UNITARY PERIODS, SPHERICAL FUNCTIONS AND JACQUET'S RELATIVE TRACE FORMULA

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Abstract. We present the main result of [LO], which is a formula for the anisotropic unitary periods of certain cusp forms in terms of special values of Rankin-Selberg L-functions. The purpose of these notes is to survey some of the recent developments that led to this formula. A new observation that we make here is that we can use the fundamental lemma of Jacquet ([Jac05]) in order to determine explicitly the correct transfer factors for the relevant orbital integrals.

Our goal in these notes is to survey some new and exciting developments in the study of periods of automorphic forms. The main result of [LO] is an application of these new developments. We will present the main formula obtained in [LO] and then explain each of the main ingredients necessary for the proof, but first let us recall in a few words our general setting for the study of period integrals. For a reductive group G defined over a number field F let θ be an involution on G defined over F and let $X = \{g \in G : g\,\theta(g) = e\}$. Then G acts on X by $(g, x) \mapsto g\,x\,\theta(g)^{-1}$. For every $x \in X_F$ let H^x be the stabilizer of x in G. Let $\mathbb{A} = \mathbb{A}_F$ denote the ring of adèles of F. A cuspidal automorphic representation π of $G_{\mathbb{A}}$ is *distinguished* by H^x if there exists a cusp form ϕ in the space of π so that

$$\int_{H^x_F \backslash H^x_{\mathbb{A}}} \phi(h)\ dh \neq 0.$$

It is expected that there is a group G' related to (G, θ) and a functorial transfer (in the sense of Langlands functoriality) of automorphic forms from G' to G, so that distinction by some H^x characterizes the functorial image. Furthermore, in many cases, for distinguished representations, the value of the period integral is expected to be related to special values of L-functions.

In this work we consider one particular case, where G is the group GL_n over a quadratic extension and the involution is defined by $\theta(g) = {}^t\bar{g}^{-1}$ where $x \mapsto \bar{x}$ is the Galois action. Thus X is the space of Hermitian matrices in G. To aline ourselves with Jacquet's notation in [Jac05], we shall use throughout the paper, the right action $(x, g) \mapsto {}^t\bar{g}xg$ of G on X. When recalling results from papers that use the left action (and sometimes a conjugate of θ rather then θ) we will adjust the results accordingly. For $x \in X_F$, the stabilizer $H^x = \{g \in G : {}^t\bar{g}xg = x\}$ is then a unitary group. The group G' is GL_n over the base field and the relevant functorial transfer from G' to G is quadratic base change. Jacquet characterized the functorial image of base change in terms of non vanishing of unitary periods.

Date: April 3, 2007.

Theorem 1 ([Jac05], Theorem 4)**.** *Let E/F be a quadratic extension of number fields. A cuspidal automorphic representation π of $GL_n(\mathbb{A}_E)$ is a base change from $GL_n(\mathbb{A}_F)$ if and only if it is distinguished by a unitary group.*

The formula obtained in [LO] relates anisotropic unitary periods of certain (distinguished) cusp forms to special values of Rankin-Selberg L-functions. The setting is the following. Let F be a totally real number field of degree d and let E be a totally imaginary quadratic extension of F. Let $G' = GL_n/F$ and let G be the restriction of scalars of GL_n from E to F. Thus $G'_F = GL_n(F)$ and $G_F = GL_n(E)$. Let $\alpha = {}^t\bar{\alpha} \in G_F$ be a Hermitian matrix which is either positive or negative definite in any real embedding of F. Consider the anisotropic unitary group

$$\mathbf{H} = \mathbf{H}^\alpha = \{g \ \in \mathbf{G} : g\alpha\, {}^t\bar{g} = \alpha\}.$$

Let $\omega = \omega_{E/F}$ be the idèle class character attached to E/F by class field theory and let $\theta = (\theta_v) \in G_{\mathbb{A}}$ be such that ${}^t\bar{\theta}_v\theta_v = \pm\alpha_v$ for every real place v of F and $\theta_v = e$ for every finite place v of F. Let π be an irreducible, everywhere unramified cuspidal representation of $G_{\mathbb{A}}$. Thus, it admits a $\mathbf{K}$-invariant, L^2-normalized automorphic form ϕ_0, where $\mathbf{K} = \prod_v K_v$ is the standard maximal compact subgroup of $G_{\mathbb{A}}$. Assume further that π is the base change from a cuspidal representation π' of $G'_{\mathbb{A}}$. The main result of [LO] is

Theorem 2 (Lapid-Offen)**.** *Under the above assumptions we have*

$$\left|\int_{H^\alpha\backslash H^\alpha_{\mathbb{A}}} \phi_0(h\theta)\ dh\right|^2 \tag{1}$$
$$= 4\cdot 2^{-2nd}\cdot \operatorname{vol}(H^e_{\mathbb{A}}\cap \mathbf{K})^2 \cdot \left|\frac{\Delta_E}{\Delta_F}\right|^{\frac{(n-1)n}{2}} \cdot \mathcal{P}_\alpha(\pi')\cdot \frac{L(1,\pi'\times\tilde{\pi}'\otimes\omega)}{\operatorname{Res}_{s=1} L(s,\pi'\times\tilde{\pi}')}.$$

Here Δ_F (resp. Δ_E) is the discriminant of F (resp. E). The Haar measure on $H^\alpha_{\mathbb{A}}$ is the pull-back of the one on $H^e_{\mathbb{A}}$ (via an inner twist). For the normalization of measure on $G_{\mathbb{A}}$ see... The term $\mathcal{P}_\alpha(\pi') = \prod_v \mathcal{P}_{\alpha_v}(\pi'_v)$ is a product over the places of F of local factors. At the archimedean places, thanks to the translate by θ the terms are independent of α and are given by

$$\mathcal{P}_{\alpha_v}(\pi'_v) = \frac{L(1,\pi'_v\times\tilde{\pi}'_v)}{L(0,\pi'_v\times\tilde{\pi}'_v\otimes\omega_v)}$$

(in particular $\mathcal{P}_{\alpha_v}(\pi'_v) = 1$ if π'_v is unramified). At the non archimedean places, the term $\mathcal{P}_{\alpha_v}(\pi'_v)$ is expressed explicitly in terms of the value at α_v of Hironaka's spherical functions if the place v is inert and in terms of Macdonald's spherical functions for GL_n if v is split. Hironaka's work is discussed in ...(maybe we should give here the general formula once Hironaka's chapter is writen). In [Hir99], Hironaka computes the spherical functions explicitly in the case of an unramified quadratic extension. In particular, $\mathcal{P}_{\alpha_v}(\pi'_v) = 1$ whenever $\alpha_v \in K_v$ and v is either split or unramified in E (hence the local factors of $\mathcal{P}_\alpha(\pi')$ are 1 at almost all places). We refer to ... for explicit formulas for Hironaka's spherical functions in all cases they are available. For convenience we summarize here the cases where $\mathcal{P}_\alpha(\pi)$ can be given explicit.

Remark 1. If E/F is unramified at all finite places (thus, in particular $d > 1$) then (1) is an explicit formula. Note further that at an archimedean place v if π'_v is unramified then $\mathcal{P}_{\alpha_v}(\pi'_v)=1$. Thus, for example assuming that E/F is unramified at all finite places, that π' is unramified at all archimidean places (and hence at all places) and that $\alpha=e$ we have

$$\left|\int_{H^e\backslash H^e_{\mathbb{A}}}\phi_0(h)\,dh\right|^2=4\cdot 2^{-2nd}\cdot\mathrm{vol}(H^e_{\mathbb{A}}\cap\mathbf{K})^2\cdot|\Delta_F|^{\frac{(n-1)n}{2}}\cdot\frac{L(1,\pi'\times\tilde{\pi}'\otimes\omega)}{\mathrm{Res}_{s=1}L(s,\pi'\times\tilde{\pi}')}.$$

Remark 2. In the case $n=2$, Hironaka computed the spherical functions for a ramified quadratic extension in the case of an odd resiual characteristic or when the base field is $\mathbb{Q}_2$. Thus, for a general CM-extension E/F under the weaker assumption that for every even place v of F either v is unramified at E or $F_v\simeq\mathbb{Q}_2$ the formula (1) is explicit for $n=2$. In particular when $F=\mathbb{Q}$, $E=\mathbb{Q}[\sqrt{-D}]$ for an integer $D>0$ and $n=2$ we have...(give the explicit formula).

Remark 3. In the non archimedean case, over an unramified quadratic extension, all Hermitian forms are integrally equivalent to diagonal forms. In the case of a ramified quadratic extension this is not the case, but every Hermitian form is integrally equivalent to a form which has diagonal blocks of size either 1 or 2. Using the partial results of Hironaka in [Hir88b], we compute in... the value of her spherical functions on Hermitian forms which are integrally equivalent to diagonal forms. Thus, in particular, if α is a diagonal matrix then again, (1) can be made explicit for a general CM-extension that is unramified at the even places.

Remark 4. The anisotropic unitary period of ϕ_0 has a more arithmetic interpretation as a certain weighted finite sum of point evaluations. If for example, $F=\mathbb{Q}$ and E is of class number one, then the sum is over classes in the genus class of α. This aspect and an interesting relation with a conjecture of Sarnak is explained in [LO] and we do not pursue it here any further (see also [CO] for a relation with certain representation numbers).

An important tool in the study of period integrals of automorphic forms is the *relative trace formula* of Jacquet (RTF). For the case at hand, this is a distribution on the space of Hermitian matrices. In order to obtain information about unitary periods, the RTF is compared with the so called *Kuznietzov trace formula*, which is a distribution on G'. Let U' (resp. U) be the subgroup of upper triangular unipotent matrices in G' (resp. G). Let ψ be a non trivial additive character on $F\backslash\mathbb{A}$. Denote by $\psi_{U'}$ the associated character on $U'_{\mathbb{A}}$ defined by

$$\psi_{U'}(u)=\psi(u_{1,2}+\cdots+u_{n-1,n})$$

and similarly denote by ψ_U the associated character on $U_{\mathbb{A}}$ defined by

$$\psi_U(u)=\psi(u_{1,2}+\bar{u}_{1,2}+\cdots+u_{n-1,n}+\bar{u}_{n-1,n}).$$

Let $X=\{g\in G:{}^t\bar{g}=g\}$ be the space of Hermitian matrices. The comparison of distributions (between the RTF and the KTF) amounts to an identity of the form

$$(2)\quad \int_{U_F\backslash U_{\mathbb{A}}}\Big(\sum_{x\in X_F}\Psi({}^t\bar{u}xu)\Big)\,\psi_U(u)\,du$$
$$=\int_{(U'_F\times U'_F)\backslash(U'_{\mathbb{A}}\times U'_{\mathbb{A}})}\Big(\sum_{g\in G'_F}\Phi({}^tu_1gu_2)\Big)\,\psi_{U'}(u_1u_2)\,du_1\,du_2$$

for suitably matching functions $\Psi\in C_c^\infty(X_{\mathbb{A}})$ and $\Phi\in C_c^\infty(G'_{\mathbb{A}})$.

The group U acts on X by $(x, u) \mapsto {}^t\bar{u}xu$. We call an element $x \in X_F$ *relevant* if ψ_U is trivial on the stabilizer $\mathrm{Stab}_{U_{\mathbb{A}}}(x)$ of x in $U_{\mathbb{A}}$. Similarly, the group $U' \times U'$ acts on G' by $(g, u_1, u_2) \mapsto {}^tu_1gu_2$ and $g \in G'_F$ is called relevant if $\psi_{U'}$ is trivial on $\mathrm{Stab}_{U'_{\mathbb{A}} \times U'_A}(g)$. Only relevant orbits contribute to the integrals in (2). The comparison in (2) is based on a natural bijection between the relevant orbits in X_F and in G'_F. Indeed, a complete common set of representatives for the relevant orbits consists of elements of the form $w_{M'}a$ where $w_{M'}$ is the longest Weyl element of a standard parabolic subgroup M' of G' and a lies in the center $A_{M'}$ of M'_F. Thus both sides admit a geometric expansion and (2) becomes

$$\begin{aligned}(3)\quad & \sum_{M'} v_{M'} \sum_{a \in A_{M'}} \int_{\mathrm{Stab}_{U_{\mathbb{A}}}(w_{M'}) \backslash U_{\mathbb{A}}} \Psi({}^t\bar{u}w_{M'}au)\ \psi_U(u)\ du \\ & = \sum_{M'} v'_{M'} \sum_{a \in A_{M'}} \int_{\mathrm{Stab}_{U'_{\mathbb{A}} \times U'_{\mathbb{A}}}(w_{M'}) \backslash (U'_{\mathbb{A}} \times U'_{\mathbb{A}})} \Phi({}^tu_1w_{M'}au_2))\ \psi_U(u_1u_2)\ du_1\ du_2\end{aligned}$$

where

$$v_{M'} = \mathrm{vol}(\mathrm{Stab}_{U_F}(w_{M'}) \backslash \mathrm{Stab}_{U_{\mathbb{A}}}(w_{M'}))$$

and

$$v'_{M'} = \mathrm{vol}(\mathrm{Stab}_{U'_F \times U'_F}(w_{M'}) \backslash \mathrm{Stab}_{U'_{\mathbb{A}} \times U'_{\mathbb{A}}}(w_{M'})).$$

The functions Ψ and Φ have matching orbital integrals if each summand on the left hand side of (3) equals the corresponding summand on the right hand side. Since the orbital integrals are decomposable distributions, the matching of orbital integrals reduces to a local linear condition at all places. The more matching functions we can find, the more useful the identity (2) becomes for applications. In §2, we overview Jacquet's results concerning local matching of orbital integrals at the finite places.

