Markus Reiß

Course *Stochastic Processes*
Winter 2013/14
Humboldt-Universität zu Berlin

Exercises: sheet 1

1. Prove: Let X be Poisson(s) and Y be Poisson(t) distributed. If X and Y are independent, then $X+Y$ is Poisson($t+s$) distributed ($t, s > 0$). This means that the property of a convolution semigroup of measures $(P(t))_{t>0}$ holds: $P(s) * P(t) = P(t+s)$, $s, t > 0$. Which measure $P(0)$ is the neutral element of such a convolution semigroup?

2. Let $(N_t,\, t \geqslant 0)$ be a Poisson process of intensity $\lambda > 0$ and let $(Y_k)_{k\geqslant 1}$ be a sequence of i.i.d. random variables, independent of N. Then $X_t := \sum_{k=1}^{N_t} Y_k$ is called *compound Poisson process* ($X_t := 0$ if $N_t = 0$).

   (a) Show that (X_t) has independent and stationary increments. Infer that the laws $P(t) = \mathbb{P}^{X_t}$ define a convolution semigroup (as in (1)).

   (b) Determine expectation and variance of X_t in the case $Y_k \in L^2$.

3. Flies and wasps land on your dinner plate in the manner of independent Poisson processes with respective intensities μ and λ. Show that the arrival of flying beasts forms a Poisson process of intensity $\lambda + \mu$ (*superposition*). The probability that an arriving fly is a blow-fly is p. Does the arrival of blow-flies also form a Poisson process? (*thinning*)

4. The number of busses that arrive until time t at a bus stop follows a Poisson process with intensity $\lambda > 0$ (in our model). Adam and Berta arrive together at time $t_0 > 0$ at the bus stop and discuss how long they have to wait in the mean for the next bus.
   Adam: Since the waiting times are Exp(λ)-distributed and the exponential distribution is memoryless, the mean is λ^{-1}.
   Berta: The time between the arrival of two busses is Exp(λ)-distributed and has mean λ^{-1}. Since on average the same time elapses before our arrival and after our arrival, we obtain the mean waiting time $\frac{1}{2}\lambda^{-1}$ (at least assuming that at least one bus had arrived before time t_0).
   What is the correct answer to this *waiting time paradoxon*?

Submit before the first lecture on Thursday, 24 October 2013
Written Exam on Thursday, February 20th 2014, 11 a.m. to 1 p.m.

Markus Reiß

Course *Stochastic Processes*
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Exercises: sheet 2

1. Let $(P(t))_{t\geqslant 0}$ be the transition matrices of a continuous-time, time-homogeneous Markov chain with finite state space. Assume that the transition probabilities $p_{ij}(t)$ are differentiable for $t \geqslant 0$. Prove:
   - (a) The derivative satisfies $p'_{ij}(0) \geqslant 0$ for $i \neq j$, $p'_{ii}(0) \leqslant 0$ and $\sum_j p'_{ij}(0) = 0$.
   - (b) With the matrix (*generator*) $A = (p'_{ij}(0))_{i,j}$ we obtain the *forward* and *backward equation*:
   $$P'(t) = P(t)A, \quad P'(t) = AP(t), \quad t \geqslant 0.$$
   - (c) The generator A defines uniquely $P(t)$: $P(t) = e^{At} := \sum_{k\geqslant 0} A^k t^k / k!$.
   - (d*) Find conditions to extend these results to general countable state space.

2. Let $(X_n,\, n \geqslant 0)$ be a discrete-time, time-homogeneous Markov chain and let $(N_t,\, t \geqslant 0)$ be a Poisson process of intensity $\lambda > 0$, independent of X. Show that $Y_t := X_{N_t}$, $t \geqslant 0$, is a continuous-time, time-homogeneous Markov chain. Determine its transition probabilities and its generator.
   Remark: Under regularity conditions this gives all continuous-time, time-homogeneous Markov chains.

3. Let $C([0,\infty))$ be equipped with the topology of uniform convergence on compacts using the metric $d(f,g) := \sum_{k\geqslant 1} 2^{-k}(\sup_{t\in[0,k]} |f(t) - g(t)| \wedge 1)$. Prove:
   - (a) $(C([0,\infty)), d)$ is Polish.
   - (b) The Borel σ-algebra is the smallest σ-algebra such that all coordinate projections $\pi_t : C([0,\infty)) \to \mathbb{R}$, $t \geqslant 0$, are measurable.
   - (c) For any continuous stochastic process $(X_t,\, t \geqslant 0)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ the mapping $\bar{X} : \Omega \to C([0,\infty))$ with $\bar{X}(\omega)_t := X_t(\omega)$ is Borel-measurable.
   - (d) The law of $\bar{X}$ is uniquely determined by the finite-dimensional distributions of X.

4. Prove the regularity lemma: Let P be a probability measure on the Borel σ-algebra $\mathfrak{B}$ of any metric (or topological) space. Then
   $$\mathcal{D} := \Big\{ B \in \mathfrak{B} \,\Big|\, P(B) = \sup_{K \subseteq B \text{ compact}} P(K) = \inf_{O \supseteq B \text{ open}} P(O) \Big\}$$
   is closed under set differences and countable unions ($\mathcal{D}$ is a σ-*ring*).

Submit before the first lecture on 31 October 2013
Written Exam on Thursday, February 20th 2014, 11 a.m. to 1 p.m.

Markus Reiß

Course *Stochastic Processes*
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Exercises: sheet 3

1. A discrete-time *Markov process* with state space $(S, \mathscr{S})$ is specified by an initial distribution μ^0 on $(S, \mathscr{S})$ and a *transition kernel* $P : S \times \mathscr{S} \to [0, 1]$ (i.e. $B \mapsto P(x, B)$ is a probability measure for all $x \in S$ and $x \mapsto P(x, B)$ is measurable for all $B \in \mathscr{S}$). Show:

   (a) If we put iteratively $P^n(x, B) := \int_S P^{n-1}(y, B)\, P(x, dy)$ for $n \geqslant 2$ and $P^1 := P$, then each P^n is again a transition kernel.

