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"Partial Differential Equations" (BMS)
– SS 10 –
Exercises No. 7

to be presented on June 9, 2010

Exercise 19:

Prove the following statements about distributions:

a) Let $f \in C(\Omega)$ and define

$$T_f[\varphi] := \int_\Omega f(x)\varphi(x)\ dx \qquad \forall \varphi \in C_0^\infty(\Omega).$$

The mapping $T : C(\Omega) \to \mathcal{D}'(\Omega)$ given by $f \mapsto T_f$ is injective, where $\mathcal{D}'(\Omega)$ is the vector space of distributions in Ω.

b) Let $a \in C^\infty(\Omega)$, $T \in \mathcal{D}'(\Omega)$. Then $(aT)[\varphi] := T[a\varphi]$ is a distribution and it holds the product rule $\partial_i(aT) = (\partial_i a)T + a(\partial_i T)$.

Exercise 20:

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(t) := \begin{cases} 1 - \frac{t}{2} & , t \in [0, 2] \\ 0 & , else \end{cases}$$

and let $f_n(t) := nf(nt)$.

a) Draft f, f_2, f_4 and f_8.

b) Determine DT_{f_n} for every $n \in \mathbb{N}$.

c) Show that $\lim_{n\to\infty} T_{f_n} = \delta_0$ and $\lim_{n\to\infty} DT_{f_n} = D\delta_0$ in $\mathcal{D}'(\mathbb{R})$.

Exercise 21:

Show that the function $W : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by

$$W(x,t) := \begin{cases} \frac{-1}{(4\pi t)^{n/2}} \exp\left(\frac{-|x|^2}{4t}\right) & , t > 0 \\ 0 & , t \leq 0 \end{cases}$$

is locally integrable in $\mathbb{R}^n \times \mathbb{R}$ and the associated regular distribution T_W is a fundamental solution to the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x,t) = 0$$

with respect to 0, i.e. it holds $\left(\Delta - \frac{\partial}{\partial t}\right) T_W = \delta_0$ in the sense of distributions.
(*This is Theorem 4.2.4 of the lecture.*)