Also crucial for applications, is a spectral expansion for the distributions in each side of (2). For the right hand side, a fine spectral expansion can be given without much difficulty, as no convergence issues occur. For the left hand side, Lapid obtains in [Lap06], the absolute convergence of the fine spectral expansion of the RTF (see §3). This is the analogue of Arthur's result in.. for the Arthur-Selberg trace formula. The results of Jacquet and Lapid combined, allow us, to compare the contribution of the discrete spectrum to each side of (2). Roughly speaking, since Jacquet obtains local matching for enough pairs of functions, he can in fact, for every cuspidal automorphic representation π of $G_{\mathbb{A}}$ find matching functions for which

$$(4)\qquad \tilde{B}^{\psi}_{\pi}(\Psi) = \sum_{\pi'} B^{\psi}_{\pi'}(\Phi)$$

where the sum is over all cuspidal representations π' of $G'_{\mathbb{A}}$ that base change to π, the *relative Bessel distribution* $\tilde{B}^{\psi}_{\pi}$ is the contribution of π to the RTF and the Bessel distribution $B^{\psi}_{\pi'}$ is the contribution of π' to the KTF. The distribution $\tilde{B}^{\psi}_{\pi}$ is not zero if and only if π is distinguished by a unitary group. If the sum on the right is not empty, we can choose matching functions so that only one summand occurs. This way Jacquet obtains Theorem 1. The identity (4) is also where we begin the computation of (1). More explicitly, the relative Bessel distribution on the symmetric space $X_{\mathbb{A}}$ captures spectral information distinguished by any unitary group. It can be expressed as a sum of distributions on G_A

$$\tilde{B}^{\psi}_{\pi}(\Psi) = \sum_{\{\xi\}} \tilde{B}^{\psi}_{\xi,\pi}(f_\xi)$$

where $\{\xi\}$ is a set of representatives for the G_F-orbits in X_F and $f_\xi \in C_c^\infty(G_{\mathbb{A}})$ is such that

$$\Psi({}^t\bar{g}\xi g) = \int_{H^\xi_{\mathbb{A}}} f_\xi(h\, g)\ dh.$$

Since we are only concerned with the period integral over H^α it is enough to consider a function Φ which is supported on the $G_{\mathbb{A}}$-orbit of α, i.e. we set $f_\xi = 0$ for every representative $\xi \neq \alpha$. For every $f = f_\alpha$ we then say that f and Φ have matching orbital integrals if Ψ and Φ do. The relative Bessel distribution on $G_{\mathbb{A}}$ is defined by

$$\tilde{B}^{\psi}_{\alpha,\pi}(f) = \sum_{\phi\in \mathrm{ob}(\pi)} \int_{H^\alpha_F\backslash H^\alpha_{\mathbb{A}}} \pi(f)\phi(h)\ dh \int_{U_F\backslash U_A} \phi(u)\psi_U(u)\ du$$

where the sum is over an orthonormal basis of π. This is independent of the choice of basis. If the cuspidal representation π is the base change of π' then it is also the base change of $\pi' \otimes \omega$. Furthermore, if the support of Φ is contained in the kernel of $\omega \circ \det$ then $B^{\psi}_{\pi'}(\Phi) = B^{\psi}_{\pi'\otimes\omega}(\Phi)$. Thus, for suitable matching functions f and Φ (4) becomes $\tilde{B}^{\psi}_{\alpha,\pi}(f) = 2B^{\psi}_{\pi'}(\Phi)$. This identity is our point of departure for (1). We may choose f to be a translate of a spherical Hecke function on G_A so that the left hand side is a unique summand, which is a product of the anisotropic unitary period we wish to compute with a Fourier coefficient of ϕ_0 and the spherical Fourier transform of the Hecke function closely related to f. The distributation on the right hand side is factorizable, thanks to results of Jacquet, up to an explicit global constant $B_{\pi'}(\Phi) = (*)\prod_v B_{\pi'_v}(\Phi_v)$. To obtain the explicit formula for the period, it remains to compute the local factors at finitely many places. The function Φ need not be a Hecke function. Thus, to compute the local terms we use a local identity of Bessel distributions that relates $B_{\pi'_v}(\Phi_v)$ to a local analogue of $B_{\alpha,\pi}(f)$ for matching functions f_v and Φ_v [Off, Theorem 3]. Since f is (essentially) a Hecke function, the local relative Bessel distribution can now become a unique summand, which we can express as a product of Hironaka's spherical function evaluated at α_v with a local Whittaker function and the spherical Fourier transform of the Hecke function related to f_v. Putting an absolute value squared on both sides, after some cancelation, we remain with the formula for the period integral in terms of Hironaka's spherical functions. As explained in the remarks after Theorem 2, whenever applicable, we then use Hironaka's explicit formulas to make (1) explicit.

To summarize, the main ingredients in the proof of (1) are:

- Local matching of orbital integrals [Jac98, Jac03b, Jac04, Jac05].
- The fine spectral expansion of the relative trace formula [Lap06].
- Explicit formulas for spherical functions on the p-adic space of invertible Hermitian matrices [Hir99].
- Local identities of Bessel distributions for principal series representations [Off].

After setting up notation in §1, we explain in §2-§5 the results of each of the main ingredients in the order appearing above. As explained in Remark 3, in §4, we also provide some new explicit formulas for the spherical functions. In §5, we refine the results in [Off, Theorem 2 and Theorem 3] by resolving the transfer factor dichotomy.

1. Notation

We shall alternate between local and global settings. Throughout, denote by E/F a quadratic extension of number fields in the global case and of local fields of characteristic zero in the local case and let $x \mapsto \bar{x}$ denote the Galois action. Locally, we shall also allow the split case where $E = F \oplus F$. Globally, we denote by $\mathbb{A} = \mathbb{A}_F$ the ring of adèles of F. Let $\mathrm{Nm}(x) = \mathrm{Nm}_{E/F}(x) x\bar{x}$ be the norm map, $\mathrm{Tr}(x) = \mathrm{Tr}_{E/F}(x) = x + \bar{x}$ be the trace map and $\omega = \omega_{E/F}$ the quadratic character associated to E/F by class field theory. In the local split case the Galois action is $(x, y) \mapsto (y, x)$ for x, $y \in F^\times$ and ω is the trivial character. In the global case, for every place v of F we let $E_v = E \otimes_F F_v$. If v is inert in E then E_v/F_v is a quadratic extension of local fields whereas if v is split in E then $E_v \simeq F_v \oplus F_v$.

We shall denote by bold letters such as $\mathbf{Y}$ an algebraic set defined over F and by the corresponding letter the set of rational points $Y = \mathbf{Y}(F)$. Globally, for every place v of F denote by $Y_v = \mathbf{Y}(F_v)$ the corresponding local space of F_v-rational points and let $Y_{\mathbb{A}} = \mathbf{Y}(\mathbb{A})$.

Let $\mathbf{G}'$ be the group GL_n regarded as an algebraic group over F and let $\mathbf{G} = R_{E/F}(GL_n)$ be the restriction of scalars of GL_n from E to F. Thus $G = GL_n(E)$ whereas $G' = GL_n(F)$. We denote by

$$\mathbf{X} = \{g \in \mathbf{G} : g = {}^t\bar{g}\}$$

the space of Hermitian matrices in G and consider it as a right G-space with the action

$$(x, g) \mapsto {}^t\bar{g} x g.$$

For every Hermitian matrix $x \in X$ let

$$\mathbf{H_x} = \{g \in \mathbf{G} : {}^t\bar{g} x g = x\}$$

be the associated unitary group.

Let ψ' be an additive character of F in the local case and respectively of $F \backslash \mathbb{A}$ in the global case and let

$$\psi = \psi' \circ \mathrm{Tr}_{E/F}\,.$$

In the rest of this section we shall fix notation and conventions with respect to the group G and the character ψ. Similar notation and convention for G' and ψ' will be appended with a prime.

In the local case we denote by K the standard maximal compact subgroup of G and in the global case we let $\mathbf{K} = \prod_v K_v$ denote the standard maximal compact subgroup of $G_{\mathbb{A}}$ where the product is over all places in F. Thus if, for example, v is a split place of F then $K_v = K'_v \times K'_v$. Let $\mathbf{B} = \mathbf{TU}$ be the subgroups of $\mathbf{G}$ so that B is the group of upper triangular matrices, T is the group of diagonal matrices and U is the group of upper triangular unipotent matrices in G. We denote by ψ_U the generic character of U in the local case and of $U_{\mathbb{A}}$ in the global case defined by

$$\psi_U(u) = \psi(u_{1,2} + \cdots + u_{n-1,n}).$$

2. Orbital integrals

The orbital integrals that we consider have been studied extensively in a long series of papers (...), culminating at the remarkable work of Jacquet [Jac05], where he obtains an explicit identity between orbital integrals that constitutes the *fundamental lemma* for a general Hecke function.

We now introduce the orbital integrals. The group $U' \times U'$ acts on G' by

$$(u_1, u_2, g) \mapsto {}^t u_1 g u_2,\ u_1,\ u_2 \in U',\ g \in G'. \tag{5}$$

An element $g \in G$ (or its orbit) is called *relevant* if the function $(u_1, u_2) \mapsto \psi_U(u_1 u_2)$ is trivial on the stabilizer Stab_g of g in $U' \times U'$. For a function $\Phi \in C_c^\infty(G')$ and a relevant $g \in G'$ let

$$\Omega[\Phi, \psi : g] = \int_{\mathrm{Stab}_g \backslash U' \times U'} \Phi({}^t u_1 g u_2) \psi_{U'}(u_1 u_2)\ du_1\ du_2. \tag{6}$$

Similarly, the group U acts on S by

$$(u, s) \mapsto {}^t \bar{u} s u,\ u \in U',\ s \in S \tag{7}$$

and s (or its orbit) is called relevant if $u \mapsto \psi_U(u)$ is trivial on Stab_s. For $\Psi \in C_c^\infty(S)$ and a relevant $s \in S$ let

$$\Omega[\Psi, \psi, E/F : s] = \int_{\mathrm{Stab}_s \backslash U} \Psi({}^t \bar{u} s u) \psi_U(u)\ du. \tag{8}$$

The *matching of orbital integrals* is based on a natural bijection between the relevant $(U' \times U')$-orbits on G' and the relevant U-orbits on G. Indeed, a complete common set of representatives for the relevant orbits consists of elements of the form $w_{M'} a$ where $w_{M'}$ is the longest Weyl element of a standard parabolic subgroup M' of G' and a lies in the center $A_{M'}$ of M'. Clearly,

$$\Omega[\Phi, \psi : {}^t u_1 g u_2] = \psi_{U'}(u_1 u_2)^{-1} \Omega[\Phi, \psi : g]$$

and

$$\Omega[\Psi, \psi, E/F : {}^t \bar{u} s u] = \psi_U(u)^{-1} \Omega[\Psi, \psi, E/F : s].$$

It is therefore enough to study the orbital integrals on the representatives $w_M a$. In a sense that we shall soon explain, the orbital integrals are determined by their values on representatives of orbits of maximal dimension, i.e when $w_M = e$ and then $a \in A'$ is any diagonal element of G'. Let $\gamma(a)$ be the transfer factor defined by

$$\gamma(a) = \omega^n(a_1) \cdots \omega^2(a_{n-1}) \omega(a_n).$$

Note that it differs from Jacquet's transfer factor in [Jac05] by a factor of $\omega(\det a)$. We say that Φ and Ψ have matching orbital integrals for ψ and write $\Phi \overset{\psi}{\longleftrightarrow} \Psi$ if

$$\Omega[\Phi, \psi : a] = \gamma(a)\ \Omega[\Psi, \psi, E/F : a],\ a \in A'.$$

Theorem 3 (Density ([Jac03a], Théorème 1.1))**.** *If $\Phi \in C_c^\infty(G')$ is such that $\Omega[\Phi, \psi : a] = 0$ for all $a \in A'$ then $\Omega[\Phi, \psi : g] = 0$ for every relevant $g \in G'$.*

Theorem 4 ([Jac03a], Théorème 2.1)**.** *There exist transfer factors $\gamma(w_{M'} a, \psi)$ such that whenever $\Phi \overset{\psi}{\longleftrightarrow} \Psi$ we also have*

$$\Omega[\Phi, \psi : w_{M'} a] = \gamma(w_{M'} a, \psi) \Omega[\Psi, \psi, E/F : w_{M'} a],\ a \in A_{M'}.$$

In fact, assuming that F is of characteristic zero and that E/F is unramified, with some mild restrictions on the residual characteristic, Theorem 3 and Theorem 4 were already proven in [Jac98]. But once developing his machinery for the study of orbital integrals Jacquet's proofs in [Jac03a] become much simpler.

For global applications, there are two major tasks in matching orbital integrals. The first is to show the existence of enough pairs $\Phi \longleftrightarrow \Psi$ of matching functions. We refer to this problem as *smooth matching.* The second, is the more difficult problem of explicitly matching a bi K'-invariant function on G' with a K-invariant function on S, which is the *fundamental lemma* for the relative trace formula... Jacquet obtained the following results:

Theorem 5 (Smooth matching [Jac03b]). *For every* $\Phi \in C_c^\infty(G')$ *there exists* $\Psi \in C_c^\infty(S)$ *so that* $\Phi \overset{\psi}{\longleftrightarrow} \Psi$ *and vice versa.*

Denote by $\Psi^{(0)}$ the characteristic function of $S \cap K$.