   (b) Put for all $n \geqslant 1$, $A \in \mathscr{S}^{\otimes n}$

   $$Q_n(A) := \int_{S^n} \mathbf{1}_A(x_0, x_1, \ldots, x_{n-1}) \mu^0(dx_0) P(x_0, dx_1) \cdots P(x_{n-2}, dx_{n-1}).$$

   Then $(Q_n)_{n \geqslant 1}$ defines a projective family on $S^{\mathbb{N}}$.

   (c) Let $(S, \mathscr{S})$ be Polish. Then for each initial distribution μ_0 and each transition kernel P there exists a stochastic process $(X_n,\, n \geqslant 0)$ satisfying $\mathbb{P}^{X_0} = \mu_0$ and $\mathbb{P}^{(X_0, \ldots, X_{n-1})} = Q_n$, $n \geqslant 1$ (the Markov process).

2. A Gaussian process $(X_t,\, t \in T)$ is a process whose finite-dimensional distributions are (generalized) Gaussian, i.e. $(X_{t_1}, \ldots, X_{t_n}) \sim N(\mu_{t_1, \ldots, t_n}, \Sigma_{t_1, \ldots, t_n})$ with $\Sigma_{t_1, \ldots, t_n} \in \mathbb{R}^{n \times n}$ positive semi-definite.

   (a) Why are the finite-dimensional distributions of X uniquely determined by the expectation function $t \mapsto \mathbb{E}[X_t]$ and the covariance function $(s, t) \mapsto \operatorname{Cov}(X_s, X_t)$?

   (b) Prove that for an arbitrary function $\mu : T \to \mathbb{R}$ and any symmetric, positive (semi-)definite function $C : T^2 \to \mathbb{R}$, i.e. $C(t, s) = C(s, t)$ and

   $$\forall n \geqslant 1;\, t_1, \ldots, t_n \in T;\ \lambda_1, \ldots, \lambda_n \in \mathbb{R} : \ \sum_{i,j=1}^{n} C(t_i, t_j) \lambda_i \lambda_j \geqslant 0,$$

   there is a Gaussian process with expectation function μ and covariance function C.

3. Let (X, Y) be a two-dimensional random vector with Lebesgue density $f^{X,Y}$.

   (a) For $x \in \mathbb{R}$ with $f^X(x) > 0$ ($f^X(x) = \int f^{X,Y}(x, \eta)\, d\eta$) consider the *conditional density* $f^{Y \mid X=x}(y) := f^{X,Y}(x, y)/f^X(x)$. Which condition on $f^{X,Y}$ ensures for any Borel set B

   $$\lim_{h \downarrow 0} \mathbb{P}(Y \in B \mid X \in [x, x+h]) = \int_B f^{Y \mid X=x}(y)\, dy \quad ?$$

   (b) Show that for $Y \in L^2$ (without any condition on $f^{X,Y}$) the function

   $$\varphi_Y(x) := \begin{cases} \int y f^{Y \mid X=x}(y)\, dy, & f^X(x) > 0 \\ 0, & \text{otherwise} \end{cases}$$

   minimizes the L^2-distance $\mathbb{E}[(Y - \varphi(X))^2]$ over all measurable functions φ. We write $\mathbb{E}[Y \mid X = x] := \varphi_Y(x)$ and $\mathbb{E}[Y \mid X] := \varphi_Y(X)$.

   (c) Prove that φ_Y is $\mathbb{P}^X$-a.s. uniquely (among all $\varphi : \mathbb{R} \to \mathbb{R}$ measurable) characterized by solving

   $$\forall A \in \mathfrak{B}_{\mathbb{R}} : \mathbb{E}[\varphi(X)\mathbf{1}_A(X)] = \mathbb{E}[Y\mathbf{1}_A(X)].$$

4. In the situation of exercise 3 prove the following properties:

   (a) $\mathbb{E}[\mathbb{E}[Y \mid X]] = \mathbb{E}[Y]$;

   (b) if X and Y are independent, then $\mathbb{E}[Y \mid X] = \mathbb{E}[Y]$ holds a.s.;

   (c) if $Y \geqslant 0$ a.s., then $\mathbb{E}[Y \mid X] \geqslant 0$ a.s.;

   (d) for all $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ we have $\mathbb{E}[\alpha Y + \beta \mid X] = \alpha\, \mathbb{E}[Y \mid X] + \beta$ a.s.;

   (e) if $\varphi : \mathbb{R} \to \mathbb{R}$ is such that $(x, y) \mapsto (x, y\varphi(x))$ is a diffeomorphism and $Y\varphi(X) \in L^2$, then $\mathbb{E}[Y\varphi(X) \mid X] = \mathbb{E}[Y \mid X]\varphi(X)$ a.s.

Submit before the first lecture on 7 November 2013

Correctors' office hour: Thursday 12h30-13h15, RUD25, 1.2.14

Markus Reiß

Course *Stochastic Processes*
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Exercises: sheet 4

1. Let $\Omega = \bigcup_{n\in\mathbb{N}} B_n$, $B_m \cap B_n = \varnothing$ for $m \neq n$, be a measurable, countable partition for given $(\Omega, \mathcal{F}, \mathbb{P})$ and put $\mathcal{B} := \sigma(B_n,\, n \in \mathbb{N})$. Show:

   (a) Every $\mathcal{B}$-measurable random variable X can be written as $X = \sum_n \alpha_n \mathbf{1}_{B_n}$ with suitable $\alpha_n \in \mathbb{R}$. For $Y \in L^1$ we have $\mathbb{E}[Y \,|\, \mathcal{B}] = \sum_{n:\, \mathbb{P}(B_n)>0} \Big(\frac{1}{\mathbb{P}(B_n)} \int_{B_n} Y \, d\mathbb{P}\Big) \mathbf{1}_{B_n}$.

