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"Partial Differential Equations" (BMS)
– SS 10 –
Exercises No. 6
to be presented on June 1, 2010

Exercise 15:

Let $A = (a_{ij})$, $B = (b_{ij})$ be symmetric positive semidefinite $n \times n$ matrices.
Prove that

$$\sum_{i,j} a_{ij} b_{ij} \geq 0.$$

(*This is Lemma 3.2.2 of the lecture.*)

Exercise 16:

a) Let $u : \mathbb{R}^n \to \mathbb{R}$ be harmonic and nonnegative. Show that u is constant.

b) Let $\Omega = \{x \in \mathbb{R}^n \mid |x| > 1\}$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ with

$$\Delta u = 0 \text{ in } \Omega \qquad \text{and} \qquad \lim_{|x|\to\infty} u(x) = 0.$$

Prove that

$$\sup_{x\in\Omega} |u(x)| = \max_{x\in\partial\Omega} |u(x)|.$$

c) Let $n \geq 2$, $\Omega = B_1(0) \setminus \{0\}$ be the n-dimensional unit ball without 0. Further let $u \in C(\overline{B_1(0)})$ be harmonic on Ω.

Show that exists some harmonic $\overline{u} : B_1(0) \to \mathbb{R}$ with $u \equiv \overline{u}$ on Ω, i.e. u can be harmonically continued to the whole ball $B_1(0)$.

(*Hint: Use the Poisson integral* $\overline{u}(x) := \int_{|y|=1} P(x,y)u(y)\, dS(y)$ *and consider Green's function on* $B_1(0)$

$$G(x) := \begin{cases} N_0(x) & \textit{for } n = 2 \\ N_0(x) + \frac{1}{(n-2)\omega_n} & \textit{for } n \geq 3 \end{cases},$$

where $N_0(x)$ *is the respective Newton potential.*)

Exercise 17:

Let $\alpha > 0$, $0 \leq |x-y| \leq R$. Show that for

$$v(x) := e^{-\alpha|x-y|^2} - e^{-\alpha R^2}$$

the second order linear differential operator $Lv(x)$ has the form

$$e^{-\alpha|x-y|^2}\left(4\alpha^2 \sum_{i,j=1}^{n} a_{ij}(x)(x_i-y_i)(x_j-y_j) - 2\alpha\sum_{i=1}^{n}(a_{ii}(x) + b_i(x)(x_i-y_i))\right) + c(x)v(x).$$

Exercise 18:

Show the following:

a) There is at most one solution $u \in C^2$ of the initial boundary-value problem

$$u_t - u_{xx} = 0 \quad \text{for } 0 < x < L \text{ and } t > 0,$$

such that for given continuous functions $f, g, h : \mathbb{R} \to \mathbb{R}$

$$u(x,0) = f(x), \quad u(0,t) = g(t), \quad u(L,t) = h(t).$$

b) The solution $u \in C^2$ depends continously on the initial and boundary data, i.e. for every $T > 0$, $\varepsilon > 0$ and every continuous $\tilde{f}, \tilde{g}, \tilde{h}$ with

$$\|f - \tilde{f}\|_{C[0,L]} \leq \varepsilon, \quad \|g - \tilde{g}\|_{C[0,T]} \leq \varepsilon, \quad \|h - \tilde{h}\|_{C[0,T]} \leq \varepsilon;$$

exists $\delta > 0$ such that

$$|u - \tilde{u}| \leq \delta \quad \text{on } [0,L] \times [0,T]$$

for solutions $\tilde{u} \in C^2$ to the respective IBVP.

c) Give bounds to the C^2-solution $u(x,t)$ of

$$u_t = 4u_{xx} \quad \text{for } 0 < x < 2 \text{ and } t > 0,$$

subject to

$$u(x,0) = x(x-2)e^{-x} \quad \text{and} \quad u(0,t) = u(2,t) = 0,$$

without solving the problem.