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Homework Problems 1

Analysis and Geometry on Manifolds WS 06/07

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Problem 1
Let $f : U \to \mathbb{R}^n$ be a smooth function on an open subset U with $d_x f \neq 0$ at some $x \in U$. Construct coordinates $(y_1, ..., y_n)$ about x such that $f(y_1, ..., y_n) = y_1$.

Problem 2
(a) Consider the set of (real) lines in $\mathbb{C}^{n+1}$,

$$\mathbb{CP}^n := \mathbb{C}^{n+1} \setminus \{0\}/\sim$$

where $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{C}$. $\mathbb{CP}^n$ is called the *n-dimensional complex projective space*. It is equipped with the quotient topology induced by the topology of $\mathbb{C}^{n+1}$ and the relation. Show that this is a manifold. Hint: Construct coordinates on $U_k := \{[x] \in \mathbb{CP}^n \mid x_k \neq 0\}$ for $k = 1, ..., n{+}1$ and show that the transformation maps are diffeomorphisms.
Show that $\mathbb{CP}^n$ is Hausdorff.
(b) Show that $\mathbb{CP}^1 \cong S^2$ are diffeomorphic manifolds.

Problem 3
Show that the following conditions are equivalent:
(i) X is Hausdorff. (ii)

$$\{x\} = \bigcap_{x \in U \subset A, U, X \setminus A \text{ open}} A$$

(iii) $\Delta := \{(x, x) \mid x \in X\} \subset X \times X$ is closed.

The following problems will be discussed in the tutorials:

Problem 4
(a) Express the Laplace operator on differentiable functions $f = f(x_1, x_2)$ on (open subsets of) $\mathbb{R}^2$,

$$\Delta f := -(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})f,$$

in terms of polar coordinates $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$. (b) How do bilinear forms on $T_x\mathbb{R}^n \cong \mathbb{R}^n$ transform under coordinate changes?
(c) How does the Hessian of a function f on $\mathbb{R}^n$,

$$\mathrm{Hess} f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j},$$

transform under change of coordinates? What is the normal form of the Hessian at a point?

Bitte wenden...

Problem 5

Show that the following topological spaces are differentiable manifolds

(a) The Klein bottle: $K^2 := [0,1] \times [0,1]/\sim$ where $\sim$ is the equivalence defined by the following relations: $(0,s) \sim (1,1-s)$ and $(t,0) \sim (t,1)$

(b) the set of un-ordered n-tuples of complex numbers $R := \mathbb{C}^n/S_n$, where the symmetric group acts via $(z_1, ..., z_n) \mapsto (z_{\sigma(1)}, ..., z_{\sigma(n)})$ for a given permutation $\sigma \in S_n$. Is the same true for un-ordered n-tuples of real numbers?

Show that both are Hausdorff.