   (b) Specify $\Omega = [0,1)$ with Borel σ-algebra and $\mathbb{P} = U([0,1))$, the uniform distribution. For $Y(\omega) := \omega$, $\omega \in [0,1)$, determine $\mathbb{E}[Y \,|\, \sigma([(k-1)/n, k/n),\, k = 1, \dots, n)]$. For $n = 1, 3, 5, 10$ plot the conditional expectations and Y itself as functions on Ω.

2. Let (X, Y) be a two-dimensional $N(\mu, \Sigma)$-random vector.

   (a) For which $\alpha \in \mathbb{R}$ are X and $Y - \alpha X$ uncorrelated?

   (b) Conclude that X and $Y - (\alpha X + \beta)$ are independent for these values α and for arbitrary $\beta \in \mathbb{R}$ such that $\mathbb{E}[Y|X] = \alpha X + \beta$ with suitable $\beta \in \mathbb{R}$.

3. For $Y \in L^2$ define the *conditional variance* of Y given X by

$$\operatorname{Var}(Y|X) := \mathbb{E}[(Y - \mathbb{E}[Y \,|\, X])^2 \,|\, X].$$

   (a) Why is $\operatorname{Var}(Y|X)$ well defined?

   (b) Show $\operatorname{Var}(Y) = \operatorname{Var}(\mathbb{E}[Y \,|\, X]) + \mathbb{E}[\operatorname{Var}(Y|X)]$.

   (c) Use (b) to prove for independent random variables $(Z_k)_{k\geqslant 1}$ and N in L^2 with (Z_k) identically distributed and N $\mathbb{N}$-valued:

$$\operatorname{Var}\Big(\sum_{k=1}^{N} Z_k\Big) = \mathbb{E}[Z_1]^2 \operatorname{Var}(N) + \mathbb{E}[N] \operatorname{Var}(Z_1).$$

4. For a convex function $\varphi : \mathbb{R} \to \mathbb{R}$ (i.e. $\varphi(\alpha x + (1-\alpha)y) \leqslant \alpha\varphi(x) + (1-\alpha)\varphi(y))$ for $x, y \in \mathbb{R}$, $\alpha \in (0,1)$) show:

   (a) $D(x,y) := \frac{\varphi(y)-\varphi(x)}{y-x}$, $x \neq y$, is non-decreasing in x and y, which implies that φ is differentiable from the right and from the left and that φ is continuous.

   (b) Using the right-derivative φ'_+, we have:

$$\begin{aligned} \forall\, x, y \in \mathbb{R} : \qquad & \varphi(y) \geqslant \varphi(x) + \varphi'_+(x)(y-x), \\ \forall\, y \in \mathbb{R} : \qquad & \varphi(y) = \sup_{x \in \mathbb{Q}} \big(\varphi(x) + \varphi'_+(x)(y-x)\big). \end{aligned}$$

   (c) Assume $Y, \varphi(Y) \in L^1$. Then $\mathbb{E}[\varphi(Y) \,|\, \mathcal{G}] \geqslant \varphi(x) + \varphi'_+(x)(\mathbb{E}[Y \,|\, \mathcal{G}] - x)$ holds for all $x \in \mathbb{R}$. Infer Jensen's inequality: $\mathbb{E}[\varphi(Y) \,|\, \mathcal{G}] \geqslant \varphi(\mathbb{E}[Y \,|\, \mathcal{G}])$.

Submit before the first lecture on 21 November 2013

Markus Reiß

Course *Stochastic Processes*
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Exercises: sheet 5

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subseteq \mathcal{F}$ a σ-algebra. Recall the definition of *conditional probability*:
$$\mathbb{P}\left(A|\mathcal{G}\right) = \mathbb{E}\left[\mathbf{1_A}|\mathcal{G}\right] .$$
Prove that
   (a) For all $A \in \mathcal{F}, B \in \mathcal{G}$: $\mathbb{P}(A \cap B) = \int_B \mathbb{P}(A|\mathcal{G})\, d\,\mathbb{P}$.
   (b) For all $A \in \mathcal{F}$: $0 \leqslant \mathbb{P}(A|\mathcal{G}) \leqslant 1$ almost surely.
   (c) $\mathbb{P}(\emptyset|\mathcal{G}) = 0$ and $\mathbb{P}(\Omega|\mathcal{G}) = 1$ almost surely.
   (d) For disjoint sets $A_1, A_2, \ldots \in \mathcal{F}$, we have almost surely:
$$\mathbb{P}\Big(A_1 \cup A_2 \cup \ldots |\mathcal{G}\Big) = \mathbb{P}(A_1|\mathcal{G}) + \mathbb{P}(A_2|\mathcal{G}) + \ldots .$$
Why does this not necessarily mean that $A \mapsto \mathbb{P}(A|\mathcal{G})(\omega)$ is a probability measure for $\mathbb{P}$-almost all $\omega \in \Omega$?

2. *Doubling strategy*: In each round a fair coin is tossed, for *heads* the player receives his double stake, for *tails* he loses his stake. His initial capital is $K_0 = 0$. At game $n \geqslant 1$ his strategy is as follows: if *heads* has appeared before, his stake is zero (he stops playing); otherwise his stake is 2^{n-1} Euro.
   (a) Argue why his capital K_n after game n can be modeled with independent (X_i) such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$ via
$$K_n = \begin{cases} -(2^n - 1), & X_1 = \cdots = X_n = -1, \\ 1, & \text{otherwise.} \end{cases}$$
   (b) Represent K_n as martingale transform.
   (c) Prove $\lim_{n\to\infty} K_n = 1$ a.s. although $\mathbb{E}[K_n] = 0$ for all $n \geqslant 0$ holds.

