Construction of a diffeomorphism of $\mathbb{CP}^1$ and S^2

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- $\mathbb{CP}^1 = (\mathbb{C}^2 \setminus (0,0))/\sim$ with $(z_1, z_2) \sim (z_1, z_2) \Leftrightarrow \exists z \in \mathbb{C}^*$, s. t. $(z_1, z_2) = (zz_1, zz_2)$.

  An atlas is given by $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ where the charts are

  $$\varphi_1 : \mathbb{CP}^1 \setminus [0:1] = U_1 \to \mathbb{C} \qquad [1:z_2] \mapsto z_2$$

  $$\varphi_2 : \mathbb{CP}^1 \setminus [1:0] = U_2 \to \mathbb{C} \qquad [z_1:1] \mapsto z_1.$$

  With $\varphi_1(U_1 \cap U_2) = \mathbb{C}^* = \varphi_2(U_1 \cap U_2)$, the transition mapping is

  $$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* \qquad z \mapsto z^{-1}$$

  since $\varphi_2 \circ \varphi_1^{-1}(z) = \varphi_2([1:z]) = \varphi_2([z^{-1}:1]) = z^{-1}$ for all $z \in \mathbb{C}^*$. Here we use charts in $\mathbb{C}$ rather than in $\mathbb{R}^2$ for convenience.

- $S^2 = \{(a,b,c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}$

  An atlas is given by $\{(V_1, \psi_1), (V_2, \psi_2)\}$ where the charts are

  $$\psi_1 : S^2 \setminus (0,0,1) = V_1 \to \mathbb{C} \qquad (a,b,c) \mapsto \frac{a}{1-c} + i\frac{b}{1-c}$$

  $$\psi_2 : S^2 \setminus (0,0,-1) = V_2 \to \mathbb{C} \qquad (a,b,c) \mapsto \frac{a}{1+c} - i\frac{b}{1+c}.$$

  With $\psi_1(V_1 \cap V_2) = \mathbb{C}^* = \psi_2(V_1 \cap V_2)$, the transition mapping also turns out to be

  $$\psi_2 \circ \psi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* \qquad z \mapsto z^{-1}.$$

  In order to see this note that for all $(a,b,c) \in S^2 \setminus (0,0,\pm 1)$ we have $(\frac{a}{1-c} + i\frac{b}{1-c})(\frac{a}{1+c} - i\frac{b}{1+c}) = 1$ and apply the fact that ψ_1 and ψ_2 are both bijections, which is well-known from the construction of the stereographic projection.

- Define a map $f : \mathbb{CP}^1 \to S^2$ by setting $f(\varphi_i^{-1}(z)) := \psi_i^{-1}(z)$ for all $z \in \mathbb{C}$.

  1. This is a well-defined mapping from $\mathbb{CP}^1$ to S^2 since

     $$\varphi_2^{-1}(z_2) = \varphi_1^{-1}(z_1) \Leftrightarrow z_2 = \varphi_2 \circ \varphi_1^{-1}(z_1) \Leftrightarrow z_2 = z_1^{-1} \Leftrightarrow z_2 = \psi_2 \circ \psi_1^{-1}(z_1) \Leftrightarrow \psi_2^{-1}(z_2) = \psi_1^{-1}(z_1).$$

  2. This is a smooth mapping since

     $$\psi_i \circ f \circ \varphi_i^{-1} : \mathbb{C} \to \mathbb{C} \qquad z \mapsto z$$

     and for $i \neq j$

     $$\psi_j \circ f \circ \varphi_i^{-1} : \mathbb{C}^* \to \mathbb{C}^* \qquad z \mapsto z^{-1}$$

     are all smooth.

  3. The map is a diffeomorphism since

     $$f^{-1} : S^2 \to \mathbb{CP}^1 \qquad \psi_i^{-1}(z) \mapsto \varphi_i^{-1}(z) \quad \forall z \in \mathbb{C}$$

     defines a smooth inverse for f.