Theorem 6 (The fundamental lemma of Jacquet [Jac05]). *Assume that* F *has odd residual characteristic and that* ψ *has conductor* $\mathcal{O}_F$. *For every Hecke function* $f \in \mathcal{H}_G(K)$ *we have*

$$b(f) \overset{\psi}{\longleftrightarrow} f * \Psi^{(0)}.$$

We now wish to explain the machinery developed by Jacquet and to very roughly explain the idea behind his proofs. It turns out to be more useful to linearize the problem and consider more general orbital integrals. The group $U' \times U'$ also acts on the linear space $M_n(F)$ by (5) and relevant orbits in $M_n(F)$ can be defined as before. For a Schwartz function $\Phi \in C_c^\infty(M_n(F))$ and a relevant $g \in M_n(F)$ we define the orbital integral $\Omega[\Phi, \psi : g]$ by the formula (6). Similarly, let

$$H_n(E/F) = \{X \in M_n(E) : {}^t\bar{X} = X\}$$

be the linear space of $n \times n$ Hermitian matrices and let U act on $H_n(E/F)$ by (7). For a Schwartz function $\Psi \in C_c^\infty(H_n(E/F))$ and a relevant $s \in H_n(E/F)$ we define the orbital integral $\Omega[\Psi, \psi : s]$ by the formula (8). A diagonal matrix $a = diag(a_1, \dots, a_n)$ with entries in F is relevant if and only if $a_1 \cdots a_{n-1} \neq 0$. Allowing all relevant diagonal matrices, we generalize the definition of matching for for Schwartz functions. In [Jac03a], Theorem 3 and Theorem 4 are in fact stated for all Schwartz functions and we therefore consider from now on only orbital integrals of relevant diagonal matrices. When $a = \mathrm{diag}(a_1, \dots, a_n)$ is relevant we sometimes also denote

$$\Omega[\Phi, \psi : a_1, \dots, a_n] = \Omega[\Phi, \psi : a] \text{ and } \Omega[\Psi, E/F, \psi : a_1, \dots, a_n] = \Omega[\Psi, \psi, E/F : a].$$

It will also be convenient to normalize the orbital integrals. We set

$$\tilde{\Omega}[\Phi, \psi : a_1, \dots, a_n] = \left|a_1^{n-1} a_2^{n-2} \cdots a_{n-1}\right|_F \Omega[\Phi, \psi : a]$$

and

$$\tilde{\Omega}[\Psi, \psi, E/F : a_1, \dots, a_n] = \eta(a_1^{n-1} a_2^{n-2} \cdots a_{n-1}) \left|a_1^{n-1} a_2^{n-2} \cdots a_{n-1}\right|_F \Omega[\Phi, \psi : a].$$

Let

$$\mathcal{O}_n^\psi(F) = \{a \mapsto \Omega[\Phi, \psi : a] : \Phi \in C_c^\infty(M_n(F))\}$$

and let

$$\mathcal{O}_n^\psi(E/F) = \{a \mapsto \Omega[\Psi, \psi, E/F : a] : \Psi \in C_c^\infty(H_n(E/F))\}.$$

Since $\bar{\psi}_U(u) = \psi_U(\epsilon u \epsilon^{-1})$ for $\epsilon = \operatorname{diag}(1,-1,1,-1,\dots)$ and similarly for $\psi_{U'}$ and since $\epsilon^2 = e$, it is easy to see that $\mathcal{O}_n^{\psi}(F) = \mathcal{O}_n^{\bar{\psi}}(F)$ and that $\mathcal{O}_n^{\psi}(E/F) = \mathcal{O}_n^{\bar{\psi}}(E/F)$. For a Schwartz function f either in $C_c^\infty(M_n(F))$ or in $C_c^\infty(H_n(E/F))$ and an integer k denote by $f[k]$ the product of f with the characteristic function of $\{X : |\det X|_F = |\varpi^k|_F\}$. Since det is fixed on the orbits, it follows that

$$\text{if } \Phi \overset{\psi}{\longleftrightarrow} \Psi \text{ then } \Phi[k] \overset{\psi}{\longleftrightarrow} \Psi[k]. \tag{9}$$

By linearity, it follows that Theorem 5 reduces to the equality

$$\mathcal{O}_n^{\psi}(F) = \mathcal{O}_n^{\psi}(E/F). \tag{10}$$

The advantage of the linearized problem of smooth matching, is that we may use Fourier analysis on the vector spaces of Schwartz functions. We define the Fourier transform $\mathcal{F} = \mathcal{F}_\psi$ as follows. For $\Phi \in C_c^\infty(M_n(F))$ let

$$\mathcal{F}(\Phi)(X) = \int_{M_n(F)} \Phi(Y)\psi(-\operatorname{Tr}(Y w_n X w_n))\, dy$$

and for $\Psi \in C_c^\infty(H_n(E/F))$ let

$$\mathcal{F}(\Psi)(X) = \int_{H_n(E/F)} \Psi(Y)\psi(-\operatorname{Tr}(Y w_n X w_n))\, dy.$$

The Fourier inversion formula is the statement that $\mathcal{F}_{\bar{\psi}} \circ \mathcal{F}_\psi = \mathrm{Id}$. To make use of the Fourier transform Jacquet introduced a transform on the spaces of orbital integrals. For a function ω on the set of relevant diagonal matrices, whenever well defined the *Jacquet transform* $\mathfrak{J} = \mathfrak{J}_\psi$ is given by the iterated integral

$$\mathfrak{J}(\omega)(a_1,\dots,a_n) = \int \omega(b_1,\dots,b_n)\psi\Big(-\sum_{i=1}^{n} b_i a_{n+1-i} + \sum_{i=1}^{n-1} \frac{1}{b_i a_{n-i}}\Big)\, db_n\, db_{n-1} \cdots db_1.$$

Not without effort, Jacquet shows that the Jacquet transform is well defined on $\mathcal{O}_n^{\psi}(F)$ and on $\mathcal{O}_n^{\psi}(E/F)$ and that the Jacquet and the Fourier transforms essentially intertwine with the operation of taking orbital integrals. More precisely, [Jac03b, Theorem 1 and Theorem 2] state that the following diagrams commute:
(11)

$$\begin{tikzcd}
C_c^\infty(M_n(F)) \arrow[r, "\Omega'_\psi"] \arrow[d, "\mathcal{F}_\psi"] & \mathcal{O}_n^{\psi}(F) \arrow[d, "\mathfrak{J}_\psi"] \\
C_c^\infty(M_n(F)) \arrow[r, "\Omega'_\psi"] & \mathcal{O}_n^{\psi}(F)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C_c^\infty(H_n(E/F)) \arrow[r, "\Omega_\psi"] \arrow[d, "\mathcal{F}_\psi"] & \mathcal{O}_n^{\psi}(E/F) \arrow[d, "\mathfrak{J}_\psi"] \\
C_c^\infty(H_n(E/F)) \arrow[r, "\Omega_\psi"] & \mathcal{O}_n^{\psi}(E/F)
\end{tikzcd}$$

where

$$\Omega'_\psi(\Phi)(a) = \Omega[\Phi,\psi : a],\ \ \Omega_\psi(\Psi)(a) = c(E/F,\psi)^{\frac{(n-1)n}{2}}\Omega[\Psi,\psi,E/F : a]$$

and $c(E/F,\psi)$ is the Weil constant defined by the identity

$$\int_E \hat{\phi}(x)\psi(ax\bar{x})\, dx = |a|_F^{-1}\,\eta(a)c(E/F,\psi)\int_E \phi(x)\psi(-a^{-1}x\bar{x})\, dx$$

for all $\phi \in C_c^\infty(E)$ and $a \in F^\times$, where the Fourier transform $\hat{\phi}$ is defined by

$$\hat{\phi}(x) = \int_E \phi(y)\psi(-(xy + \overline{xy})).$$

Applying this identity twice and the Fourier inversion formula we see that

$$c(E/F,\psi)c(E/F,\bar{\psi}) = 1.$$

Thus, the Fourier inversion formula and the commutative diagrams in (11) imply the inversion formula of the Jacquet transform on $\mathcal{O}_n^{\psi}(F)$ and on $\mathcal{O}_n^{\psi}(E/F)$. Namely,

$$\mathfrak{J}_{\bar{\psi}} \circ \mathfrak{J}_{\psi} = \mathrm{Id}\,.$$

The proof is using Weil's formula for the integral of the Fourier transform of a Schwartz function on a vector space against a character of second order. It amounts to an elementary yet complicated computation based on certain intermediate orbital integrals and an inductive argument. The inversion formula implies that

$$\Phi \overset{\psi}{\longleftrightarrow} \Psi \text{ if and only if } \mathcal{F}_{\psi}(\Phi) \overset{\bar{\psi}}{\longleftrightarrow} c(E/F,\psi)^{\frac{(n-1)n}{2}} \mathcal{F}_{\psi}(\Psi). \tag{12}$$

This equivalence is the main reason for linearizing the problem and introducing the Jacquet transform. Another useful and much more elementary formula is obtained in [Jac03b, Proposition 4]. For $\Phi \in C_c^{\infty}(M_n(F))$ the function $a \mapsto \Omega[\Phi,\psi : w_n a]$ on $F^{\times}$ is smooth and of compact support, furthermore it satisfies the identity:

$$\Omega[\Phi,\psi : w_n a] = |a|_F^{1-n^2} \int_F \Omega[\mathcal{F}_{\psi}(\Phi),\bar{\psi} : \begin{pmatrix} -w_{n-1}a & 0 \\ 0 & b \end{pmatrix}]. \tag{13}$$

There is an analogue of (13) for $\Psi \in C_c^{\infty}(H_n(E/F))$. Theorem 3 and Theorem 4 follow from (12) and (13). Indeed, for the decomposable representatives, i.e. those of the form $w_{M'}a = \mathrm{diag}(w_1 a_1, w_2 a_2)$ where $n = n_1 + n_2$ with $n_i > 0$ and $w_i a_i$ is one of our relevant representatives for $GL_{n_i}(F)$, $i = 1, 2$, both the vanishing stated in Theorem 3 and the existence of transfer factors as in Theorem 4 follow by induction using certain intermediate orbital integrals. Once this is granted both theorems are straightforward consequences of (12), (13) and its analogue for Ψ. The equality (10) and therefore Theorem 5 also follow from (12) and (13) with the help of the intermediate orbital integrals. The argument, given in [Jac03b, §8], is a little more elaborate.

Remark 5. In [Off05], the Jacquet transform is defined on an analogue space of orbital integrals for the space of $n \times n$ symmetric matrices over F, it is proved that the diagram for this case analogue to (11) commutes and therefore that a similar inversion formula holds for the Jacquet transform. An analogue of the simpler identity (13) is also provided. See remark ... for the relevant trace formula for this setting.

Though extremely useful, the inversion formula for the Jacquet transform is still far from enough machinery in order to face the fundamental lemma. Again the problem is linearized. Let Φ_0 be the characteristic function of $M_n(\mathcal{O}_F)$ and similarly let Ψ_0 be the characteristic function of the lattice $H_n(E/F) \cap M_n(\mathcal{O}_E)$. We assume from now on that ψ has conductor $\mathcal{O}_F$. The linearized version of the fundamental lemma is the explicit matching [Jac05, Theorem 1]:

$$b(f) * \Phi_0 \overset{\psi}{\longleftrightarrow} (f * \Psi_0)_{\omega} \tag{14}$$

where for a function Ψ on X we let $\Psi_{\omega}(x) = \omega(\det x)\Psi(x)$. One can use (9) to deduce Theorem 6 from (14). In fact, the simple argument given in [Jac05, p. 613] provides more explicit pairs of matching functions then Jacquet admits in his paper to have given. For all $k \geq 0$ let $\Phi^{(k)} = \Phi_0[k]$ and $\Psi^{(k)} = \Psi_0[k]$. Assume that

$f \in \mathcal{H}_G(K)$ is supported on $\{g \in G : |\det g|_E = |\varpi^m|_E\}$. Then $b(f)$ is supported on $\{g \in G' : |\det g|_F = \left|\varpi^{2m}\right|_F\}$. Note then that

$$(b(f)*\Phi_0)[k-2m] = b(f)*\Phi^{(k)} \text{ and } (f*\Psi_0)_\omega[k-2m] = (f*\Psi^{(k)})_\omega = (-1)^k f*\Psi^{(k)}.$$

Applying (9), (14) and linearity of the orbital integrals we therefore get that

$$b(f) * \Phi^{(k)} \overset{\psi}{\longleftrightarrow} (-1)^k f * \Psi^{(k)} \tag{15}$$

for all $f \in \mathcal{H}_G(K)$ and all $k \geq 0$.

Remark 6. This is a more general explicit matching then stated in Theorem 6 for K-invariant functions on G and bi K' invariant functions on G' (which are not necessarily in the image of base change). In particular, the fact that we have a pair of matching functions supported on matrices with determinant of odd valuation (e.g. $\Phi^{(k)} \overset{\psi}{\longleftrightarrow} (-1)^k \Psi^{(k)}$ for k odd) is enough in order to determine the dichotomy of the transfer factor explained in ...