3. Let T be an $\mathbb{N}_0$-valued random variable and $S_n := \mathbf{1}_{\{n \geqslant T\}}$, $n \geqslant 0$. Show:
   (a) The natural filtration satisfies $\mathcal{F}_n^S = \sigma(\{T = k\},\, k = 0, \ldots, n)$.
   (b) (S_n) is a submartingale with respect to $(\mathcal{F}_n^S)$ and
$$\mathbb{E}[S_{n+1} \,|\, \mathcal{F}_n^S] = \mathbf{1}_{\{S_n=1\}} + \mathbb{P}(T = n+1 \,|\, T \geqslant n+1)\mathbf{1}_{\{S_n=0\}} \ \mathbb{P}\text{-a.s.}$$
   (c) Determine the Doob decomposition of (S_n). Sketch for geometrically distributed T ($\mathbb{P}(T = k) = (1-p)p^k$) the sample paths of (S_n), its compensator and their difference.

Submit before the first lecture on 21 November 2013

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Exercises: sheet 6

1. Let $(\mathcal{F}_n^X)_{n\geqslant 0}$ be the natural filtration of a process $(X_n)_{n\geqslant 0}$ and consider a finite stopping time τ with respect to $(\mathcal{F}_n^X)$.

   (a) Prove $\mathcal{F}_\tau = \sigma(\tau,\, X_{\tau\wedge n},\, n\geqslant 0)$.
   Hint: for '$\subseteq$' write $A\in\mathcal{F}_\tau$ as $A=\bigcup_n A\cap\{\tau=n\}$.

   (b*) Do we even have $\mathcal{F}_\tau = \sigma(X_{\tau\wedge n},\, n\geqslant 0)$?

2. Let $(X_n)_{n\geqslant 0}$ be an $(\mathscr{F}_n)$-adapted family of random variables in L^1. Show that $(X_n)_{n\geqslant 0}$ is a martingale if and only if for all bounded $(\mathscr{F}_n)$-stopping times τ the identity $\mathbb{E}[X_\tau]=\mathbb{E}[X_0]$ holds.

3. Let $(S_n)_{n\geqslant 0}$ be a simple random walk with $\mathbb{P}(S_n - S_{n-1} = 1) = p$, $\mathbb{P}(S_n - S_{n-1} = -1) = q = 1-p$, $p\in(0,1)$. Prove:

   (a) Put $M(\lambda) = pe^\lambda + qe^{-\lambda}$, $\lambda\in\mathbb{R}$. Then the process
   $$Y_n^\lambda := \exp\Big(\lambda S_n - n\log(M(\lambda))\Big),\quad n\geqslant 0,$$
   is a martingale (w.r.t. its natural filtration).

   (b) For $a,b\in\mathbb{Z}$ with $a<0<b$ and the stopping time(!) $\tau := \inf\{n\geqslant 0\,|\,S_n\in\{a,b\}\}$ we have
   $$e^{a\lambda}\,\mathbb{E}[M(\lambda)^{-\tau}\mathbf{1}_{\{S_\tau=a\}}] + e^{b\lambda}\,\mathbb{E}[M(\lambda)^{-\tau}\mathbf{1}_{\{S_\tau=b\}}] = 1 \text{ if } M(\lambda)\geqslant 1.$$

   (c) This implies for all $s\in(0,1]$ (put $s=M(\lambda)^{-1}$)
   $$\mathbb{E}[s^\tau\mathbf{1}_{\{S_\tau=a\}}] = \frac{\lambda_+(s)^b-\lambda_-(s)^b}{\lambda_+(s)^b\lambda_-(s)^a-\lambda_+(s)^a\lambda_-(s)^b},$$
   $$\mathbb{E}[s^\tau\mathbf{1}_{\{S_\tau=b\}}] = \frac{\lambda_-(s)^a-\lambda_+(s)^a}{\lambda_+(s)^b\lambda_-(s)^a-\lambda_+(s)^a\lambda_-(s)^b}$$
   with $\lambda_\pm(s) = (1\pm\sqrt{1-4pqs^2})/(2ps)$.

   (d) Now let $a\downarrow -\infty$ and infer that the generating function of the first passage time $\tau_b := \inf\{n\geqslant 0\,|\,S_n=b\}$ is given by
   $$\varphi_{\tau_b}(s) := \mathbb{E}[s^{\tau_b}\mathbf{1}_{\{\tau_b<\infty\}}] = \Big(\frac{1-\sqrt{1-4pqs^2}}{2qs}\Big)^b,\quad s\in(0,1].$$
   In particular, we have $\mathbb{P}(\tau_b<\infty) = \varphi_{\tau_b}(1) = \min(1,p/q)^b$.

4. Your winnings per unit stake on game n are ε_n, where (ε_n) are independent random variables with $\mathbb{P}(\varepsilon_n = 1) = p$, $\mathbb{P}(\varepsilon_n = -1) = 1 - p$ for $p > 1/2$. Your stake X_n on game n must lie between zero and C_{n-1}, your capital at time $n-1$. For some $N \in \mathbb{N}$ and $C_0 > 0$ your objective is to maximize the expected interest rate $\mathbb{E}[\log(C_N/C_0)]$.
   Show that for any predictable strategy X the process $\log(C_n) - n\alpha$ is a supermartingale with respect to $\mathscr{F}_n := \sigma(\varepsilon_1, \ldots, \varepsilon_n)$ where
   $$\alpha := p \log p + (1-p)\log(1-p) + \log 2 \ (\textit{entropy}).$$
   Hence, $\mathbb{E}[\log(C_N/C_0)] \leqslant N\alpha$ always holds. Find an optimal strategy such that $\log(C_n) - n\alpha$ is even a martingale.
   Remark: This is the martingale approach to optimal control.

