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"Partial Differential Equations" (BMS)
– SS 10 –
Exercises No. 5
to be presented on May 25, 2010

Exercise 12:

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Show that for $u \in C^2(\Omega) \cap C(\overline{\Omega})$ holds:

a) If $-\Delta u < 0$ in Ω, then u does not possess a local maximum in Ω.

b) If $-\Delta u \leq 0$ in Ω, then $\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x)$.

Exercise 13:

Show:

a) The Newton potentials $N_a : \mathbb{R}^n \setminus \{a\} \to \mathbb{R}$ given by

$$N_a(x) = \begin{cases} \frac{1}{2\pi} \ln(|x-a|), & \text{for } n = 2 \\ \frac{-1}{(n-2)\omega_n} \frac{1}{|x-a|^{n-2}}, & \text{for } n \neq 2 \end{cases}$$

are harmonic functions in $\mathbb{R}^n \setminus \{a\}$.

b) For every fixed $y \in \mathbb{R}^n$ the function $x \mapsto P(x, y)$ is harmonic in $\mathbb{R}^n \setminus \{y\}$, where

$$P(x, y) = \frac{1}{\omega_n} \frac{|y|^2 - |x|^2}{|y - x|^n}$$

is the Poisson's integral kernel $P : (\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x, x) \mid x \in \mathbb{R}^n\} \to \mathbb{R}$.
(*This is Lemma 3.1.8(i) of the lecture.*)

Here ω_n denotes the surface measure of the n-dimensional unit ball.

Exercise 14:

a) Prove the following converse to the mean value property for harmonic functions: If $\varphi \in C^2(\Omega)$ satisfies

$$\varphi(a) = \frac{1}{\omega_n} \int_{|\xi|=1} \varphi(a + r\xi)\, dS(\xi)$$

for each ball $\overline{B_r(a)} \subset \Omega$, then φ is harmonic.

b) Let Ω be a bounded domain, $f \in C(\Omega)$, $g \in C(\partial\Omega)$ and let $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ be solutions of the Poisson equation $\Delta u = f$ in Ω. Show:

(i) If $u_1(x) \leq u_2(x)$ for every $x \in \partial\Omega$, then this holds for every $x \in \overline{\Omega}$.

(ii) If u_1 and u_2 additionally fulfill the boundary condition $u = g$ on $\partial\Omega$, then $u_1 \equiv u_2$ on $\overline{\Omega}$.

(*This means there is at most one solution in* $C^2(\Omega) \cap C(\overline{\Omega})$ *for the Poisson problem with boundary-value condition and (for* $f = 0$*) harmonic solutions are unique for given boundary values.*)

c) Let $n \geq 3$ and $\psi \in C_0^2(\mathbb{R}^n)$. Then any bounded solution of $\Delta u = \psi$ in $\mathbb{R}^n$ has the form

$$u(x) = \int_{\mathbb{R}^n} N_y(x)\psi(y)\, dy + C, \qquad x \in \mathbb{R}^n,$$

for some constant C.