The technical heart of the proof of the fundamental lemma is a certain *uncertainty principle* for the space $\mathcal{O}_n^\psi(F)$. The standard uncertainty principal for a function $f \in C_c^\infty(F)$ and its Fourier transfor $\mathcal{F}(f)$ can be formulated as follows. If the support of f lies in $\mathfrak{p}^k$ and the support of $\mathcal{F}(f)$ lies in $\mathfrak{p}^{-k}$ then f lies in the one dimensional space spanned by the characteristic function of $\mathfrak{p}^k$. Jacquet's generalization for the space of normalized orbital integrals is described as follows. Let

$$\Lambda_n = \{(m_1, \ldots, m_n) \in \mathbb{Z}^n : m_1 \geq \cdots \geq m_n\}.$$

For $m = (m_1, \ldots, m_n)$ let $\tilde{m} = (-m_n, \ldots, -m_1) \in \Lambda_n$ and let $m \preceq m'$ be the standard partial order on Λ_n defined by

$$m_1 + \cdots + m_i \leq m'_1 + \cdots + m'_i,\ i = 1, \ldots, n-1,\ m_1 + \cdots + m_n = m'_1 + \cdots + m'_n.$$

For $m \in \Lambda_n$ let $\mathcal{F}(m)$ be the space of functions $\omega \in \mathcal{O}_n^\psi(F)$ such that the support of ω lies in the set of all relevant $a = \operatorname{diag}(a_1, \ldots, a_n)$ such that

$$|a_1 \cdots a_i|_F \leq \left|\varpi^{-(m_1+\cdots+m_i)}\right|_F,\ i = 1, \ldots, n \tag{16}$$

and the support of $\mathfrak{J}(\omega)$ lies in the set of all a such that

$$|a_1 \cdots a_i|_F \leq \left|\varpi^{m_n+\cdots+m_{n+1-i}}\right|_F,\ i = 1, \ldots, n. \tag{17}$$

Since $m \preceq m'$ implies that $\tilde{m} \preceq \tilde{m}'$ we then also have $\mathcal{F}(m) \subset \mathcal{F}(m')$. We also let

$$\Phi_m = \Phi_0(\cdot\ \varpi^m) \text{ and } \Psi_m = \Psi_0(\varpi^m\ \cdot\ \varpi^m)$$

where $\varpi^m = \operatorname{diag}(\varpi^{m_1}, \ldots, \varpi^{m_n})$. Note that

$$\mathcal{F}(\Phi_m) = q^{n(m_1+\cdots+m_n)}\Phi_{\tilde{m}} \text{ and } \mathcal{F}(\Psi_m) = q^{2n(m_1+\cdots+m_n)}\Psi_{\tilde{m}}.$$

Since the diagrams in (11) commute it is therefore not hard to see that the function $a \mapsto \tilde{\Omega}[\Phi_m, \psi : a]$ lies in $\mathcal{F}(m)$ and that $a \mapsto \tilde{\Omega}[\Psi_m, \psi, E/F : a]$ lies in $\mathcal{F}(2m)$.

Theorem 7 (The uncertainty principle ([Jac05], Proposition 4)). *The functions*

$$a \mapsto \tilde{\Omega}[\Phi_{m'}, \psi : a],\ m' \preceq m$$

form a basis of the space $\mathcal{F}(m)$.

We remark first that for $m = (0, \dots, 0)$ this statement was already proved in [Jac04] and it implies the matching

$$\Phi_0 \overset{\psi}{\longleftrightarrow} (\Psi_0)_\omega.$$

The proof of Theorem 7 is of combinatorial nature and is rather long. We shall not explain it here. We simply narrate that to describe conditions such as (16) and (17) on the support of functions in $\mathcal{O}_n^\psi(F)$, Jacquet introduces the terminology of box diagrams and proves a series of lemmas concerning the diagrams in [Jac05, §7-§9]. If $m_1 - m_2 \geq 2$ then the existence of $m' \prec m$ with $m'_i = m_i$, $i \geq 3$ (e.g. $m' = (m_1 - 1, m_2 + 1, m_3, \dots, m_n)$) simplifies the proof of Theorem 7 using induction. When $m_1 - m_2 \leq 1$ the proof is more delicate and requires the full use of the machinery of box diagrams developed.

It follows from the uncertainty principle that there exist unique constants $\xi_m^{m'}$ for all $m' \preceq 2m$ such that

$$(\Psi_m)_\omega \overset{\psi}{\longleftrightarrow} \sum_{m' \preceq 2m} \xi_m^{m'} q^{\langle 2m' - m, \rho \rangle} \Phi_m. \tag{18}$$

For every $m \in \Lambda_n$ let $\sigma_m \in \mathbb{S}_n$ be the associated Schur polynomial. There exist constants $\theta_m^{m'}$, $m' \preceq 2m$ such that

$$\sigma_m(x^2) = \sum_{m' \preceq 2m} \theta_m^{m'} \sigma_{m'}(x).$$

Denote by $f_m^F \in \mathcal{H}_{G'}(K')$ the Hecke function such that $\hat{f}_m^F(x) = q^{\langle m, \rho \rangle} \sigma_m(x)$ and similarly let $f_m^E \in \mathcal{H}_G(K)$ be such that $\hat{f}_m^E(x^2) = q^{2\langle m, \rho \rangle} \sigma_m(x^2)$. Thus $f_m^F, m \in \Lambda_n$ is a basis for $\mathcal{H}_{G'}(K')$, $f_m^E, m \in \Lambda_n$ is a basis for $\mathcal{H}_G(K)$ and based on the Shintani, Casselman-Shalika formula for the spherical Whittaker function, Jacquet showed that

$$b(f_m^E) = \sum_{m' \preceq 2m} \theta_m^{m'} q^{\langle 2m - m', \rho \rangle} f_{m'}^F.$$

By linearity, it is enough to prove (14) for $f = f_m^E$. Jacquet also showed that the orbital integral of $(f_m^E * \Psi_0)_\omega$ is the same as the orbital integral of $q^{2\langle m, 2\rho \rangle} (\Psi_m)_\omega$ and that the orbital integral of $f_m^F * \Phi_0$ is the same as the orbital integral of $q^{\langle m, 2\rho \rangle} \Phi_m$. It follows, that in order to prove (14) it is enough to show that for every $m \in \Lambda_n$ we have

$$(\Psi_m)_\omega \overset{\psi}{\longleftrightarrow} \sum_{m' \preceq 2m} \theta_m^{m'} q^{\langle m' - 2m, \rho \rangle} \Phi_{m'}. \tag{19}$$

With this in mind, Jacquet defines the linear map $\beta : \mathcal{H}_G(K) \to \mathcal{H}_{G'}(K')$ by

$$\beta(f_m^E) = \sum_{m' \preceq 2m} \xi_m^{m'} q^{\langle 2m - m', \rho \rangle} f_{m'}^F.$$

The constants $\xi_m^{m'}$ were so defined that

$$(\Psi_m)_\omega \overset{\psi}{\longleftrightarrow} \sum_{m' \preceq 2m} \xi_m^{m'} q^{\langle m' - 2m, \rho \rangle} \Phi_{m'}.$$

Using the above arguments we then see that

$$\beta(f) * \Phi_0 \overset{\psi}{\longleftrightarrow} (f * \Psi_0)_\omega$$

for every $f \in \mathcal{H}_G(K)$. To prove the fundamental lemma it is therefore left to show that $\beta = b$ or what amounts to the same that $\xi_m^{m'} = \theta_m^{m'}$ whenever $m' \preceq 2m$. Computing the constants explicitly, Jacquet shows that $\xi_m^{m'} = \theta_m^{m'}$ whenever $m_1 - m_n \leq 1$. This amounts to saying that β agrees with b on a set of generators for the Hecke algebra $\mathcal{H}_G(K)$. A global argument is then used in order to prove the identity on the entire Hecke algebra. Indeed, applying the map β at almost every inert place to a simple version of the relative trace formula Jacquet shows that β is an algebra homomorphism.

3. The fine spectral expansion of the relative trace formula

The relative trace formula relevant to us is the distribution on $X_{\mathbb{A}}$ given by

$$RTF(\Psi) = \int_{U_F \backslash U_{\mathbb{A}}} \Big(\sum_{x \in X_F} \Psi({}^t\bar{u}xu) \Big) \, \theta(u) \, du.$$

It can be expressed as a sum

$$\sum_{\{\xi\}} RTF_\xi(f_\xi) \tag{20}$$

over a set of representatives $\{\xi\}$ of the G_F-orbits in X_F of the distributions on $G_{\mathbb{A}}$

$$RTF_\xi(f) = \int_{H_F^\xi \backslash H_{\mathbb{A}}^\xi} \int_{U_F \backslash U_{\mathbb{A}}} K_f(h,u)\theta(u) \, du \, dh$$

where

$$\Psi(g \cdot \xi) = \int_{H_{\mathbb{A}}^\xi} f_\xi(hg) \, dh$$

and

$$K_f(x,y) = \sum_{\gamma \in G_F} f(x^{-1}\gamma y)$$

is the standard kernel function associated to the test function f acting on $L^2(G_F \backslash G_{\mathbb{A}}^1)$. For a fixed compact subset C of $X_{\mathbb{A}}$ there is a finite set Γ of representatives ξ of G_F-orbits, so that for any test function $\Psi \in C_c^\infty(X_{\mathbb{A}})$ with support contained in C and for any representative $\xi \notin \Gamma$ we have $RTF_\xi(f_\xi) = 0$ [Jac95, Lemma 1.1]. In particular, the sum (20) involves only finitely many non zero terms. From now on we focus on an individual term. Thus we fix $\xi \in X_F$, let $H = H^\xi$ and denote by $RTF = RTF_\xi$ the associated distribution on $G_{\mathbb{A}}$.

According to Langlands spectral decomposition

$$L^2(G_F \backslash G_{\mathbb{A}}^1) = \bigoplus_{\chi \in \mathfrak{X}} L^2(G_F \backslash G_{\mathbb{A}}^1)_\chi$$

as a direct sum over cuspidal data (see [Art78, §3]) Arthur expanded in [Art78, §4] the kernel function as

$$K_f(x,y) = \sum_\chi K_\chi(x,y)$$

where

$$K_\chi(x,y) = \sum_M \frac{|W_M|}{|W|} \sum_\pi \int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi \in \mathrm{ob}(\mathcal{A}_P^\pi)} E(x, I(f,\lambda)\varphi, \lambda)\overline{E(y,\varphi,\lambda)} \, d\lambda \tag{21}$$

where the sum is over all standard parabolic subgroups P of G (with standard Levi subgroup M), over the (finitely many) irreducible, discrete spectrum representations π of $L^2(M_F\backslash M^1_{\mathbb{A}})_\chi$ and over an orthonormal basis of the space $\mathcal{A}^\pi_P$ of automorphic forms on $U_{P,\mathbb{A}}M_F\backslash G^1_{\mathbb{A}}$ in the induced representation $\mathrm{ind}^{G_{\mathbb{A}}}_{P_{\mathbb{A}}}(\pi)$. In [Art82], Arthur obtained the fine spectral expansion for the Arthur-Selberg trace formula, i.e. an explicit expression for the contribution of each cuspidal data χ. The analogue for the RTF we consider in this work was obtained by Lapid in [Lap06]. In the rest of this section we recall his results.

If π is a cuspidal representation of $G_{\mathbb{A}}$ its contribution to the RTF is the relative Bessel distribution defined by

$$B_\pi(f) = \sum_{\phi\in\mathrm{ob}(\pi)} \Pi^H(\pi(f)\phi)\overline{\mathcal{W}^\psi(\phi)}$$

where

$$\Pi^H(\phi) = \int_{H_F\backslash H_{\mathbb{A}}} \phi(h)\ dh$$

is the unitary period integral (it converges by [AGR93]) and

$$\mathcal{W}^\psi(\phi) = \int_{U_F\backslash U_{\mathbb{A}}} \phi(u)\bar{\psi}(u)\ du$$

is the Whittaker functional. For a general automorphic form, the unitary period integral does not converge, and in order to define similar distributions one uses a regularization of the period integrals. The regularized period integrals were studied in [LR03] in the context of a quasi split Galois pair (G,θ), i.e. when θ is a Galois involution on the reductive group G (both defined over F) so that G has a θ stable minimal parabolic subgroup. The regularized integral

$$\Pi^H(\phi) = \int^*_{H_F\backslash H_{\mathbb{A}}} \phi(h)\ dh$$

is defined for essentially all automorphic forms (with a non trivial closed condition on the exponents of ϕ) and is an $H(\mathbb{A})$-invariant linear functional that agrees with the period integral whenever it converges. For a cuspidal representation π of $M^1_{\mathbb{A}}$ we can now define (at least for generic λ) the relative Bessel distribution

$$B^G_{(M,\pi)}(f,\lambda) = B_{(M,\pi)}(f,\lambda) = \sum_{\varphi\in\mathrm{ob}(\mathcal{A}^\pi_P)} \Pi^H(E(I(f,\lambda)\varphi,\lambda))\overline{\mathcal{W}^\psi(E(\varphi,-\bar{\lambda}))}.$$

In fact, we need to consider the analog distributions with respect to θ stable Levi subgroups of G. Note that H is the fixed point group of the involution $\theta_\xi(g) = \xi\theta(g)\xi^{-1}$. One of the technical difficulties in [LR03] is that the pair (G,θ_ξ) may no longer be a quasi split Galois pair. This motivates Lapid and Rogawsky to introduce the defect in [LR03, §4.4]. Essentially, this is the standard Levi M^{00} of a minimal θ_x stable parabolic for some x in the G_F-orbit of ξ. It is convenient (and always possible) to choose $\xi\in M^{00}$ ([LR03, §4.5]). Once we do so, every θ_ξ-stable Levi subgroup L of G is also θ stable, and $(L,\theta_{|L})$ is a quasi split Galois pair. Thus, for an automorphic form φ on $V_{\mathbb{A}}L_F\backslash G_{\mathbb{A}}$ that satisfies $\varphi(ag) = e^{\langle\rho_Q,H(a)\rangle}\varphi(g)$ for $a\in A_Q$ we can define the regularized integral

$$\int^*_{(Q_H)_F\backslash H_{\mathbb{A}}} \varphi(h)\ dh = \int_{\mathbf{K_H}}\int^*_{(L_H)_F\backslash(L_H)_{\mathbb{A}}} \varphi(lk)\ dl\ dk.$$

Accordingly, we define the relative Bessel distribution

$$B^L_{(M,\pi)}(f,\lambda) = \sum_{\varphi\in \mathrm{ob}(\mathcal{A}^\pi_P)} [\int^*_{(Q_H)_F\backslash H_{\mathbb{A}}} E^Q(h, I(f,\lambda)\varphi,\lambda)\ dh]\overline{\mathcal{W}^\theta(E(\varphi,-\bar\lambda))}.$$

Remark 7. The regularized integrals are defined using the *mixed truncation* operators $\Lambda^{T,Q}_m$ for any θ_ξ stable parabolic subgroup Q. These are certain relative variants of Arthur's truncation operator, well adapted to the setting of the RTF.