Submit before the first lecture on Tuesday, 28 November 2013

Markus Reiß

Course *Stochastic Processes*
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Exercises: sheet 7

1. Prove that a family $(X_i)_{i\in I}$ of random variables is uniformly integrable if and only if $\sup_{i\in I}\|X_i\|_{L^1} < \infty$ holds as well as

$$\forall \varepsilon > 0\, \exists \delta > 0 : \mathbb{P}(A) < \delta \Rightarrow \sup_{i\in I} \mathbb{E}[|X_i|\mathbf{1}_A] < \varepsilon.$$

2. Show for an L^p-bounded martingale (M_n) (i.e. $\sup_n \mathbb{E}[|M_n|^p] < \infty$) with $p \in (1,\infty)$:

   (a) (M_n) converges a.s. and in L^1 to some $M_\infty \in L^1$.

   (b) Use $|M_\infty| \leqslant \sup_{n\geqslant 0}|M_n|$ (why?) and Doob's inequality to infer $M_\infty \in L^p$.

   (c) Prove with dominated convergence that (M_n) converges to M_∞ in L^p.

3. Give a martingale proof of Kolmogorov's 0-1 law:

   (a) Let $(\mathscr{F}_n)$ be a filtration and $\mathscr{F}_\infty = \sigma(\mathscr{F}_n, n \geqslant 0)$. Then for $A \in \mathscr{F}_\infty$ we have $\lim_{n\to\infty} \mathbb{E}[\mathbf{1}_A \,|\, \mathscr{F}_n] = \mathbf{1}_A$ a.s.

   (b) For a sequence $(X_k)_{k\geqslant 1}$ of independent random variables consider the natural filtration $(\mathscr{F}_n)$ and the terminal σ-algebra $\mathscr{T} := \bigcap_{n\geqslant 1}\sigma(X_k,\, k \geqslant n)$. Then for $A \in \mathscr{T}$ we deduce $\mathbb{P}(A) = \mathbb{E}[1_A \,|\, \mathscr{F}_n] \to \mathbf{1}_A$ a.s. such that $P(A) \in \{0,1\}$ holds.

4. A monkey types at random the 26 capital letters of the Latin alphabet. Let τ be the first time by which the monkey has completed the sequence ABRACADABRA. Prove that τ is almost surely finite and satisfies

$$\mathbb{E}[\tau] = 26^{11} + 26^4 + 26.$$

   How much time does it take on average if one letter is typed every second?
   Hint: You may look at a fair game with gamblers G_n arriving before times $n = 1, 2, \ldots$ G_n bets 1 Euro on 'A' for letter n; if he wins, he puts 26 Euro on 'B' for letter $n+1$, otherwise he stops. If he wins again, he puts 26^2 Euro on 'R', otherwise he stops etc.

Submit before the first lecture on Thursday, 5 December 2009

Markus Reiß

Course *Stochastic Processes*
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Exercises: sheet 8

1. Suppose $\mathbb{P}_0$, $\mathbb{P}_1$, $\mathbb{P}_2$ are probability measures on $(\Omega, \mathscr{F})$. Show:
   (a) If $\mathbb{P}_2 \ll \mathbb{P}_1 \ll \mathbb{P}_0$ holds, then $\frac{d\,\mathbb{P}_2}{d\,\mathbb{P}_0} = \frac{d\,\mathbb{P}_2}{d\,\mathbb{P}_1}\frac{d\,\mathbb{P}_1}{d\,\mathbb{P}_0}$ holds $\mathbb{P}_0$-a.s.
   (b) $\mathbb{P}_0$ and $\mathbb{P}_1$ are *equivalent* if and only if $\mathbb{P}_1 \ll \mathbb{P}_0$ and $\frac{d\,\mathbb{P}_1}{d\,\mathbb{P}_0} > 0$ holds $\mathbb{P}_0$-a.s. In that case we have $\frac{d\,\mathbb{P}_0}{d\,\mathbb{P}_1} = \big(\frac{d\,\mathbb{P}_1}{d\,\mathbb{P}_0}\big)^{-1}$ $\mathbb{P}_0$-a.s. and $\mathbb{P}_1$-a.s.

2. Prove in detail for $\mathbb{Q} \ll \mathbb{P}$, $Z = \frac{d\,\mathbb{Q}}{d\,\mathbb{P}}$ and $Y \in L^1(\mathbb{Q})$ the identity $\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[YZ]$. Give an example where $Y \in L^1(\mathbb{Q})$, but not $Y \in L^1(\mathbb{P})$ holds.

3. Let $(X_k)_{k\geqslant 1}$ be a sequence of i.i.d. $\{-1,+1\}$-valued random variables. Under the null hypothesis H_0 we have $\mathbb{P}_0(X_k = +1) = p_0$ with $p_0 \in (0,1)$, while under the alternative H_1 we have $\mathbb{P}_1(X_k = +1) = p_1$ with $p_1 \neq p_0$.
   (a) Explain why the likelihood quotient $\frac{d(\otimes_{i=1}^n \mathbb{P}_1^{X_i})}{d(\otimes_{i=1}^n \mathbb{P}_0^{X_i})}$ after n observations $X_1, \dots, X_n$ is given by
   $$L_n = \frac{p_1^{(n+S_n)/2}(1-p_1)^{(n-S_n)/2}}{p_0^{(n+S_n)/2}(1-p_0)^{(n-S_n)/2}} \text{ with } S_n = \textstyle\sum_{k=1}^n X_k.$$
   (b) Show that the *likelihood process* $(L_n)_{n\geqslant 0}$ (put $L_0 := 1$) forms a non-negative martingale under the hypothesis H_0 (i.e. under $\mathbb{P}_0$) w.r.t. its natural filtration.
   (c) A *sequential likelihood-quotient test*, based on $0 < A < B$ and the stopping time
   $$\tau_{A,B} := \inf\{n \geqslant 1 \,|\, L_n \geqslant B \text{ or } L_n \leqslant A\},$$
   rejects H_0 if $L_{\tau_{A,B}} \geqslant B$, and accepts H_0 if $L_{\tau_{A,B}} \leqslant A$. Determine the probability for errors of the first and second kind (i.e., $\mathbb{P}_0(L_{\tau_{A,B}} \geqslant B)$ and $\mathbb{P}_1(L_{\tau_{A,B}} \leqslant A)$) in the case $p_0 = 0.4$, $p_1 = 0.6$, $A = (2/3)^5$, $B = (3/2)^5$. Calculate $\mathbb{E}[\tau_{A,B}]$.

