widetilde{M}^2$ is a fundamental domain of $\widetilde{M}^1 \setminus \{\text{a minimal stratum}\}$, which is a fundamental domain of M with two strata removed. We continue until $N = N^r$ is a G-manifold with all orbits of type $[G_0]$ and is a 2^r-fold branched cover of M, branched over $M \setminus M_0$. We set $\widetilde{M} = \widetilde{M}^r$, which is a fundamental domain of M_0 in N.

Further, one may independently desingularize $M_{\geq j}$, since this submanifold is itself a closed G-manifold. If $M_{\geq j}$ has more than one connected component, we may desingularize all components simultaneously. The isotropy type of all points of $\widetilde{M_{\geq j}}$ is $[G_j]$, and $\widetilde{M_{\geq j}}\diagup G$ is a smooth (open) manifold.

We now more precisely describe the desingularization. If M is equipped with a G-equivariant, transversally elliptic differential operator on sections of an equivariant vector bundle over M, then this data may be pulled back to the desingularization $\widetilde{M}$. Given the bundle and operator over N^j, simply form the invertible double of the operator on N^{j+1}, which is the double of the manifold with boundary $N^j \setminus T_\varepsilon(\Sigma)$, where Σ is a minimal stratum on N^j.

Specifically, we modify the metric equivariantly so that there exists $\varepsilon > 0$ such that the tubular neighborhood $B_{2\varepsilon}\Sigma$ of Σ in N^j is isometric to a ball of radius 2ε in the normal bundle $N\Sigma$. In polar coordinates, this metric is $ds^2 = dr^2 + d\sigma^2 + r^2 d\theta_\sigma^2$, with $r \in (0, 2\varepsilon)$, $d\sigma^2$ is the metric on Σ, and $d\theta_\sigma^2$ is the metric on $S(N_\sigma\Sigma)$, the unit sphere in $N_\sigma\Sigma$; note that $d\theta_\sigma^2$ is isometric to the Euclidean metric on the unit sphere. We simply choose the horizontal metric on $B_{2\varepsilon}\Sigma$ to be the pullback of the metric on the base Σ, the fiber metric to be Euclidean, and we require that horizontal and vertical vectors be orthogonal. We do not assume that the horizontal distribution is integrable.

Next, we replace r^2 with $f(r) = [\tilde{g}(r)]^2$ in the expression for the metric, where $\tilde{g}$ is defined so that the metric is cylindrical for small r.

In our description of the modification of the differential operator, we will need the notation for the (external) product of differential operators. Suppose that $F \hookrightarrow X \xrightarrow{\pi} B$ is a fiber bundle that is locally a metric product. Given an operator $A_{1,x} : \Gamma\left(\pi^{-1}(x), E_1\right) \to \Gamma\left(\pi^{-1}(x), F_1\right)$ that is locally given as a differential operator $A_1 : \Gamma(F, E_1) \to \Gamma(F, F_1)$ and $A_2 : \Gamma(B, E_2) \to \Gamma(B, F_2)$ on Hermitian bundles, we have the product

$$A_{1,x} * A_2 : \Gamma\left(X, (E_1 \boxtimes E_2) \oplus (F_1 \boxtimes F_2)\right) \to \Gamma\left(X, (F_1 \boxtimes E_2) \oplus (E_1 \boxtimes F_2)\right)$$

as in K-theory (see, for example, [2], [27, pp. 384ff]), which is used to define the Thom Isomorphism in vector bundles.

Let $D = D^+ : \Gamma\left(N^j, E^+\right) \to \Gamma\left(N^j, E^-\right)$ be the given first order, transversally elliptic, G-equivariant differential operator. Let Σ be a minimal stratum of N^j. Here we assume that Σ has codimension at least two. We modify the metrics and bundles equivariantly so that there exists $\varepsilon > 0$ such that the tubular neighborhood $B_\varepsilon(\Sigma)$ of Σ in M is isometric to a ball of radius ε in the normal bundle $N\Sigma$, and so that the G-equivariant bundle E over $B_\varepsilon(\Sigma)$ is a pullback of the bundle $E|_\Sigma \to \Sigma$. We assume that near Σ, after a G-equivariant homotopy D^+ can be written on $B_\varepsilon(\Sigma)$ locally as the product

$$D^+ = (D_N * D_\Sigma)^+ ,$$

where D_Σ is a transversally elliptic, G-equivariant, first order operator on the stratum Σ, and D_N is a G-equivariant, first order operator on $B_\varepsilon(\Sigma)$ that is elliptic on the fibers. If r is the distance from Σ, we write D_N in polar coordinates as

$$D_N = Z\left(\nabla^E_{\partial_r} + \frac{1}{r} D^S\right)$$

where $Z = -i\sigma(D_N)(\partial_r)$ is a local bundle isomorphism and the map D^S is a purely first order operator that differentiates in the unit normal bundle directions tangent to $S_x\Sigma$.

