SOLUTION FOR EXERCISE 9 (C)

Given the PDE

$$2u_x(x,y,z) + 3u_y(x,y,z) + 4u_z(x,y,z) = e^{x+y+z}$$

for some smooth $u : \mathbb{R}^3 \to \mathbb{R}$. This equation is linear. Hence, we can use the system characterstic ODEs from the lucture (just generalizing them to three dimensions). Hence, we get:

$$\begin{aligned} x'(t) &= 2 \\ y'(t) &= 3 \\ z'(t) &= 4 \\ u'(t) &= e^{x(t)+y(t)+z(t)} \end{aligned}$$

Itegration yields:

$$\begin{aligned} x(t) &= 2t + C_1 \\ y(t) &= 3t + C_2 \\ z(t) &= 4t + C_3 \end{aligned}$$

Now we have to choose the "initial conditions" for $(x(0), y(0), z(0))$. But since the curve $(x(t), y(t), z(t))$ is a line in $\mathbb{R}^3$ we choose $(x(0), y(0), z(0))$ to lie on a hyperplane orthogonal to $(x(t), y(t), z(t))$. Hence, we choose:

$$\begin{pmatrix} x(0,s,k) \\ y(0,s,k) \\ z(0,s,k) \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} s + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} k$$

This gives rise for the characteristics:

$$\begin{aligned} x(t,s,k) &= 2t - 3s + 2k \\ y(t,s,k) &= 3t + 2s \\ z(t,s,k) &= 4t - k \end{aligned}$$

Moreover, we have:

$$u(t,s,k) = \frac{1}{9} e^{9t-s+k} + C(s,k)$$

This is actually the solution but in the new coordinates (t, s, k). Now, calculating the inverse for

$$A = \begin{pmatrix} 2 & -3 & 2 \\ 3 & 2 & 0 \\ 4 & 0 & -1 \end{pmatrix}$$

gives the coordinate transformation $(t,s,k) \to (x,y,z)$:

$$\begin{aligned} t &= \frac{1}{29}(2x+3y+4z) \\ s &= \frac{1}{29}(-3x+10y-6z) \\ k &= \frac{1}{29}(8x+12y-13z) \end{aligned}$$

After change of coordinates, we get

$$u(x,y,z) = \frac{1}{9}e^{x+y+z} + C(\frac{1}{29}(-3x+10y-6z), \frac{1}{29}(8x+12y-13z))$$

Applying the equation to $C(\cdot,\cdot)$, gives the necessary and suffcient condition $\partial_1 C(\cdot,\cdot) = 0$.