4. Let $Z_n(x) = (3/2)^n \sum_{k\in\{0,2\}^n} \mathbf{1}_{I(k,n)}(x)$, $x \in [0,1]$, with intervals $I(k,n) := [\sum_{i=1}^n k_i 3^{-i}, \sum_{i=1}^n k_i 3^{-i} + 3^{-n}]$. Show:

   (a) $(Z_n)_{n\geqslant 0}$ with $Z_0 = 1$ forms a martingale on $([0,1], \mathfrak{B}_{[0,1]}, \lambda, (\mathscr{F}_n))$ with Lebesgue measure λ on $[0,1]$ and $\mathscr{F}_n := \sigma(I(k,n),\ k \in \{0,1,2\}^n)$.

   (b) (Z_n) converges λ-a.s., but not in $L^1([0,1], \mathfrak{B}_{[0,1]}, \lambda)$ (Sketch!).

   (c) Interpret Z_n as the density of a probability measure $\mathbb{P}_n$ with respect to λ. Then $(\mathbb{P}_n)$ converges weakly to some probability measure $\mathbb{P}_\infty$ ($\mathbb{P}_\infty$ is called *Cantor measure*). There is a Borel set $C \subseteq [0,1]$ with $\mathbb{P}_\infty(C) = 1$, $\lambda(C) = 0$.
   Hint: Show that the distribution functions converge to a limit distribution function, which is λ-a.e. constant.

Submit before the first lecture on Thursday, 12 December 2013

Markus Reiß

Course *Stochastic Processes*
Winter 2013/14
Humboldt-Universität zu Berlin

Exercises: sheet 9

1. Let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain with arbitrary initial distribution μ and transition matrix

$$P = \begin{pmatrix} 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 & 0 & 0 \end{pmatrix}.$$

Draw a graph representing the transition probabilities with states as vertices and edges for transitions to clarify the structure of the Markov chain. Find all recurrent, transient, closed and irreducible sets of states.

2. Let $S = \mathbb{N}_0$ and define the transition probabilities

$$\begin{aligned} i \geq 1: &\quad p_{i,i+1} = p_i > 0, \quad p_{i,i-1} = q_i > 0, \quad p_{i,i} = s_i, \\ i = 0: &\quad p_{0,1} = p_0 > 0, \quad p_{0,0} = s_0 = 1 - p_0 \end{aligned}$$

for some birth and death probabilities $p_i, q_i > 0$, as well as $s_i = 1 - p_i - q_i$ for $i \geq 1$. This defines a so called *birth and death process* X with transition probabilities $p_{i,j}$ as given above and arbitrary initial distribution μ. Our goal is to prove that the process X is recurrent if and only if $\sum_{l=1}^{\infty} \prod_{k=1}^{l-1} \frac{q_k}{p_k} = \infty$.

(a) Observe (i.e. prove) that X is irreducible.

(b) Show that irreducible Markov chains on discrete state spaces are either transient or recurrent.

(c) Conclude that it is enough to show that 0 is recurrent if and only if $\sum_{l=1}^{\infty} \prod_{k=1}^{l-1} \frac{q_k}{p_k} = \infty$. Moreover, it is enough to prove that $r_{1,0} = P_1(T_0 < \infty) = 1$ with $T_y = \inf\{n > 0 : X_n = y\}$.

(d) Find a function $\varphi : \mathbb{N}_0 \to \mathbb{R}$ such that $(\varphi(X_{T_0 \wedge n}))_{n \geq 0}$ is a martingale with respect to the natural filtration of X under P_i for $i > 0$.

(e) Let $0 < i < b$ be integers and observe that $(\varphi(X_{T_0 \wedge T_b \wedge n}))_{n \geq 0}$ is a bounded martingale under P_i for $i > 0$.

(f) Conclude by the stopping theorem and the martingale convergence theorem that $T_0 \wedge T_b < \infty$ P_i-almost surely for $i > 0$ and that $\mathbb{E}_i[\varphi(X_{T_0 \wedge T_b})] = \varphi(i)$.

(g) Obtain the claim that X is recurrent if and only if $\sum_{l=1}^{\infty}\prod_{k=1}^{l-1}\frac{q_k}{p_k}=\infty$.

Discuss different choices of $(p_i)_{i\geq 0}, (q_i)_{i\geq 0}$ for which the condition $\sum_{l=1}^{\infty}\prod_{k=1}^{l-1}\frac{q_k}{p_k}=\infty$ is satisfied.

3. ($*$) Prove that if the two families $(X_i)_{i\in I}$, $(Y_j)_{j\in J}$ of random variables are uniformly integrable, then also the family $\{X_i+Y_j:\ i\in I, j\in J\}$ is uniformly integrable.

   Hint: Review problem 1 of exercise sheet 7.