On first glance, the product form above appears to be restrictive, but as we show in [12], the product assumption is satisfied by most operators under fairly weak topological conditions on the stratum.

We modify the operator D_N on each Euclidean fiber of $N\Sigma \xrightarrow{\pi} \Sigma$ by converting the conical metric to a cylindrical metric via a radial blow-up; the result is a G-manifold $\widetilde{M}^j$ with boundary $\partial\widetilde{M}^j$, a G-vector bundle $\widetilde{E}^j$, and the induced operator $\widetilde{D}^j$, all of which locally agree with the original counterparts outside $B_\varepsilon(\Sigma)$. We may double $\widetilde{M}^j$ along the boundary $\partial\widetilde{M}^j$ and reverse the chirality of $\widetilde{E}^j$ as described in [8, Ch. 9]. Doubling produces a closed G-manifold N^j, a G-vector bundle E^j, and a first-order transversally elliptic differential operator D^j. This process may be iterated until all orbits of the resulting G-manifold are principal.

In the technical proof of the main theorem in [12], we carefully track the changes to the heat kernel integral (2) throughout the desingularization process. When the radial blowup occurs, the manifold is replaced with a manifold with boundary and nonlocal boundary conditions; this is the reason that eta invariants appear in the formula for the equivariant index multiplicities. In spite of this, every part of the formula is explicitly computable from the principal transverse symbol of the operator restricted to small saturated neighborhoods.

The crucial formula is as follows. In calculating the small t asymptotics of

$$\int \mathrm{str} K\left(t, z_p, z_p\right)^\rho = \int \mathrm{str}\left(E_t\left(D\right)^\rho\right)\left(z_p, z_p\right) \ ,$$

with $E_t\left(D\right)^\rho = \exp\left(-tD^*D\right)^\rho$, it suffices to calculate the right hand side of the formula above over a small tubular neighborhood $B_\varepsilon(U) \subset M$ of a saturated open set $U \subset \Sigma_\alpha \subseteq \Sigma$ in a most singular stratum. We then sum over fine components $b \in B$ (see Definition 3), using the heat kernel coming from the blown up manifold $\widetilde{B_\varepsilon(U)}$. As $t \to 0$,

$$\begin{aligned}\int_{B_\varepsilon(U)} \mathrm{str} K\left(t, z_p, z_p\right)^\rho &\sim \int_{\widetilde{B_\varepsilon(U)}} \mathrm{str} K\left(t, z_p, z_p\right)^\rho \\ &+ \sum_b \frac{1}{2n_b \mathrm{rank}\left(W^b\right)}\left(-\eta\left(D^{S+,b}\right) + h\left(D^{S+,b}\right)\right)\int_{p\in U} \mathrm{str} K_\Sigma^b\left(t,p,p\right)^\rho , \qquad (3)\end{aligned}$$

with $\mathrm{str} K_\Sigma^b\left(t,p,p\right)^\rho = \mathrm{str}\left(E_t\left(\mathbf{1}^b \otimes D_\Sigma\right)^\rho\right)(p,p)$ is the local heat supertrace corresponding to the operator $\mathbf{1}^b \otimes D_\Sigma$ on $\Gamma\left(U, W^b \otimes E_\Sigma\right)^\rho$. The eta invariant $\eta\left(D^{S+,b}\right)$ is the equivariant eta invariant of D^{S+} restricted to isotropy representation types present in W^b; since the eigenvalues of D^{S+} are integers, this quantity is constant over components of the stratum relative to G. Similarly, the dimension $h\left(D^{S+,b}\right)$ of the kernel of D^{S+} restricted to those sections is locally constant.