We now explain which triples (M,π,L) contribute to the spectral expansion.

Definition 1. A Levi subgroup M is called *θ-elliptic* in G, if $w_M M w_M^{-1} = \theta(M)$ and $w_M\theta$ acts as -1 on $\mathfrak{a}^G_M$. For a cuspidal representation π of $M^1_{\mathbb{A}}$ we say that (M,π) is θ_ξ-elliptic with respect to G if M is θ-elliptic in G and π is distinguished by M^{θ_x} for some $x \in G\cdot\xi \cap Mw_M^{-1}$.

The contribution to the RTF comes only from triples (M,π,L) so that (M,π) is θ_ξ-elliptic with respect to L. The θ_ξ-stable Levi L is determined uniquely by (M,π). That other terms do not contribute, can be seen from

Theorem 8 (Theorem 9.1.1, [LR03])**.** *Let π be a cuspidal representation of $M^1_{\mathbb{A}}$ then $\Pi^H(E(\varphi,\lambda)) = 0$ unless (M,π) is θ_ξ-elliptic in G, in which case*

$$\Pi^H(E(\varphi,\lambda) = J(w_{\theta(M)},\varphi,\lambda)$$

where the right hand side is the intertwining period *defined by the sum over M_F-orbits $\mathcal{O}$ in $G_F\cdot\xi\cap Mw_M^{-1}$ of*

$$\int_{H_F\cap\eta^{-1}P_F\eta\backslash H_{\mathbb{A}}} e^{\langle\lambda,H_P(\eta h)\rangle}\varphi(\eta h)\ dh$$

and $\eta\in G_F$ is a representative such that $\eta\cdot\xi\in\mathcal{O}$ (this is independent on the choice of η).

The intertwining periods were first introduced in [Jac95] for GL_3 and studied further in [JLR99] for split Galois pairs. More generally, we have that

$$\int^*_{(Q_H)_F\backslash H_{\mathbb{A}}} E^Q(h, I(f,\lambda)\varphi,\lambda)\ dh = 0$$

unless (M,π) is θ_ξ elliptic with respect to L in which case

$$\int^*_{(Q_H)_F\backslash H_{\mathbb{A}}} E^Q(h, I(f,\lambda)\varphi,\lambda)\ dh = J(w^L_{\theta(M)},\varphi,\lambda).$$

Analyzing these results explicitly for the case at hand Lapid obtained a very explicit fine spectral expansion [Lap06, Theorem 10.4]. We can always choose the representative ξ to be of the form

$$\xi = \begin{pmatrix} & & D \\ & x & \\ {}^t\bar{D} & & \end{pmatrix}$$

where D is anti diagonal in $GL_t(F)$ (t is the Witt index of ξ) and x is an anisotropic hermitian form of size $d = n-2t$.

Theorem 9 (Fine spectral expansion of the RTF [Lap06]). *For every θ_ξ-elliptic pair (M,π) with respect to L the relative Bessel distribution $B^L_{(M,\pi)}(f,\lambda)$ is holomorphic on $i(\mathfrak{a}_M^{L_H})^*$ where*

$$(\mathfrak{a}_M^{L_H})^* = (\mathfrak{a}_M^L)^* \oplus ((\mathfrak{a}_L^G)^*)^-_\theta .$$

There are combinatorial constants $\mathfrak{c}(M,\pi)$, that can be computed explicitly, so that

$$RTF(f) = \sum_{(M,\pi)} \mathfrak{c}(M,\pi) \int_{i(\mathfrak{a}_M^{L_H})^*} B^L_{(M,\pi)}(f,\lambda)\, d\lambda \tag{22}$$

where the sum is over all Levi subgroups M of G of type $(n_1,\dots,n_k,m_1,\dots,m_l,n_k,\dots,n_1)$ and cuspidal representations π of $M^1_{\mathbb{A}}$ of the form

$$\pi = \sigma_1 \otimes \cdots \otimes \sigma_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \bar\sigma_k \otimes \cdots \otimes \bar\sigma_1 \tag{23}$$

where $\sigma_i \not\simeq \bar\sigma_i$, $i=1,\dots,k$ and each τ_j is distinguished by some unitary group. For such a pair (M,π) the Levi subgroup L is then of type $(n_1,\dots,n_k,m_1+\cdots+m_l,n_k,\dots,n_1)$ and $m_1+\cdots+m_l \ge d$. Furthermore, the integral-sum (22) *is absolutely convergent.*

Very roughly speaking, the main technical difficulty is to interchange between 2 integrals where the inner integral is over the imaginary axis of a certain vector space and the outer integral is a unitary period. This is achieved using a shift of contour and coming back to the unitary access after interchanging the integrals. Lapid's approach, using complex analysis, is new. The formal manipulations are justified by a majorization of Eisenstein series, which is the technical heart of the paper [Lap06, Proposition 6.1]. The combinatorics of (G,M)-families, introduced by Arthur in [Art81], is applied to reduce the problem to lower bounds of Rankin-Selberg L-functions at the edge of the critical strip, which appear in the normalization of intertwining operators. Such lower bounds were obtained by Brumeley [Bru06] for GL_n. For the absolute convergence, the uniform bound of Lou, Rudnick and Sarnak towards the Ramanujan conjecture is also applied [LRS99].

We will not get here into the deep analytic problems involved. However, in order to introduce the reader with the complexity of the problem, we wish to roughly explain the main 3 steps in the proof. Assume first that f is K_∞-finite. The first step was already obtained by Jacquet in [Jac95]. Jacquet obtained in [Jac95, Proposition 2.1] for any integer N and for y in a fixed compact set the bound

$$\sum_{\chi\in\mathfrak{X}} |K_\chi(x,y)| \le c\|x\|^{-N}.$$

Since $U_F\backslash U_{\mathbb{A}}$ is compact, using (21), this bound enables us to write

$$\begin{aligned} &RTF(f) \\ &= \sum_{\chi\in\mathfrak{X}} \sum_{[(M,\pi)]} \frac{|W_M|}{|W|} \int_{H_F\backslash H_{\mathbb{A}}} \int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi\in \mathrm{ob}(\mathcal{A}_P^\pi)} E(h, I(f,\lambda)\varphi,\lambda)\overline{\mathcal{W}^\theta(E(\varphi,\lambda))}\, d\lambda\, dh \end{aligned} \tag{24}$$

where the inner sum is over equivalence classes of pairs such that π is in the discrete spectrum of $L(M_F\backslash M^1_{\mathbb{A}})_\chi$. Based on the classification of the discrete spectrum of GL_n [MW89], representations in the residual spectrum are not generic. This way Jacquet showed that for a non cuspidal Eisenstein series $\mathcal{W}^\theta(E(\varphi,\lambda))=0$, i.e. that the only terms in (24) that contribute to the RTF are associated with pairs (M,π)

where π is a cuspidal representation of $M_{\mathbb{A}}^1$ (see also [Lap06, Lemma 9.1]). Thus we obtain

$$\begin{aligned} &RTF(f) \\ &\quad = \sum_{[(M,\pi)]} \frac{|W_M|}{|W|} \int_{H_F \backslash H_{\mathbb{A}}} \int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi \in \mathrm{ob}(\mathcal{A}_P^\pi)} E(h, I(f,\lambda)\varphi, \lambda) \overline{\mathcal{W}^\theta(E(\varphi,\lambda))} \, d\lambda \, dh \end{aligned} \tag{25}$$

where the sum is now over all pairs (M, π) up to conjugation, where π is a cuspidal representation of $M(\mathbb{A})^1$. At this point, in order to expand $RTF(f)$ as a sum of relative Bessel distributions, we would formally want to exchange order of integration. Of course, the problem is that the unitary period of an Eisenstein series does not converge. The shift of contour uses an inversion formula for automorphic forms [LR03, Lemma 8.2.1] based on the mixed truncation. At the end of the day, the summand of (25) associated to (M, π) can be written as a sum of integrals of the form

$$\int_{\mathrm{Re}\,\lambda = \lambda_0} \frac{e^{\langle \lambda, T \rangle} F(\lambda)}{\prod_{\varpi \in \tilde{\hat{\Delta}}_{Q_H}} \langle -\lambda, \varpi^\vee \rangle} \, d\lambda \tag{26}$$

where Q ranges over certain parabolic subgroups, λ_0 is a generic point sufficiently close to zero in the negative Weyl chamber of $\mathfrak{a}_L^*$ with respect to Q and F is holomorphic and rapidly decreasing in an appropriate domain. Thus, these integrals converge and roughly speaking, the interchange of integrals with the unitary period is at this stage already been performed. Though the estimates obtained in [Lap06, §6] are used, arguments similar to those in [Mül02] suffice to get to this point. The full power of the majorization of Eisenstein series is used in the next step, getting back to an integral on the unitary axis. The main problem with directly getting back to the unitary axis is that the integrands may have singularities there. For this reason, Lapid introduces in [Lap06, §3] certain improper integrals for a family of meromorphic functions on a vector space. Let V is a real vector space and Λ a set of linearly independent linear functionals on V. For a *tame*, complex valued function F on $V_{\mathbb{C}} = V \otimes \mathbb{C}$ in the sense of [Lap06, §3] and a generic vector $v \in V$ (outside the kernel of each $\lambda \in \Lambda$) the improper integral

$$\int_{\nearrow v} \frac{F(u)}{\prod_{\lambda \in \Lambda} \lambda(u)}$$

is defined. If F is holomorphic and rapidly decreasing then it equals

$$\int_{\mathrm{Re}\,u = v} \frac{F(u)}{\prod_{\lambda \in \Lambda} \lambda(u)}$$

(the improper integrals then regularize the latter for a wider family of functions). Thus (26) can be expressed as an improper integral in $v = \lambda_0$. The improper integrals do not quite depend on the vector v but rather on its connected component with respect to the hyperplanes defined by the kernels of $\lambda \in \Lambda$. Thus, in [Lap06, Lemma 3.3], the relation between the improper integrals for two generic vectors v and v' is given precisely. This is used in [Lap06, §9.3] to express (26) as a sum of improper integrals with respect to a fixed generic point in the positive Weyl chamber of $\mathfrak{a}_M^*$ with respect to P, sufficiently close to zero. Using the majorization of Eisenstein series, Lapid shows in [Lap06, Lemma 7.4] that as a function of λ

$B^L_{(M,\pi)}(f,\lambda)$ is tame. Based on this and using [Lap06, Lemma 3.3] repeatedly he finally expresses (26) as a sum of expressions of the form

$$\int_{i(\mathfrak{a}^{L_H}_{M'})^*} B^L_{(M',\pi)}(f,\lambda)\ d\lambda.$$

Collecting together the terms associated to (M,π,L) whenever (M,π) is θ_ξ elliptic in L the fine spectral expansion is obtained for every K_∞-finite test function f. In fact, in order to compute the combinatorial constants $\mathfrak{c}(M,\pi)$ one has to carefully follow the use of [Lap06, Lemma 3.3]. This seems to be rather complicated and is not carried out in the paper. However, once M is fixed the dependence is only on the type of π, i.e. for π in the form (23), $\mathfrak{c}(M,\pi)$ only depends on the integers k and l. Thus it only receives finitely many possible values and to prove the absolute convergence it is enough to show that

$$\sum_{[(M,\pi)]} \int_{i(\mathfrak{a}^{L_H}_{M})^*} \left| B^L_{(M,\pi)}(f,\lambda) \right|\ d\lambda < \infty.$$

The absolute convergence then follows from bounds obtained in [Mül02, §6]. The fact that the expansion holds for any $f \in C_c^\infty(G_{\mathbb{A}})$ (dropping the K_∞-finiteness assumption) now follows from Lebesgue's dominant convergence theorem, using all the deep analytic bounds mentioned above.

Remark 8. Of course, in order to understand the actual analytic difficulties that occur one will have to read [Lap06]. We do hope, however, that this somewhat metaphorical description of the proof will make the paper of Lapid more approachable to the reader.

4. Spherical functions on Hermitian matrices

In this section F is a non archimedean local field. The symmetric space

$$X = \{g \in G : {}^t\bar{g} = g\}$$

is the space of Hermitian matrices with respect to the quadratic extension E/F. The Hecke algebra $\mathcal{H}_G(K)$ acts on the space $C^\infty(K\backslash X)$ of K-invariant functions on X by the convolution

$$f * \phi(x) = \int_G f(g)\phi(g^{-1}\cdot x)\ dg.$$

Definition 2. A *spherical function* on X is a function $\Omega \in C^\infty(K\backslash X)$ which is an $\mathcal{H}_G(K)$ eigenfunction.