4. ($*$) Let $\left\{X_k^{(n)}: n,k\geq 1\right\}$ be an iid family of $\mathbb{Z}^+$-valued random variables. We define a *branching process* $(Z_n)_{n\geq 0}$ by $Z_0=1$ and

$$Z_{n+1}=X_1^{(n+1)}+\cdots+X_{Z_n}^{(n+1)}\quad (n\geq 0)\,,$$

   recursively. Assume that if X denotes any of $X_k^{(n)}$, then $\mu:=\mathbb{E}\left[X\right]<\infty$ and $0<\sigma^2:=Var\left(X\right)<\infty$.

   (a) Read about branching processes and, in particular, about Galton-Watson processes on Wikipedia and describe briefly how they are used in applications.

   (b) If possible, read (or at least skim through) Chapter 0 of David Williams' book *Probability with martingales* in order to understand why they are interesting as mathematical objects.

   (c) Prove that $M_n:=Z_n/\mu^n$ defines a martingale M relative to the filtration $\mathscr{F}_n=\sigma\left(Z_0,Z_1,\ldots,Z_n\right)$. Show that

$$\mathbb{E}\left[\left.Z_{n+1}^2\right|\mathscr{F}_n\right]=\mu^2 Z_n^2+\sigma^2 Z_n$$

   and deduce that M is bounded in L^2 if and only if $\mu>1$. Show that when $\mu>1$,

$$Var\left(M_\infty\right)=\sigma^2\left(\mu\left(\mu-1\right)\right)^{-1},$$

   where M_∞ is the almost-sure limit of M_n.

Exercises *without* ($*$) are regular exercises. Submit them before the first lecture on Thursday, 19 December 2013.
Exercises *with* ($*$) are extra. You can use them to get additional homework points. Submit those before the first lecture on Thursday, 9 January 2014.

Markus Reiß

Course *Stochastic Processes*
Winter 2013/14
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Exercises: sheet 10

1. Prove von Neumann's ergodic theorem: For measure-preserving T and $X \in L^p$, $p \geqslant 1$, we have that $A_n := \frac{1}{n}\sum_{i=0}^{n-1} X \circ T^i$ converges to $\mathbb{E}[X \mid \mathscr{I}_T]$ in L^p.
Hint: Show that $|A_n|^p$ is uniformly integrable.

2. Show that a measure-preserving map T on $(\Omega, \mathscr{F}, \mathbb{P})$ is ergodic if and only if for all $A, B \in \mathscr{F}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}(A \cap T^{-k}B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

Hint: For one direction apply an ergodic theorem to $\mathbf{1}_B$.
(*) Extension: If even $\lim_{n\to\infty} \mathbb{P}(A \cap T^{-n}B) = \mathbb{P}(A)\,\mathbb{P}(B)$ holds, then T is called *mixing*. Show that T mixing implies T ergodic, but not conversely (e.g., consider rotation by an irrational angle).

3. *Gelfand's Problem:* Does the decimal representation of 2^n ever start with the initial digit 7? Study this as follows:
   (a) Determine the relative frequencies of the initial digits of $(2^n)_{1\leqslant n\leqslant 30}$.
   (b) Let $A \sim U([0,1])$. Prove that the initial digit k in $(10^A 2^n)_{1\leqslant n\leqslant m}$ converges as $m \to \infty$ a.s. to $\log_{10}(k+1) - \log_{10}(k)$ (consider $X_n = A + n\log_{10}(2) \mod 1$!).
   (c) Prove that the convergence in (b) even holds everywhere. In particular, the relative frequency of the initial digit 7 in the powers of 2 converges to $\log_{10}(8/7) \approx 0,058$.
   Hint: Show for trigonometric polynomials $p(a) = \sum_{|m|\leqslant M} c_m e^{2\pi i m a}$ that $\frac{1}{n}\sum_{k=0}^{n-1} p(a+k\eta) \to \int_0^1 p(x)dx$ holds for all $\eta \in \mathbb{R}\setminus\mathbb{Q}$, $a \in [0,1]$ (calculate explicitly for monomials!) and approximate.

4. Consider the Ehrenfest model, i.e. a Markov chain on $S = \{0, 1, \ldots, N\}$ with transition probabilities $p_{i,i+1} = (N-i)/N$, $p_{i,i-1} = i/N$.
   (a) Show that $\mu(\{i\}) = \binom{N}{i} 2^{-N}$, $i \in S$, is an invariant initial distribution.
   (b) Is the Markov chain starting in μ ergodic?
   (*c) Simulate the Ehrenfest model with initial value $i_0 \in \{N/2; N\}$, $N = 100$ for $T \in \{100; 100{,}000\}$ time steps. Plot the relative frequencies of visits to each state in S and compare with μ. Explain what you see!

Submit before the first lecture on Thursday, 16 January 2010

Markus Reiß

Course *Stochastic Processes*
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Exercises: sheet 11

1. Let $(X_n)_{n\geqslant 1}$ be a real stationary process and define $S_n = \sum_{k=1}^n X_k$. The range of S is defined as $R_n := \#\{S_1, \dots, S_n\}$. Then with the invariant σ-algebra $\mathcal{I}$ and $A := \{\forall n \geqslant 1 : S_n \neq 0\}$ prove $\lim_{n\to\infty} \frac{1}{n} R_n = \mathbb{P}(A \,|\, \mathcal{I})$ a.s.

