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Introduction to Spectral Theory of Differential Operators, SS 2011

Exercises 8, week of June 7, 2011

1

a) Prove that vector bundle (E, π, B) of rank k is trivial if and only if it has k sections $s_1, \ldots, s_k$ such that $s_1(b), \ldots, s_k(b)$ are linearly independent for each $b \in B$.

b) Show that the tangent bundle TS^1 over the circle S^1 is trivial.

2

Let (TM, τ_M, M) be the tangent bundle over an m-dimensional manifold M and denote by T^2M the subset of elements in $\xi \in TTM$ which satisfy $\tau_{TM}(\xi) = d(\tau_M)(\xi)$. Denote by $\tau_M^{2,1}$ the map $\tau_{TM}|_{T^2M}$.
Show that $(T^2M, \tau_M^{2,1}, TM)$ is not a vector bundle in any natural way, although the typical fibre is $\mathbb{R}^m$. Compare T^2M and $\tau_M^\star TM$.
Is $(TTM, \tau_M \circ \tau_{TM}, M)$ a vector bundle?

3

a) Consider the Möbius strip $\mathbb{R}^2/\sim$ with the relation

$$(x, y) \sim (x + n, (-1)^n y) \text{ for } n \in \mathbb{Z}.$$

Let $p : \mathbb{R}^2/\sim \ \to \ \mathbb{R}/\mathbb{Z}$ be the projection on the first component. Show that $(\mathbb{R}^2/\sim,\ p,\ \mathbb{R}/\mathbb{Z})$ is a non-trivial vector bundle.

b) Let E be the total space of the canonical line bundle over $\mathbb{RP}^1$. Show that the map

$$g :\ \mathbb{R}^2/\sim\ \ni\ [(x, \lambda)] \mapsto \left(\left[\begin{pmatrix} \cos \pi x \\ \sin \pi x \end{pmatrix} \right], \lambda \begin{pmatrix} \cos \pi x \\ \sin \pi x \end{pmatrix} \right) \ \in\ E$$

is a bundle isomorphism between the Möbius strip as a bundle over $\mathbb{R}/\mathbb{Z}$ and the canonical line bundle over $\mathbb{RP}^1$.