5. The Equivariant Index Theorem. To evaluate $\mathrm{ind}^\rho(D)$ as in Equation (2), we apply formula (3) repeatedly, starting with a minimal stratum and then applying to each double of the equivariant desingularization. After all the strata are blown up and doubled, all of the resulting manifolds have a single stratum, and the G-manifold is a fiber bundle with homogeneous fibers. We obtain the following result. In what follows, if U denotes an open subset of a stratum of the action of G on M, U' denotes the equivariant desingularization of U, and $\widetilde{U}$ denotes the fundamental domain of U inside U', as in Section 4. We also refer the reader to Definitions 1 and 9. For the sake of simplicity of exposition, we assume that the codimension of each stratum is at least two.

Theorem 11. *(Equivariant Index Theorem) Let M_0 be the principal stratum of the action of a compact Lie group G on the closed Riemannian M, and let Σ_{α_1},...,Σ_{α_r} denote all the components of all singular strata relative to G. Let $E \to M$ be a Hermitian vector bundle on which G acts by isometries. Let $D : \Gamma\left(M, E^+\right) \to \Gamma\left(M, E^-\right)$ be a first order, transversally elliptic, G-equivariant differential operator. We assume that near each Σ_{α_j}, D is G-homotopic to the product $D_N * D^{\alpha_j}$, where D_N is a G-equivariant, first order differential operator on $B_\varepsilon\Sigma$ that is elliptic and has constant coefficients on the fibers and D^{α_j} is a global transversally elliptic, G-equivariant, first order operator on the Σ_{α_j}. In polar coordinates*

$$D_N = Z_j\left(\nabla^E_{\partial_r} + \frac{1}{r}D^S_j\right) \ ,$$

where r is the distance from Σ_{α_j}, where Z_j is a local bundle isometry (dependent on the spherical parameter), the map D^S_j is a family of purely first order operators that differentiates in directions tangent to the unit *normal bundle of* Σ_j. *Then the equivariant index* $\operatorname{ind}^{\rho}(D)$ *is given by the formula*

$$\begin{aligned} \operatorname{ind}^{\rho}(D) &= \int_{G\diagdown \widetilde{M_0}} A_0^{\rho}(x)\ \widetilde{|dx|}\ + \sum_{j=1}^{r} \beta\left(\Sigma_{\alpha_j}\right)\ , \\ \beta\left(\Sigma_{\alpha_j}\right) &= \frac{1}{2\dim V_\rho} \sum_b \frac{1}{n_b \text{rank } W^b} \left(-\eta\left(D_j^{S+,b}\right) + h\left(D_j^{S+,b}\right)\right) \int_{G\diagdown \widetilde{\Sigma_{\alpha_j}}} A_{j,b}^{\rho}(x)\ \widetilde{|dx|}\ , \end{aligned}$$

where the sum is over all canonical isotropy bundles W^b, *a finite number of which yield nonzero* $A_{j,b}^{\rho}$, *and where*

(1) $A_0^{\rho}(x)$ *is the Atiyah-Singer integrand, the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from* D' *(blown-up and doubled from* D*) on the quotient* $M_0'\diagup G$, *where the bundle* E *is replaced by the finite-dimensional space of sections of type* ρ *over an orbit.*

(2) *Similarly,* $A_{i,b}^{\rho}$ *is the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from* $(\mathbf{1}\otimes D^{\alpha_j})'$ *(blown-up and doubled from* $\mathbf{1}\otimes D^{\alpha_j}$, *the twist of* D^{α_j} *by the canonical isotropy bundle* $W^b \to \Sigma_{\alpha_j}$ *) on the quotient* $\Sigma_{\alpha_j}'\diagup G$, *where the bundle is replaced by the space of sections of type* ρ *over each orbit.*

(3) $\eta\left(D_j^{S+,b}\right)$ *is the eta invariant of the operator* D_j^{S+} *induced on any unit normal sphere* $S_x\Sigma_{\alpha_j}$, *restricted to sections of isotropy representation types in* W_x^b, *which is constant on* Σ_{α_j}.

(4) $h\left(D_j^{S+,b}\right)$ *is the dimension of the kernel of* $D_j^{S+,b}$, *restricted to sections of isotropy representation types in* W_x^b, *again constant on on* Σ_{α_j}.

(5) n_b *is the number of different inequivalent* G_x*-representation types present in each* W_x^b, $x\in\Sigma_{\alpha_j}$.

