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"Partial Differential Equations" (BMS)
– SS 10 –
Exercises No. 10
to be presented on June 29, 2010

Exercise 28:

Let $\Omega \subset \mathbb{R}^n$ be bounded, $f, g \in W^{1,p}(\Omega)$ $(1 \le p \le \infty)$. Prove the following:

a) $|f| \in W^{1,p}(\Omega)$ (where $|f|(x) := |f(x)|$) and

$$D|f| = \begin{cases} Df & \text{, a.e. on } \{f > 0\} \\ -Df & \text{, a.e. on } \{f < 0\} \\ 0 & \text{, a.e. on } \{f = 0\} \end{cases} .$$

In particular it is $|D|f|| = |Df|$ a.e.
(*Hint: Consider for* $\varepsilon > 0$ *the function* $G_\varepsilon : \mathbb{R} \to \mathbb{R}$ *defined by* $G_\varepsilon(z) := \sqrt{z^2 + \varepsilon^2} - \varepsilon$.)

b) $\max\{f, g\}, \min\{f, g\} \in W^{1,p}(\Omega)$ and

$$D\max\{f, g\} = \begin{cases} Df & \text{, a.e. on } \{f > g\} \\ Dg & \text{, a.e. on } \{f < g\} \\ Df = Dg & \text{, a.e. on } \{f = g\} \end{cases} ;$$

$$D\min\{f, g\} = \begin{cases} Dg & \text{, a.e. on } \{f > g\} \\ Df & \text{, a.e. on } \{f < g\} \\ Df = Dg & \text{, a.e. on } \{f = g\} \end{cases} .$$

Exercise 29:

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 1 - |x|$. Show:

a) $f \in \mathring{W}^{1,p}(-1, 1)$ for every $p \in [1, \infty)$, $f \in W^{1,\infty}(-1, 1)$ but $f \notin \mathring{W}^{1,\infty}(-1, 1)$.

b) $f'' = 0$ a.e. on $[-1, 1]$ and $f(-1) = f(1) = 0$.

c) f is not a weak solution to

$$u'' = 0 \text{ in } (-1, 1) \quad \text{and} \quad u(-1) = u(1) = 0.$$