Hironaka studied the spherical functions on X in a series of papers [Hir88a, Hir88b, Hir89, Hir90, Hir99]. When E/F is unramified she obtained explicit formulas for all spherical functions. For a ramified quadratic extension there are only partial results. We will recall her results.

The explicit computation of spherical functions on a reductive p-adic group was first obtained by Macdonald in [Mac71]. His formulas were reproved by Casselman in [Cas80] using the theory of unramified principal series representations. With this new approach Casselman and Shalika obtained explicit formulas for Whittaker spherical functions [CS80] (generalizing Shintani's explicit formulas for GL_n). The method of Casselman-Shalika was then used to obtain explicit formulas for the spherical functions for various other cases of p-adic spaces, e.g.

[HS88, Off04, Hir05b, Sak06]. In a recent work of Sakellaridis [Sak], much of the theory is developed in the general setting of a quasi affine p-adic spherical G-variety for a split reductive group G, although the problem of computing the spherical functions explicitly is not addressed there. Roughly speaking, once the Casselman-Shalika method is applied, there are still three main obstacles to obtaining explicit formulas for the spherical functions. The first obstacle is to obtain an analog of the Cartan decomposition, i.e. a K-orbit decomposition on X. In the case of a p-adic symmetric space, Delorme and Sécherre provided recently a description of the K-orbits [DS]. The second obstacle is to explicitly describe certain functional equations satisfied by the spherical functions. In [Hir05a, Hir06], Hironaka suggests a strategy to reduce the computation of the functional equations to some low rank cases under some assumptions on X in the setting of a spherical G-variety (with G not necessarily split). It follows from [Sak] that some of her assumptions are always satisfied in the setting considered by Sakellaridis. The third obstacle is an explicit computation of certain integrals over a Iwahori subgroup. In many examples (but not in general) those are easy to compute.

Denote by $\overline{B} = T\overline{U}$ the Borel subgroup of lower triangular matrices in G with its standard levi decomposition. Let $d_i(x)$ be the determinant of the $i \times i$ upper left block of $x \in X$. Thus d_i is a regular function on X which is $\overline{B}$-equivariant with respect to the rational character $b \mapsto \mathrm{Nm}(b_1 b_2 \cdots b_i)$ on $\overline{B}$, where $b = \mathrm{diag}(b_1, \dots, b_n)u$ and $u \in \overline{U}$, i.e. $d_i(b \cdot x) = \mathrm{Nm}(b_1 b_2 \cdots b_i) d_i(x)$. Note that the lattice spanned by these n rational characters is of finite index in $X^*(T)$ and that they provide a basis to $\mathfrak{a}^*_{\mathbb{C}}$. There is a unique open $\overline{\mathbf{B}}$-orbit $\mathbf{X}^\circ$ in $\mathbf{X}$ given by

$$\mathbf{X}^\circ = \{x \in \mathbf{X} : d_i(x) \neq 0,\ i = 1, \dots, n\}.$$

The set of rational points $X^\circ = \mathbf{X}^\circ(F)$ consists of finitely many $\overline{B}$ orbits parameterized by the abelian group

$$\Gamma = T' / \operatorname{Nm} T \simeq (F^\times / \operatorname{Nm} E^\times)^n$$

(this is not a coincidence, see [Sak, Corollary 3.3.2] when G is split and X is a quasi affine spherical G-variety). For $a = \mathrm{diag}(a_1, \dots, a_n) \in \Gamma$ we denote by

$$X_a = \{x \in X^\circ : d_i(x) \in a_1 a_2 \cdots a_i,\ i = 1, \dots, n\}$$

the associated $\overline{B}$-orbit. For $a \in \Gamma$ and $s = (s_1, \dots, s_n) \in \mathbb{C}^n$ let

$$\omega_a(x; s) = \int_K \mathbf{1}_{X_a}(k \cdot x) \prod_{i=1}^n |d_i(k \cdot x)|^{s_i} \ dk.$$

Hironaka's spherical functions $\{\omega_a(\cdot; s)\}_{a \in \Gamma}$ form a basis of the space of spherical functions on X with a fixed Hecke eigenvalue depending on s. Let $\lambda = \lambda(s) = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ be such that

$$f * \omega_a(\cdot; s) = \hat{f}(\lambda) \omega_a(\cdot; s)$$

and let

$$\omega_a^\lambda = \omega_a(\cdot; s).$$

Thus, for every Weyl element $w \in W$ the set $\{\omega_a^{w\lambda}\}_{a \in \Gamma}$ forms another basis for the same space of spherical functions and there are therefore matrices $B(w, \lambda) = (B_{a,a'}(w, \lambda))_{a,a' \in \Gamma} \in M_{2^n}(\mathbb{C}(q^\lambda))$ such that

$$(\omega_a^\lambda)_{a \in \Gamma} = B(w, \lambda)(\omega_a^{w\lambda})_{a \in \Gamma}. \tag{27}$$

Applying [Hir99, Theorem 1.9] to this setting Hironaka obtains that

$$(\omega_a^\lambda(x))_{a\in\Gamma} = \frac{1}{Q}\sum_{w\in W} c(w\lambda)B(w,\lambda)(I_a^{w\lambda}(x))_{a\in\Gamma}. \tag{28}$$

Here

$$c(\lambda) = \prod_{i<j} \frac{1-q^{-(\lambda_i-\lambda_j+1)}}{1-q^{-(\lambda_i-\lambda_j)}}$$

and if we denote by $\overline{\mathcal{I}}$ the Iwahori subgroup of K associated with $\overline{B}$ then

$$Q = \sum_{w\in W} \frac{1}{[\overline{\mathcal{I}}w\overline{\mathcal{I}}:\overline{\mathcal{I}}]} = \prod_{i=1}^n \frac{1-q^{-2i}}{1-q^{-2}}$$

and

$$I_a^\lambda(x) = \int_{\overline{\mathcal{I}}} \mathbf{1}_{X_a}(k\cdot x)\prod_{i=1}^n |d_i(k\cdot x)|^{s_i}\ dk.$$

Remark 9. In fact, an analog of formula (28) is obtained in a more general context for certain spherical varieties by using the Casselman-Shalika method. It reduces the explicit computation of spherical functions to the 3 obstacles discussed above. It is enough to compute the spherical functions on a choice of representatives of the K-orbits in X. In many cases, an explicit choice of representatives is made so that the integrals over the Iwahori subgroup are easy to compute explicitly. Hironaka also suggests a general method to reduce the functional equations (27) for a simple reflection w to the case of small prehomogeneous spaces.

Remark 10. In the general context of spherical varieties, it is observed in [Sak, §3.3], (at least when G is split) that the B-orbits in X° are naturally parameterized by an abelian group Γ. Roughly speaking, if $\{\omega_a^\lambda\}_{a\in\Gamma}$ forms a basis of spherical functions of a given Hecke eigenvalue $f\mapsto \hat{f}(\lambda)$, with ω_a^λ supported on the B-orbit associated to a then we can define the stable spherical functions $\omega_\eta^\lambda = \sum_{a\in\Gamma}\eta(a)\omega_a^\lambda$ for every character η on Γ (more precisely, in general the stabilization should be done with respect to a certain subgroup of Γ, see [Sak, §4.4.2]). It then follows from [Sak, Theorem 5.3.1] that the stable spherical functions ω_η^λ and $\omega_\eta^{w\lambda}$ differ only by a scalar, thus for the stable basis of spherical functions, the functional equations should be much simpler. As we shall soon see, Hironaka indeed considered this stabilization in order to obtain the functional equations in the case at hand when X is the space of Hermitian matrices.

The K-orbit decomposition of X was obtained in [Jac62]. If E/F is an unramified quadratic equation then the functional equations (27) are given explicitly in [Hir88b, §2]. If E/F is ramified then the functional equation is given explicitly only for a simple reflection w (see [Hir88b, §3] for odd residual characteristic and [Hir90] when $F=\mathbb{Q}_2$), but note that $B(w_1w_2,\lambda)=B(w_2,\lambda)B(w_1,w_2\lambda)$. In the unramified case the integral $I_a^\lambda(x)$ is easy to compute for a special choice of representative for each of the K-orbits in X. Thus collecting all this information Hironaka obtains the explicit formulas for the spherical functions. In the ramified case, we discuss in...which explicit formulas are available.

In order to describe the functional equations it is more convenient however to introduce a different basis for the spherical functions. Let $\chi=(\chi_1,\dots,\chi_n)$ be a character of T' which is trivial on $\mathrm{Nm}\, T$. Hironaka introduced the spherical functions

$$L(x;\chi;s) = \int_{K'} \prod_{i=1}^{n} |d_i(k\cdot x)|^{s_i}\, \chi_i(d_i(k\cdot x))\; dk$$

where $K' = \{k \in K : k\cdot x \in X^{\circ}\}$. We can think of χ as an element of the dual $\hat{\Gamma}$ of Γ. Yet more convenient is to make a change in the variable χ. For χ as above we let $\nu = (\nu_1, \dots, \nu_n) \in \hat{\Gamma}$ be such that

$$\nu_i = \omega^i \chi_{n+1-i} \cdots \chi_n. \tag{29}$$

For $s \in \mathbb{C}^n$ we can define $\lambda = \lambda(s) = (\lambda_1, \dots, \lambda_n)$ by

$$\lambda_i = \frac{n+1}{2} - i - (s_{n+1-i} + \cdots + s_n). \tag{30}$$

We then set

$$\mathcal{L}(x;\nu;\lambda) = L(x;\chi;s)$$

where λ is related to s by (30) and ν is related to χ by (29). Note then that

$$\mathcal{L}(x;\nu;\lambda) = \sum_{a\in\Gamma} {}^{w_0}(\nu_0\nu)(a)\omega_a^{\lambda}(x)$$

where $\nu_0 = (\omega, \omega^2, \dots, \omega^n)$ and ${}^{w_0}\nu = (\nu_n, \dots, \nu_0)$ and therefore that the basis $\{\mathcal{L}(\cdot;\nu;\lambda)\}_{\nu\in\hat{\Gamma}}$ consists of the stabilized spherical functions in the sense of Remark 10. Note that our change of variables from s to λ is slightly different then Hironaka's from s to z. In what follows we adjust her results accordingly.

4.1. The unramified Hermitian space.

We assume here that E/F is an unramified quadratic extension of p-adic fields. Note then that $\omega = |\cdot|^{\epsilon_0}$ is an unramified character with $\epsilon_0 = \frac{\pi i}{\log q}$ and that

$$L(\mu\omega, s) = L(\mu, s+\epsilon_0) \tag{31}$$

for any character μ of $F^{\times}$. Note further that the spherical functions $\omega_a^{\lambda}(x)$ (and hence also $\mathcal{L}(x;\nu;\lambda)$) depends only on $\lambda \mod 2\epsilon_0\mathbb{Z}^n$ whereas their common Hecke eigenvalue $\hat{f}(\lambda)$ for $f \in \mathcal{H}_K(G)$ depends only on $\lambda \mod \epsilon_0\mathbb{Z}^n$.

The K-orbit decomposition on S is given by the disjoint union

$$S = \sum_{m\in\Lambda_n} K\cdot\varpi^m$$

where

$$\Lambda_n = \{m = (m_1, \dots, m_n) \in \mathbb{Z}^n : m_1 \geq \cdots \geq m_n\}$$

and $\varpi^m = \mathrm{diag}(\varpi^{m_1}, \dots, \varpi^{m_n})$. The integrand for $I_a^{\lambda}(w_0\cdot\varpi^m)$ is in fact constant on $\overline{\mathcal{I}}$ and we therefore get that for $m \in \Lambda_n$ we have

$$I_a^{\lambda}(w_0\cdot\varpi^m) = \mathbf{1}_{X_a}(w_0\cdot\varpi^m)q^{\langle m,\lambda-\rho\rangle}. \tag{32}$$

This is [Hir99, Lemma 2.1]. Note that

$$L(x;\chi,s) = L(x;\mathbf{1}_{\Gamma};s+\epsilon(\chi))$$

where $\epsilon(\chi) = (\epsilon(\chi_1), \dots, \epsilon(\chi_n))$ and

$$\epsilon(\mu) = \begin{cases} 0 & \mu = 1 \\ \epsilon_0 & \mu = \omega \end{cases}$$

and that $\lambda(s+\epsilon(\chi)) \equiv \lambda(s) - \epsilon(\nu\nu_0) \mod 2\epsilon_0\mathbb{Z}^n$ where ν is related to χ by (29). Thus

$$\mathcal{L}(x;\nu;\lambda) = \mathcal{L}(x;\nu_0;\lambda - \epsilon(\nu\nu_0)) \tag{33}$$

and it is therefore enough to compute $\mathcal{L}(x;\nu_0;\lambda) = \sum_{a\in\Gamma} \omega_a^\lambda$. Let

$$\tau(\nu;\lambda) = \prod_{i<j} \frac{L(\nu_i\nu_j^{-1}, \lambda_i - \lambda_j + 1)}{L(\nu_i\nu_j^{-1}\omega, \lambda_i - \lambda_j)}.$$

With our notation Hironaka obtains in [Hir88b, §2] the functional equation

$$\tau(\nu_0;\lambda)\mathcal{L}(x;\nu_0,\lambda) = \tau(w\nu_0;w\lambda)\mathcal{L}(x;w\nu_0,w\lambda)$$

for every $w \in W$. Applying (31) and (33) this functional equation generalizes. For any $\nu \in \hat{\Gamma}$, $\lambda \in \mathbb{C}^n$ and $w \in W$ we have

$$\tau(\nu;\lambda)\mathcal{L}(x;\nu,\lambda) = \tau(w\nu;w\lambda)\mathcal{L}(x;w\nu,w\lambda). \tag{34}$$

