Prof. Dr. J. Sprekels
HUMBOLDT-UNIVERSITÄT ZU BERLIN
Institut für Mathematik

"Partial Differential Equations" (BMS)
– SS 10 –
Exercises No. 2

to be presented on Tuesday 4 Mai, 2010

Exercise 4:

a) Calculate the outer unit normal $n(x)$ for the sphere of the n-dimensional ball

$$B_r(a) := \{x \in \mathbb{R}^n \mid \|x - a\|_2 \leq r\}$$

with center a and radius r.

b) Confirm Gauss's theorem for the vector field $F(x) = x$ on the 3-dimensional unit ball $B = B_1(0) \subset \mathbb{R}^3$, i.e. calculate

$$\int_B \mathrm{div} F(x)\; dx \quad \text{and} \quad \int_{\partial B} F(\xi) \cdot n(\xi)\; dS(\xi)$$

directly and compare them.

c) Prove that Laplace's equation $\Delta u = 0$ is rotation invariant, i.e. if A is an orthogonal $n \times n$ matrix and we define

$$v(x) := u(Ax) \qquad (x \in \mathbb{R}^n),$$

then

$$\Delta v = 0.$$

Exercise 5:

In the following let $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ sufficiently smooth be a solution for the heat equation

$$(*) \qquad u_t = \Delta u$$

where $\Delta u = \Delta_x u = \sum_{i=1}^n u_{x_i x_i}$.

a) Show that $u^\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves $(*)$ for each $\lambda \in \mathbb{R}$.

b) The scaling a) indicates that the ratio $\frac{|x|^2}{t}$ (where $|x| := \|x\|_2 =$ length of the vector x) has some importance for the heat equation. Thus it seems to be appropriate to search for solutions of the form $u(x,t) = v(\frac{|x|^2}{t})$ with $v : \mathbb{R}_0^+ \to \mathbb{R}$.

So let's assume $n = 1$, $u(x,t) = v(\frac{x^2}{t})$ and show:

(i)

$$u_t = u_{xx}$$

if and only if

$$(**) \qquad 4zv''(z) + (2+z)v'(z) = 0 \quad (z > 0).$$

(ii) The general solution of $(**)$ is

$$v(z) = c\int_0^z e^{-s/4}s^{-1/2}ds + d.$$

c) Show that $\sum_{j=1}^n u_{x_j}$ and Δu are solutions of $(*)$ too and use this to show that

$$\frac{\alpha}{\sqrt{t}}e^{-\frac{x^2}{4t}} \quad (\alpha \in \mathbb{R})$$

solves the heat equation for $n = 1$.