We now give a simple application of our result. It is well known that if M is a Riemannian manifold and $f: M\to M$ is an isometry that is homotopic to the identity, then the Euler characteristic of M is the sum of the Euler characteristics of the fixed point sets of f. We generalize this result as follows. We consider the de Rham operator

$$d+d^*:\Omega^{\text{even}}(M)\to\Omega^{\text{odd}}(M)$$

on a G-manifold, and the invariant index of this operator is the equivariant Euler characteristic $\chi^G(M)$, the Euler characteristic of the elliptic complex consisting of invariant forms. If G is connected and the Euler characteristic is expressed in terms of its ρ-components, only the invariant part $\chi^G(M)=\chi^{\rho_0}(M)$ appears. This is a consequence of the homotopy invariance of de Rham cohomology. Thus $\chi^G(M)=\chi(M)$ for connected Lie groups G. In general the Euler characteristic is a sum of components

$$\chi(M)=\sum_{[\rho]}\chi^{\rho}(M),$$

where $\chi^{\rho}(M)$ is the alternating sum of the dimensions of the $[\rho]$-parts of the cohomology groups (or spaces of harmonic forms). Since the connected component G_0 of the identity in G acts trivially on the harmonic forms, the only nontrivial components $\chi^{\rho}(M)$ correspond to representations induced from unitary representations of the finite group $G\diagup G_0$.

Using Theorem 11 and the formula $\operatorname{ind}^{\rho}(d+d^*)=\frac{1}{\dim V_\rho}\chi^{\rho}(M)$, we obtain the following result.

Theorem 12. *Let* M *be a compact* G*-manifold, with* G *a compact Lie group and principal isotropy subgroup* H_{pr}. *Let* M_0 *denote the principal stratum, and let* $\Sigma_{\alpha_1},\ldots,\Sigma_{\alpha_r}$ *denote all the components of all singular strata relative to* G. *Then*

$$\begin{aligned} \chi^{\rho}(M) &= \chi^{\rho}(G\diagup H_{\text{pr}})\,\chi(G\diagdown M, G\diagdown \textit{singular strata}) \\ &\quad + \sum_j \chi^{\rho}\left(G\diagup G_j\ ,\mathcal{L}_{N_j}\right)\chi\left(G\diagdown\overline{\Sigma_{\alpha_j}}, G\diagdown \textit{lower strata}\right), \end{aligned}$$

where $\mathcal{L}_{N_j}$ *is the orientation line bundle of normal bundle of the stratum component* Σ_{α_j}.

In the formula above, the $\chi(X,Y)$ refers to the relative Euler characteristic (see [12] for details and examples).

6. The Basic Index Theorem for Riemannian foliations. The content of this section is discussed and proved in detail in [13]. Let M be an n-dimensional, closed, connected manifold, and let $\mathcal{F}$ be a codimension q foliation on M. Let Q denote the quotient bundle $TM\diagup T\mathcal{F}$ over M. Such a foliation is called a *Riemannian foliation* if it is endowed with a metric on Q (called the transverse metric) that is *holonomy-invariant*; that is, the Lie derivative of that transverse metric with respect to every leafwise tangent vector is zero. The metric on Q can always be extended to a Riemannian metric on M; the extended metric restricted to the normal bundle $N\mathcal{F} = (T\mathcal{F})^{\perp}$ agrees with the transverse metric via the isomorphism $Q \cong N\mathcal{F}$. We refer the reader to [28], [31], and [32] for introductions to the geometric and analytic properties of Riemannian foliations.

Let $\widehat{M}$ be the transverse orthonormal frame bundle of $(M, \mathcal{F})$, and let p be the natural projection $p : \widehat{M} \to M$. The Bott connection is a natural connection on Q that induces a connection on $\widehat{M}$ (see [28, pp. 80ff]). The manifold $\widehat{M}$ is a principal $O(q)$-bundle over M. Given $\hat{x} \in \widehat{M}$, let $\hat{x}g$ denote the well-defined right action of $g \in G = O(q)$ applied to $\hat{x}$. Associated to $\mathcal{F}$ is the lifted foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$; the distribution $T\widehat{\mathcal{F}}$ is the horizontal lift of $T\mathcal{F}$. By the results of Molino (see [28, pp. 105-108, p. 147ff]), the lifted foliation is transversally parallelizable (meaning that there exists a global basis of the normal bundle consisting of vector fields whose flows preserve $\widehat{\mathcal{F}}$), and the closures of the leaves are fibers of a fiber bundle $\widehat{\pi} : \widehat{M} \to \widehat{W}$. The manifold $\widehat{W}$ is smooth and is called the basic manifold. Let $\overline{\widehat{\mathcal{F}}}$ denote the foliation of $\widehat{M}$ by leaf closures of $\widehat{\mathcal{F}}$, which is shown by Molino to be a fiber bundle. The leaf closure space of $(M, \mathcal{F})$ is denoted $W = M\diagup\overline{\mathcal{F}} = \widehat{W}\diagup G$.

