Klaus Mohnke
Institut für Mathematik
Rudower Chaussee 25
Haus 1 Raum 306

Homework Problems 5

Analysis and Geometry on Manifolds WS 06/07

due 30.11.2006

The following 3 *problems are your homework assignment.*

Problem 1
(1) Consider the action of $G := \mathbb{Z}/2\mathbb{Z}$ on S^2 given by $z \mapsto -z$ for the group element different from the neutral element. Explain why S^2/G is a differentiable manifold. Find a a diffeomorphism $\Phi : S^2/G \cong \mathbb{RP}^2$, where $\mathbb{RP}^2$ is the real projective plane.
(2) Show that the map $F : S^2/G \to \mathbb{R}^4$ given by $F([x, y, z]) := (x^2 - y^2, xy, xz, yz) \in \mathbb{R}^4$ is well-defined and defines an embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$ (by this we mean an injective immersion which is a topological embedding). Here $(x, y, z) \in S^2$ and $[x, y, z]$ denotes the equivalence class in S^2/G.

Problem 2
(1) Show that $\mathbb{RP}^2$ is not orientable. (Hint: First show that for any parametrization of an oriented manifold (U, ϕ) with $U \subset \mathbb{R}^n$ connected, the bases $\{d_x\phi(e_1); ...; d_x\phi(e_n)\}$ are either oriented for all $x \in U$ or not oriented for all x. Then consider the two parametrizations $\phi, \psi : \mathbb{R}^2 \to \mathbb{RP}^2$ given by $\phi(s, t) := [1, s, t]$ and $\psi(x, y) := [x, 1, y]$.)
(2) Show that for any differentiable manifold M its tangent space TM is an orientable manifold.

Problem 3
(1) Show that a sphere S^n never admits an atlas with one element.
(2) Show that S^n is simply connected for $n \geq 2$. *Hint: Exploit the atlas consisting of the two stereographic projections discussed in the lecture*
(3)* Show that a non simply-connected compact manifold without boundary of dimension bigger than 2 never admits an atlas with less than three elements

The following problems will be discussed in the tutorials.

Problem 4
Let M be a differentiable manifold and consider the following set:

$$\tilde{M} := \{(x, o) \mid x \in M, o \text{ is an orientation of } T_xM\}.$$

(1) Show that $\tilde{M}$ is a differentiable, orientable manifold.
(2) Show that $\pi : \tilde{M} \to M$, $\pi(x, o) = x$ is a differentiable map whose differential is a linear isomorphism everywhere.
(3) Show that $G := \mathbb{Z}/2\mathbb{Z}$ acts on $\tilde{M}$ such that $\tilde{M}/G \cong M$, and the projection is given by the map defined in (2).
(4) Let M be connected. Show that M is orientable if and only if $\tilde{M}$ is NOT connected. *Hint:* Use Satz 1.14. of the lecture.