This functional equation is the one obtained in [Hir99, p. 570] for $L(x;\chi;s)$. Our normalization of variables is more natural and simplifies the functional equation. Let

$$\Delta^w = ({}^{w_0}(\nu_0{}^w\nu)(a))_{\nu\in\hat{\Gamma}\,;\,a\in\Gamma} \text{ and } T^w(\lambda) = (\delta_{\nu,\nu'}\tau({}^w\nu, w\lambda))_{\nu,\,\nu'\in\hat{\Gamma}}$$

and denote $\Delta = \Delta^1$ and $T(\lambda) = T^1(\lambda)$. From the functional equation (34) we get that

$$\Delta B(w,\lambda) = T(\lambda)^{-1}T^w(\lambda)\Delta^w.$$

It follows from (32) that for $m \in \Lambda_n$ we have

$$\Delta^w(I_a(w_0\cdot\varpi^m))_{a\in\Gamma} = ((\nu_0{}^w\nu)(\varpi^m)q^{\langle m,w\lambda-\rho\rangle})_{\nu\in\hat{\Gamma}}.$$

It is now convenient to denote by $Z = (Z_1,\dots,Z_n)$ the variable given by

$$Z_i = Z_i(\nu;\lambda) = (-1)^{\epsilon(\nu_i)}q^{\lambda_i}.$$

Applying Δ to both sides of (28) after some cancelation we obtain

$$\mathcal{L}(\varpi^m;\nu;\lambda) = \frac{1}{Q}\nu_0(\varpi^m)q^{-\langle m,\rho\rangle}\prod_{i<j}\frac{Z_i - q^{-1}Z_j}{Z_i + Z_j}\sum_{w\in W} w\left(Z^m \prod_{i<j}\frac{Z_i + q^{-1}Z_j}{Z_i - Z_j}\right)$$

where $Z^m = \prod_{i=1}^n Z_i^{m_i}$. This can be expressed in terms of the mth Hall-Littlewood polynomial:

$$P_m(Z_1,\dots,Z_n;t) = \frac{(1-t)^n}{V_m(t)}\sum_{w\in W} w\left(Z^m\prod_{i<j}\frac{Z_i - tZ_j}{Z_i - Z_j}\right).$$

It is well known that this is a symmetric Laurant polynomial in Z. Here the combinatorial factor $V_m(t)$ is determined by the requirement that $P_m(Z;t)$ is monic (i.e. that the leading monomial symmetric polynomial, that associated to m, has coefficient one). Explicitly, let $v_n(t) = \prod_{i=1}^n (1-t^i)$ thus,

$$V_m(t) = \prod_j v_{N_i(\lambda)}(t)$$

where $N_i(\lambda) = \#\{j : 1 \le j \le n,\, m_j = i\}$. In fact $\{P_m(Z;t) : m \in \Lambda_n\}$ forms a basis of $\mathbb{Z}[t][Z_1^{\pm 1}, \dots, Z_n^{\pm 1}]^W$. We then have

$$\begin{gathered}\mathcal{L}(\varpi^m;\nu;\lambda) = \\ \nu_0(\varpi^m) q^{-\langle m,\rho\rangle}(1-q^{-1})^n \frac{V_m(-q^{-1})}{v_n(q^{-2})}\left[\prod_{i<j}\frac{Z_i - q^{-1}Z_j}{Z_i+Z_j}\right] P_m(Z_1,\dots,Z_n;-q^{-1}).\end{gathered} \tag{35}$$

Next Hironaka defines the spherical Fourier transform. Again, we define the transform slightly differently so that it will be more convenient to apply with our notation. We consider the normalized spherical function

$$\Omega(x;\nu;\lambda) = \frac{\mathcal{L}(x;\nu;\lambda)}{\mathcal{L}(e;\nu;\lambda)}.$$

Thus for $m \in \Lambda_n$ we have

$$\Omega(\varpi^m;\nu;\lambda) = \nu_0(\varpi^m) q^{-\langle m,\rho\rangle}\frac{V_m(-q^{-1})}{v_n(-q^{-1})} P_m(Z;-q^{-1}).$$

Let $\mathcal{H}_K(X) = C_c^\infty(K\backslash X)$ be the $H_K(G)$ module of compactly supported K-invariant functions on X. The analog for X of the Satake transform is defined by

$$\hat{\varphi}(\nu;\lambda) = \int_X \varphi(x)\Omega(x^{-1};\nu;\lambda)\, dx. \tag{36}$$

Theorem 10 (Theorem 2, [Hir99])**.** *The spherical Fourier transform* (36) *defines an isomorphism of* $\mathcal{H}_K(G)$*-modules*

$$\mathcal{H}_K(X) \simeq \mathbb{C}[Z_1^{\pm 1}, \dots, Z_n^{\pm 1}]^W.$$

For later reference we note that the characteristic functions $\mathbf{1}_{K\cdot\varpi^m}$, $m \in \Lambda_n$ form a basis of $\mathcal{H}_K(X)$ and

$$\hat{\mathbf{1}}_{K\cdot\varpi^m}(\lambda) = \mathrm{vol}(K\cdot\varpi^m)\Omega_\lambda(\varpi^{\tilde{m}})$$

where we recall that as in §2 $\tilde{m} = (-m_n, \dots, -m_1)$. Hironaka computed the volume of every K-orbits using explicit formulas for certain local densities and obtained that

$$\mathrm{vol}(K\cdot\varpi^m) = q^{2\langle m,\rho\rangle}\frac{v_n(-q^{-1})}{V_m(-q^{-1})}.$$

Since $P_{\tilde{m}}(Z;t) = P_m(Z^{-1};t)$ where $Z^{-1} = (Z_1^{-1}, \dots, Z_n^{-1})$ we therefore obtain

$$\hat{\mathbf{1}}_{K\cdot\varpi^m}(\lambda) = ({}^{w_n}\nu_0)(\varpi^m) q^{\langle m,\rho\rangle} P_m(Z^{-1};-q^{-1}). \tag{37}$$

For $m \in \Lambda_n$ we denote by σ_m the associated monomial symmetric polynomial

$$\sigma_m(Z_1,\dots,Z_n) = \sum_{w\in W} Z^{wm}.$$

Recall that the functions $\Phi^{(k)} \in \mathcal{H}_{G'}(K')$ and $\Psi^{(k)} \in \mathcal{H}_X(K')$ were introduced in §2. For $m \in \mathbb{Z}^n$ we let $|m| = m_1 + \cdots + m_n$.

Lemma 1. *For every integer* $k \ge 0$ *we have*

$$\widehat{(-1)^k\Psi^{(k)}}(\nu;\lambda) = \widehat{\Phi^{(k)}}(\nu;\lambda) = q^{k\frac{n-1}{2}}\sum_{m\in\Lambda_n^+(k)}\sigma_m(Z^{-1})$$

where $\Lambda_n^+(k) = \{m \in \Lambda_n : m_n \ge 0$ *and* $|m| = k\}$.

Proof. We first note that it is easy to compute $\hat{\Phi}^{(k)}$ explicitly. Using the Iwasawa decomposition $G = UTK$ we get that

$$\widehat{\Phi^{(k)}}(\nu;\lambda) = \sum_{m\in\mathbb{Z}^n} q^{\langle\rho,m\rangle} \int_U \Phi^{(k)}(u\varpi^m)\ du Z^{-m}.$$

Let

$$M^+(k) = \{m \in \mathbb{Z}^n : m_1, \dots, m_n \geq 0 \text{ and } |m| = k\}.$$

Note that

$$\int_U \Phi^{(k)}(u\varpi^m) = \begin{cases} q^{\sum_{i=1}^n (i-1)m_i} & m \in M^+(k) \\ 0 & \text{else} \end{cases}$$

and that

$$q^{\langle\rho,m\rangle + \sum_{i=1}^n (i-1)m_i} = q^{|m|\frac{n-1}{2}}.$$

We therefore indeed see that

$$\widehat{\Phi^{(k)}}(\nu;\lambda) = q^{k\frac{n-1}{2}} \sum_{m\in\Lambda_n^+(k)} \sigma_m(Z^{-1}).$$

On the other hand we have

$$\Phi^{(k)} = \sum_{m\in\Lambda_n^+(k)} \mathbf{1}_{K\varpi^m K}.$$

From Macdonald's computation of the spherical functions (e.g. [Mac95, p. 299]) we have

$$\hat{\mathbf{1}}_{K\varpi^m K}(\nu;\lambda) = q^{\langle\rho,m\rangle} P_m(Z^{-1};q^{-1}) = q^{|m|\frac{n-1}{2}} (q^{-1})^{\sum_{i=1}^n (i-1)m_i} P_m(Z^{-1};q^{-1}).$$

We see that the identity

$$\sum_{m\in\Lambda_n^+(k)} \sigma_m(Z^{-1}) = \sum_{m\in\Lambda_n^+(k)} (q^{-1})^{\sum_{i=1}^n (i-1)m_i} P_m(Z^{-1};q^{-1})$$

holds for infinitely many values of q and therefore that the algebraic identity

$$\sum_{m\in\Lambda_n^+(k)} \sigma_m(Z^{-1}) = \sum_{m\in\Lambda_n^+(k)} t^{\sum_{i=1}^n (i-1)m_i} P_m(Z^{-1};t) \tag{38}$$

holds. Since

$$\Psi^{(k)} = \sum_{m\in\Lambda_n^+(k)} \mathbf{1}_{K\cdot\varpi^m}$$

using (37) we see that

$$\widehat{\Psi^{(k)}}(\nu;\lambda) = q^{k\frac{n-1}{2}} \sum_{m\in\Lambda_n^+(k)} ({}^{w_n}\nu_0)(\varpi^m)(q^{-1})^{\sum_{i=1}^n (i-1)m_i} P_m(Z^{-1};-q^{-1}).$$

Since $\omega(\varpi) = -1$ we see that

$$({}^{w_n}\nu_0)(\varpi^m)(q^{-1})^{\sum_{i=1}^n (i-1)m_i} = (-1)^k(-q^{-1})^{\sum_{i=1}^n (i-1)m_i}.$$

It follows from (38) that

$$\widehat{\Psi^{(k)}}(\nu;\lambda) = (-1)^k q^{k\frac{n-1}{2}} \sum_{m\in\Lambda_n^+(k)} \sigma_m(Z^{-1}).$$

□

5. BESSEL IDENTITIES FOR PRINCIPAL SERIES REPRESENTATIONS

In this section we recall and refine the results of [Off]. The main result is a local identity between a distribution on X and a distribution on G'. The notation in [Off] was slightly different. The space

$$Y = \{g \in G : gw_n{}^t\bar{g}^{-1}w_n^{-1} = e\}$$

was considered as a left G-space with the action $g \cdot y = gyw_n{}^t\bar{g}w_n^{-1}$. The map $x \mapsto (w_n x)^{-1}$ defines an isomorphism of the right G space X with the left G space Y and $H^x = \{g \in G : g \cdot y = y\}$ for x and y corresponding by this isomorphism. With this isomorphism in mind, we shall present the results of [Off] using the notation of this work without making any further remarks.

We start with the main local result. Assume then that E/F is a quadratic extension of local fields of characteristic zero. For a character $\nu = (\nu_1, \dots, \nu_n)$ of T' and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ we let

$$\gamma(\nu, \lambda, \psi) = \prod_{i<j} \gamma(\nu_i \nu_j^{-1}\omega, \lambda_i - \lambda_j, \psi)$$

where for a character μ of $F^\times$ and $s \in \mathbb{C}$ we let

$$\gamma(\mu, s, \psi) = \frac{L(\mu, s)}{\epsilon(\mu, s, \psi)L(\mu^{-1}, 1-s)}$$

be the Tate γ-factor. Let χ be a unitary character of T_F which is a base change from T'_F, i.e. it factors through the norm map. Thus the set $\mathcal{B}(\chi)$ of characters ν of T'_F such that $\chi = \nu \circ \mathrm{Nm}$ is not empty. Recall that $\Gamma = T'_F / \mathrm{Nm}(T_F)$. Let $x \in X_F$, $a \in \Gamma$ and $\nu \in \mathcal{B}(\chi)$. If the G_F orbit of x does not contain a we set $J_\nu^x(a, \varphi, \lambda) = 0$. Otherwise, let $\eta \in G_F$ and $t \in a$ be such that ${}^t\bar{\eta}x\eta = t$ and let $H_\eta^x = H^x \cap \eta B \eta^{-1}$. The linear functional $J_\nu^x(a, \varphi, \lambda)$ defined for $\varphi \in I(\chi)$ and $\lambda \in \mathbb{C}^n$ so that $\mathrm{Re}\,\lambda$ lies in some positive cone by the convergent integral

$$J_\nu^x(a, \varphi, \lambda) = (\nu_0\nu)(t)e^{\frac{1}{2}\langle\lambda+\rho, H(t)\rangle} \int_{H_\eta^x \backslash H^x} e^{\left\langle\lambda, H(\eta^{-1}h)\right\rangle}\varphi(\eta^{-1}h)\, dh$$

is independent of the choices of η and t. We define the stable linear functional

$$J^{st,x}(\nu, \varphi, \lambda) = \sum_{a \in \Gamma} J_\nu^x(a, \varphi, \lambda).$$

