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"Partial Differential Equations" (BMS)
– SS 10 –
Exercises No. 8

to be presented on June 16, 2010

Exercise 22:

Let $1 \le p \le \infty$, $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$. The convolution $f * g$ is defined by

$$(f * g)(x) := \begin{cases} \int_{\mathbb{R}^n} f(x-y)g(y)\ dy & \text{, if } y \mapsto f(x-y)g(y) \in L^1(\mathbb{R}^n) \\ 0 & \text{, else} \end{cases}.$$

Show the following properties of $f * g$:

(i) $f * g \in L^p(\mathbb{R}^n)$;

(ii) $\|f * g\|_{L^p} \le \|f\|_{L^1}\|g\|_{L^p}$;

(iii) $(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\ dy$ for almost every $x \in \mathbb{R}^n$;

(iv) $\operatorname{supp}(f*g) \subseteq \operatorname{supp} f + \operatorname{supp} g := \{x \in \mathbb{R}^n \mid \exists\, y \in \operatorname{supp} f, z \in \operatorname{supp} g \text{ with } y+z = x\}$.

Exercise 23:

Let $T \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $(T * \varphi)(x) := T[\tau_x \varphi]$. Then for every $i = 1, \ldots, n$

$$\frac{\partial}{\partial x_i}(T * \varphi) = \left(\frac{\partial}{\partial x_i} T\right) * \varphi.$$

Exercise 24:

Let $I \subset \mathbb{R}$ be an open interval and $T \in \mathcal{D}'(I)$ such that $T' := DT \equiv 0$. Prove that T is constant in the sense of distributions, i.e. there exists a constant $c \in \mathbb{R}$ such that $T = cT_{\mathbf{1}}$, where $\mathbf{1} : \mathbb{R} \to \mathbb{R}$ is defined by $\mathbf{1}(x) = 1$ for every $x \in \mathbb{R}$.
(*Hint: First show that for* $\varphi \in \mathcal{D}(I)$ *with* $T_{\mathbf{1}}[\varphi] = 0$ *holds* $T[\varphi] = 0$ *and then that* $c = T[\psi]$ *for some* $\psi \in \mathcal{D}(I)$ *with* $T_{\mathbf{1}}[\psi] = 1$.)