$$
\begin{array}{ccccc}
p^{*}E & & & & \mathcal{E} \\
 & \searrow & & & \downarrow \\
O(q) & \hookrightarrow & \left(\widehat{M}, \widehat{\mathcal{F}}\right) & \xrightarrow{\widehat{\pi}} & \widehat{W} \\
 & & \downarrow^{p} & \circlearrowleft & \downarrow \\
E & \to & (M, \mathcal{F}) & \xrightarrow{\pi} & W
\end{array}
$$

Endow $(\widehat{M}, \widehat{\mathcal{F}})$ with the transverse metric $g^{Q} \oplus g^{O(q)}$, where g^{Q} is the pullback of metric on Q, and $g^{O(q)}$ is the standard, normalized, biinvariant metric on the fibers. We require that vertical vectors are orthogonal to horizontal vectors. This transverse metric gives each of $(\widehat{M}, \widehat{\mathcal{F}})$ and $(\widehat{M}, \overline{\widehat{\mathcal{F}}})$ the structure of a Riemannian foliation. The transverse metric on $(\widehat{M}, \overline{\widehat{\mathcal{F}}})$ induces a well–defined Riemannian metric on $\widehat{W}$. The action of $G = O(q)$ on $\widehat{M}$ induces an isometric action on $\widehat{W}$.

For each leaf closure $\overline{\widehat{L}} \in \overline{\widehat{\mathcal{F}}}$ and $\widehat{x} \in \overline{\widehat{L}}$, the restricted map $p : \overline{\widehat{L}} \to \overline{L}$ is a principal bundle with fiber isomorphic to a subgroup $H_{\widehat{x}} \subset O(q)$, which is the isotropy subgroup at the point $\widehat{\pi}(\hat{x}) \in \widehat{W}$. The conjugacy class of this group is an invariant of the leaf closure $\overline{L}$, and the strata of the group action on $\widehat{W}$ correspond to the strata of the leaf closures of $(M, \mathcal{F})$.

A *basic form* over a foliation is a global differential form that is locally the pullback of a form on the leaf space; more precisely, $\alpha \in \Omega^{*}(M)$ is basic if for any vector tangent to the foliation, the interior product with both α and $d\alpha$ is zero. A *basic vector field* is a vector field V whose flow preserves the foliation. In a Riemannian foliation, near any point it is possible to choose a local orthonormal frame of Q represented by basic vector fields.

A vector bundle $E \to (M, \mathcal{F})$ that is *foliated* may be endowed with a basic connection ∇^{E} (one for which the associated curvature forms are basic – see [21]). An example of such a bundle is the normal bundle Q. Given such a foliated bundle, a section $s \in \Gamma(E)$ is called a *basic section* if for every $X \in T\mathcal{F}$, $\nabla^{E}_{X} s = 0$. Let $\Gamma_{b}(E)$ denote the space of basic sections of E. Note that the basic sections of Q correspond to basic normal vector fields.

An example of another foliated bundle over a component of a stratum M_{j} is the bundle defined as follows. Let $E \to M$ be any foliated vector bundle. Let $\Sigma_{\alpha_{j}} = \widehat{\pi}\left(p^{-1}(M_{j})\right)$ be the corresponding stratum on the basic manifold $\widehat{W}$, and let $W^{\tau} \to \Sigma_{\alpha_{j}}$ be a canonical isotropy bundle (Definition 9). Consider the bundle $\widehat{\pi}^{*}W^{\tau} \otimes p^{*}E \to p^{-1}(M_{j})$, which is foliated and basic for the lifted foliation restricted to $p^{-1}(M_{j})$. This defines a new foliated bundle $E^{\tau} \to M_{j}$ by letting E^{τ}_{x} be the space of $O(q)$-invariant sections of $\widehat{\pi}^{*}W^{\tau} \otimes p^{*}E$ restricted to $p^{-1}(x)$. We call this bundle **the W^{τ}-twist of $E \to M_{j}$**.