2. For probability measures $\mathbb{P}$ and $\mathbb{Q}$ on a measurable space $(\Omega, \mathcal{F})$ their total variation distance is given by
$$\|\mathbb{P} - \mathbb{Q}\|_{TV} = \sup_{A\in\mathcal{F}} |\mathbb{P}(A) - \mathbb{Q}(A)|.$$
Decide whether for $n \to \infty$ the probabilities $\mathbb{P}_n$ with the following Lebesgue densities f_n on $\mathbb{R}$ converge in total variation distance, weakly or not at all:
$$f_n(x) = ne^{-nx}\mathbf{1}_{[0,\infty)}(x), \quad f_n(x) = \tfrac{n+1}{n} x^{1/n}\mathbf{1}_{[0,1]}(x), \quad f_n(x) = \tfrac{1}{n}\mathbf{1}_{[0,n]}(x).$$

3. We say that a family of real-valued random variables $(X_i)_{i\in I}$ is *stochastically bounded*, notation $X_i = O_{\mathbb{P}}(1)$, if
$$\lim_{R\to\infty} \sup_{i\in I} \mathbb{P}(|X_i| > R) = 0.$$
   (a) Show $X_i = O_{\mathbb{P}}(1)$ if and only if the laws $(\mathbb{P}^{X_i})_{i\in I}$ are uniformly tight.
   (b) Prove that any L^p-bounded family of random variables is stochastically bounded, hence has uniformly tight laws.
   (c) If $X_n \xrightarrow{\mathbb{P}} 0$ holds, then we write $X_n = o_{\mathbb{P}}(1)$. Check the symbolic rules $O_{\mathbb{P}}(1) + o_{\mathbb{P}}(1) = O_{\mathbb{P}}(1)$ and $O_{\mathbb{P}}(1) o_{\mathbb{P}}(1) = o_{\mathbb{P}}(1)$.

4. Prove: Every relatively (weakly) compact family $(\mathbb{P}_i)_{i\in I}$ of probability measures on a Polish space $(S, \mathfrak{B}_S)$ is uniformly tight. Proceed as follows (cf. proof of Ulam's Theorem):
   (a) For $k \geqslant 1$ consider open balls $(A_{k,m})_{m\geqslant 1}$ of radius $1/k$ that cover S. If $\limsup_{M\to\infty} \inf_i \mathbb{P}_i(\bigcup_{m=1}^M A_{k,m}) < 1$ were true, then by assumption and by the Portmanteau Theorem we would have $\limsup_{M\to\infty} \mathbb{Q}(\bigcup_{m=1}^M A_{k,m}) < 1$ for some limiting probability measure $\mathbb{Q}$, which is contradictory.
   (b) Conclude that for any $\varepsilon > 0$, $k \geqslant 1$ there are indices $M_{k,\varepsilon} \geqslant 1$ such that $\inf_i \mathbb{P}_i(K) > 1 - \varepsilon$ holds with $K := \bigcap_{k\geqslant 1} \bigcup_{m=1}^{M_{k,\varepsilon}} A_{k,m}$. Moreover, K is relatively compact in S, which suffices.

Submit before the first lecture on Thursday, 23 January 2014

Markus Reiß

Course *Stochastic Processes*
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Exercises: sheet 12

1. Show for a sequence $(\mathbb{P}_n)$ of probability measures on $C([0,T])$:

$$\forall \varepsilon > 0: \lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{t\in[0,T-\delta]} \delta^{-1}\,\mathbb{P}_n\big(\max_{s\in[t,t+\delta]} |f(s)-f(t)| \geqslant \varepsilon\big) = 0$$

   implies for the modulus of continuity $\omega_\delta(f)$

$$\forall \varepsilon > 0: \lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}_n(\omega_\delta(f) \geqslant \varepsilon) = 0.$$

2. Let the random vectors $X_n \in \mathbb{R}^{d_1}$ be independent of the random vectors $Y_n \in \mathbb{R}^{d_2}$ for all $n \geqslant 1$ and $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{d} Y$. Deduce that $(X_n, Y_n) \xrightarrow{d} \mathbb{P}^X \otimes \mathbb{P}^Y$ on $\mathbb{R}^{d_1+d_2}$, the law of (X,Y) for independent X and Y.
   Hint: Check that $(X_n, Y_n)_{n\geqslant 1}$ has tight laws and identify the limiting laws on cartesian products.

3. Let $(S, \mathscr{S})$ be a measurable space, T an uncountable set.
   (a) Show that for each $B \in \mathscr{S}^{\otimes T}$ there is a countable set $I \subseteq T$ such that

$$\forall x \in S^T,\ y \in B:\ (x(t) = y(t) \text{ for all } t \in I) \Rightarrow x \in B.$$

   Hint: Check first that sets B with this property form a σ-algebra.
   (b) Conclude for a metric space S with at least two elements that the set $C := \{f : [0,1] \to S \,|\, f \text{ continuous}\}$ is not product-measurable, i.e. $C \notin \mathscr{S}^{\otimes [0,1]}$.

4. The *Brownian bridge* $(X_t, t \in [0,1])$ is a centered and continuous Gaussian process with $\operatorname{Cov}(X_s, X_t) = s(1-t)$ for $0 \leqslant s \leqslant t \leqslant 1$. Show that it has the same law on $C([0,1])$ as $(B_t - tB_1,\ t \in [0,1])$, B a Brownian motion.
   Optional: Simulate 100 trajectories of a Brownian bridge. Use conditional densities to show that X is the process obtained from $(B_t,\ t \in [0,1])$ conditioned on $\{B_1 = 0\}$.

Submit before the first lecture on Thursday, 23 January 2014

Markus Reiß

Course *Stochastic Processes*
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Exercises: sheet 13

1. Use Kolmogorov's criterion to give a simple tightness proof for Donsker's invariance principle in the case of $S_n = \sum_{k=1}^n X_k$ with $(X_k)_{k\geqslant 1}$ i.i.d., $X_k \in L^4$, $\mathbb{E}[X_k] = 0$, $\mathrm{Var}(X_k) = 1$.

2. Let $(B_t,\, t \geqslant 0)$ be a Brownian motion. Show that for $a > 0$ also $X_t = a^{-1/2} B_{at}$, $t \geqslant 0$, (*scaling*) and $Y_0 = 0$, $Y_t = tB_{1/t}$, $t > 0$, (*time reversal*) are Brownian motions.

Submit before the first lecture on Thursday, 6 February 2014