Suppose that E is a foliated $\mathbb{Cl}(Q)$ module with basic $\mathbb{Cl}(Q)$ connection ∇^E over a Riemannian foliation $(M,\mathcal{F})$. Then it can be shown that Clifford multiplication by basic vector fields preserves $\Gamma_b(E)$, and we have the operator

$$D_b^E : \Gamma_b\left(E^+\right) \to \Gamma_b\left(E^-\right)$$

defined for any local orthonormal frame $\{e_1,...,e_q\}$ for Q by

$$D_b^E = \sum_{j=1}^{q} c\left(e_j\right) \nabla_{e_j}^E \Big|_{\Gamma_b(E)} .$$

Then D_b^E can be shown to be well-defined and is called the basic Dirac operator corresponding to the foliated $\mathbb{Cl}(Q)$ module E (see [17]). We note that this operator is not symmetric unless a zero$^{\text{th}}$ order term involving the mean curvature is added; see [22], [23], [24], [17], [30], [19], [13] for more information regarding essential self-adjointness of the modified operator and its spectrum. In the formulas below, any lower order terms that preserve the basic sections may be added without changing the index.

Definition 13. The *analytic basic index* of D_b^E is

$$\operatorname{ind}_b\left(D_b^E\right) = \dim\ker D_b^E - \dim\ker\left(D_b^E\right)^* .$$

It is well-known that these dimensions are finite (see [16], [24], [15], [13]), and it is possible to identify $\operatorname{ind}_b\left(D_b^E\right)$ with the invariant index of a first order, $O(q)$-equivariant differential operator $\widehat{D}$ over a vector bundle over the basic manifold $\widehat{W}$. By applying the equivariant index theorem (Theorem 11), we obtain the following formula for the index. In what follows, if U denotes an open subset of a stratum of $(M,\mathcal{F})$, U' denotes the desingularization of U very similar to that in Section 4, and $\widetilde{U}$ denotes the fundamental domain of U inside U'.

Theorem 14. *(Basic Index Theorem for Riemannian foliations* [13]*) Let M_0 be the principal stratum of the Riemannian foliation $(M,\mathcal{F})$, and let M_1, ... , M_r denote all the components of all singular strata, corresponding to $O(q)$-isotropy types $[G_1]$, ... ,$[G_r]$ on the basic manifold. With notation as in the discussion above, we have*

$$\begin{aligned}
\operatorname{ind}_b\left(D_b^E\right) &= \int_{\widetilde{M_0}\diagup\overline{\mathcal{F}}} A_{0,b}(x)\ \widetilde{|dx|} + \sum_{j=1}^{r} \beta\left(M_j\right) \\
\beta\left(M_j\right) &= \frac{1}{2}\sum_{\tau} \frac{1}{n_\tau \operatorname{rank} W^\tau} \left(-\eta\left(D_j^{S+,\tau}\right) + h\left(D_j^{S+,\tau}\right)\right) \int_{\widetilde{M_j}\diagup\overline{\mathcal{F}}} A_{j,b}^\tau(x)\ \widetilde{|dx|},
\end{aligned}$$

where the sum is over all components of singular strata and over all canonical isotropy bundles W^τ, only a finite number of which yield nonzero terms $A_{j,b}^\tau$, and where

1. *$A_{0,b}(x)$ is the Atiyah-Singer integrand, the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $\widetilde{D_b^E}$ (a desingularization of D_b^E) on the quotient $\widetilde{M_0}\diagup\overline{\mathcal{F}}$, where the bundle E is replaced by the space of basic sections of over each leaf closure;*
2. *$\eta\left(D_j^{S+,b}\right)$ and $h\left(D_j^{S+,b}\right)$ are defined in a similar way as in Theorem 11, using a decomposition $D_b^E = D_N * D_{M_j}$ at each singular stratum;*
3. *$A_{j,b}^\tau(x)$ is the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $\left(\mathbf{1}\otimes D_{M_j}\right)'$ (blown-up and doubled from $\mathbf{1}\otimes D_{M_j}$, the twist of D_{M_j} by the canonical isotropy bundle W^τ) on the quotient $\widetilde{M_j}\diagup\overline{\mathcal{F}}$, where the bundle is replaced by the space of basic sections over each leaf closure; and*
4. *n_τ is the number of different inequivalent G_j-representation types present in a typical fiber of W^τ.*

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