Using Bernstein's principle of analytic continuation, the proof of [LR00, Proposition 2] shows that if E/F is a quadratic extension of p-adic fields then $J^{st,x}(\nu, \varphi, \lambda)$ admits a meromorphic continuation to a rational function in q^λ, where q is the size of the residual field of F. In the archimedean case, the meromorphic continuation then follows from an analog global statement that we shall soon come to. We also denote by $W^\psi(g, \varphi, \lambda)$ the Whittaker function defined by

$$W^\psi(g, \varphi, \lambda) = \int_U \varphi(w_n u g)\bar{\psi}_U(u)\, du$$

and let $\mathcal{W}^\psi(\varphi, \lambda) = W^\psi(g, \varphi, \lambda)$. The local stable relative Bessel distribution is defined for $\Psi \in C_c^\infty(X)$ by

$$B^{st}(\Psi, \nu, \lambda) = \sum_{\varphi \in \mathrm{ob}(I(\chi))} \int_X \Psi(x) J^{st,x}(\nu, \varphi, \lambda)\, dx \cdot \overline{\mathcal{W}^\psi(\varphi, -\bar{\lambda})}.$$

For any $\nu \in \mathcal{B}(\chi)$ and $\varphi' \in I'(\nu)$ we denote by ${W'}^{\psi}(g', \varphi', \lambda)$ the Whittaker function

$$ {W'}^{\psi}(g, \varphi, \lambda) = \int_{U'} \varphi'(w_n u g) \bar{\psi}_{U'}(u) \ du. $$

The local Bessel distribution is defined for $\Phi \in C_c^\infty(G')$ by

$$ B'(\Phi, \nu, \lambda) = \sum_{\varphi' \in \mathrm{ob}(I(\nu))} W^{\psi}(w_n, I'(\Phi, \lambda)\varphi', \lambda)\overline{{\mathcal{W}'}^{\psi}(\varphi, -\bar{\lambda})}. $$

Theorem 11. *There exists a root of unity $\kappa_{E/F} = \kappa_{E/F}(\psi, n)$ so that for any unitary character ν of T' and matching functions $\Phi \overset{\psi}{\longleftrightarrow} \Psi$ we have*

$$ B^{st}(\Psi, \nu, \lambda) = \kappa_{E/F}\gamma(\nu, \lambda, \psi) B'(\Phi, \nu, \lambda). $$

If $\psi' = \psi(a\cdot)$ for some $a \in F^\times$ then

$$ \kappa_{E/F}(\psi', n) = \omega(a)^{\frac{(n-1)n}{2}} \kappa_{E/F}(\psi, n). $$

Furthermore, if E/F is unramified and ψ has conductor $\mathcal{O}_F$ then $\kappa_{E/F}(\psi, n) = 1$.

As already explained in the introduction, this local identity enables us to express the left hand side of (1) explicitly in terms of Hironaka's spherical functions. When $n = 3$ the result was first obtained in [LR00]. The results of Jacquet, Lapid and Hironaka in §2-§4 enable us to apply the method of Lapid and Rogawski and prove Theorem 11 for general n. The proof uses global methods and we also prove an analog global identity. Before explaining the method of proof, let us first recall the global analog identity of Bessel distributions.

We turn to a global setting. Thus E/F denotes a quadratic extension of number fields. Let χ be a unitary character of $T' \backslash T'_{\mathbb{A}}$ which is in the image of base change, i.e. so that the set $\mathcal{B}(\chi)$ of characters ν of $T \backslash T_{\mathbb{A}}$ such that $\chi = \nu \circ \mathrm{Nm}$ is not empty. For $x \in X_{\mathbb{A}}$, $\nu \in \mathcal{B}(\chi)$ and $\varphi = \otimes_v \varphi_v \in I(\chi)$ we define the stable intertwining period $J^{x,st}(\varphi, \nu, \lambda)$ by the meromorphic continuation of

$$ J^{x,st}(\varphi, \nu, \lambda) = \prod_v J^{x_v, st}(\varphi_v, \nu_v, \lambda). $$

We shall soon see that the right hand side is an Euler product convergent in some positive cone and admits a meromorphic continuation in λ. The global stable relative Bessel distribution is defined for $\Psi \in C_c^\infty(X_{\mathbb{A}})$ by

$$ B^{st}(\Psi, \nu, \lambda) = \sum_{\varphi \in \mathrm{ob}(I(\chi))} \int_{X_{\mathbb{A}}} \Psi(x) J^{x,st}(\varphi, \nu, \lambda) \ dx \cdot \overline{\mathcal{W}^{\psi}(\varphi, -\bar{\lambda})}. $$

The Bessel distribution is defined for $\Phi \in C_c^\infty(G'_{\mathbb{A}})$ by

$$ B'(\Phi, \nu, \lambda) = \sum_{\varphi' \in \mathrm{ob}(I(\nu))} W^{\psi}(w_n, I(\Phi, \lambda)\varphi', \lambda)\overline{\mathcal{W}^{\psi}(\varphi, -\bar{\lambda})} $$

Theorem 12. *Let ν be a unitary character of $T \backslash T_{\mathbb{A}}$. For $\Phi \in C_c^\infty(G'_{\mathbb{A}})$ and $\Psi \in C_c^\infty(X_{\mathbb{A}})$ such that $\Phi \overset{\psi}{\longleftrightarrow} \Psi$ we have*

$$ B^{st}(\Psi, \nu, \lambda) = B'(\Phi, \nu, \lambda). $$

The stable relative Bessel distribution $B^{st}(\Psi,\nu,\lambda)$ contributes to the most continuous part of Lapid's spectral expansion for the RTF, i.e. to the summands in (22) where the integration is over the $(n-1)$-dimensional space $i(\mathfrak{a}_0^G)^*$. Let $\{\xi\}$ be a set of representatives for the G orbits in X. Recall that for a function $\Psi \in C_c^\infty(X_{\mathbb{A}})$ we associate a family $\{f^\xi\}$ of functions in $C_c^\infty(G_{\mathbb{A}})$ so that

$$\Psi({}^t\bar{g}\xi g) = \int_{H_{\mathbb{A}}^\xi} f^\xi(hg)\ dh. \tag{39}$$

As already mentioned in §3, the distribution RTF on $X_{\mathbb{A}}$ can be expressed as a sum of the distributions RTF_ξ on $G_{\mathbb{A}}$, the spectral expansion of which is described in §3. The most continuous part of the spectrum is the distribution

$$\sum_\chi \int_{i(\mathfrak{a}_0^G)^*} B(\Psi,\chi,\lambda)\ d\lambda$$

where the sum is over all unitary Hecke characters χ of $T_{\mathbb{A}}$ that lie in the image of base change from $T'_{\mathbb{A}}$. The relative Bessel distribution $B(\Psi,\chi,\lambda)$ is defined by

$$B(\Psi,\chi,\lambda) = \sum_\xi \sum_{\varphi\in \mathrm{ob}(I(\chi))} \Pi^{H^\xi}(E(I(f^\xi,\lambda)\varphi,\lambda)))\cdot \overline{\mathcal{W}^\psi(\varphi,-\bar{\lambda})}$$

where for an automorphic form ϕ on $G_{\mathbb{A}}$ we denote by $\Pi^{H^\xi}(\phi)$ the regularized period integral over $H^\xi\backslash H^\xi_{\mathbb{A}}$ as defined in [LR03]. We also recall the definition of certain global intertwining periods from [LR03]. We now denote by Γ the group $T'/\operatorname{Nm}T$. For every place v of F we denote by Γ_v the analog local group $T'_v/\operatorname{Nm}T_v$ and let $\Gamma_{\mathbb{A}} = \oplus_v\Gamma_v$. The group Γ naturally embeds in $\Gamma_{\mathbb{A}}$ in the diagonal and it follows from class field theory that $\#(\Gamma_{\mathbb{A}}/\Gamma) = 2^n$. Let $x\in X$ and let $\Gamma(x) = \{a\in\Gamma : \exists g\in G,\ {}^t\bar{g}xg\in a\}$. For $a\in\Gamma(x)$ and λ such that $\operatorname{Re}\lambda$ is sufficiently large we define

$$J^x(a,\varphi,\lambda) = \int_{(H^x_\eta)_{\mathbb{A}}\backslash H^x_{\mathbb{A}}} e^{\langle\lambda,H(\eta^{-1}h)\rangle}\varphi(\eta^{-1}h)\ dh$$

where $\eta\in G$ is such that ${}^t\bar{g}xg\in a$. This integral converges and is independent of the choice of η but is not expected to have a meromorphic continuation in λ. In [LR03, Theorem 9.1.1] Lapid and Rogawski express the regularized integral of an Eisenstein series in terms of intertwining periods. In our setting their result gives

$$\Pi^{H^x}(E(\varphi,\lambda)) = 2^n \sum_{a\in\Gamma(x)} J^x(a,\varphi,\lambda). \tag{40}$$

Thus the (infinite) sum of intertwining periods on the right hand side does admit a meromorphic continuation in λ. A simple argument in Fourier analysis of finite groups gives that for any factorizable, absolutely sumable function $g(a) = \prod_v g_v(a_v)$ on $\Gamma_{\mathbb{A}}$ we have

$$2^n\sum_{a\in\Gamma} g(a) = \sum_{\kappa\in(\Gamma_{\mathbb{A}}/\Gamma)^*}\prod_v \hat{g}_v(\kappa_v)$$

where the sum on the right is over the 2^n characters κ of $\Gamma_{\mathbb{A}}/\Gamma$, κ_v is the restriction of κ to Γ_v and $\hat{g}_v(\kappa_v) = \sum_{a\in\Gamma_v} g_v(a_v)$. Applying this to the function

$$g(a) = \begin{cases} J^x(a,\varphi,\lambda) & a\in\Gamma(x)\\ 0 & \text{else}\end{cases}$$

the equality (40) becomes

$$\Pi^{H^x}(E(\varphi,\lambda))=\sum_{\nu\in\mathcal{B}(\chi)}J^{st,x}(\nu,\varphi,\lambda). \tag{41}$$

Thus we express the regularized period of an Eisenstein series associated to a principal series representation as a finite sum of factorizable linear functionals. Using the fact that the local period $J^{st,x_v}(\nu_v,\varphi_v,\lambda)$ in a p-adic place v is meromorphic it is not difficult to deduce from (41) that the stable intertwining period $J^{st,x}(\nu,\varphi,\lambda)$ (and hence also each archimedean factor) admits a meromorphic continuation to $\lambda\in\mathbb{C}^n$ (see [Off, Lemma 6]). For $x\in X_{\mathbb{A}}$ and $f\in C_c^\infty(G_{\mathbb{A}})$ we also define the stable relative Bessel distribution

$$B^{st,x}(f,\nu,\lambda)=\sum_{\varphi\in\mathrm{ob}(I(\chi))}J^{st,x}(I(f,\lambda)\varphi,\nu,\lambda)\cdot\overline{\mathcal{W}(\varphi,-\bar{\lambda})}.$$

Using formal manipulation of integrals and following the definitions we get that

$$B^{st}(\Psi,\nu,\lambda)=\sum_{[\xi]_{G_{\mathbb{A}}}\in X_{\mathbb{A}}/G_{\mathbb{A}}}B^{st,\xi}(f^\xi,\nu,\lambda)$$

where Ψ and $\{f^\xi\}$ are related by (39). If $\nu=(\nu_1,\dots,\nu_n)\in\mathcal{B}(\chi)$ we let $\omega\nu=(\omega\nu_1,\dots,\omega\nu_n)\in\mathcal{B}(\chi)$. It is also easy to see that for $x\in X_{\mathbb{A}}$ we have

$$J^{st,x}(\varphi,\omega\nu,\lambda)=\omega(\det x)J^{st,x}(\varphi,\nu,\lambda).$$

It follows from the local to global principle for Hermitian forms that $\omega(\det x)=1$ if and only if the $G_{\mathbb{A}}$-orbit of x contains a rational point, i.e. that $[x]_{G_{\mathbb{A}}}\cap X\neq\emptyset$. We therefore see that

$$\sum_{\nu\in\mathcal{B}(\chi)}B^{st}(\Psi,\nu,\lambda)=\sum_{\nu\in\mathcal{B}(\chi)}\sum_{[\xi]_G\in X/G}B^{st,\xi}(f^\xi,\nu,\lambda).$$

Combined with (41) we now get that

$$B(\Psi,\chi,\lambda)=\sum_{\nu\in\mathcal{B}(\chi)}B^{st}(\Psi,\nu,\lambda).$$

Thus, for a principal series representation $I(\chi)$ of $G_{\mathbb{A}}$ in the image of quadratic base change from $G'_{\mathbb{A}}$, its contribution is expressed as a finite sum of factorizable distributions. Note that $B'(\Phi,\nu,\lambda)$ is the contribution of the principal series representation $I(\nu)$ of $G'_{\mathbb{A}}$ to the KTF. The content of Theorem 12 is that the contribution of $I(\chi)$ to the RTF can be written as a finite sum of stable distributions parameterized by $\mathcal{B}(\chi)$ so that for every ν we obtain a term wise comparison of distributions.

We now recall form [Off] some related local results that will help us explain how the Bessel identities are proved. We start with the split case, i.e. when $E=F\otimes F$. In this case $G=G'\times G'$ and $X=\{({}^tg,g):g\in G'\}$ is a unique G-orbit. A unitary character of $T=T'\times T'$ is in the image of base change if and only if it is of the form $\chi=(\nu,\nu)$ for a character ν on T' and in this case $\mathcal{B}(\chi)=\{\nu\}$. For $\Psi\in C_c^\infty(X)$ let $\Phi\in C_c^\infty(G')$ be defined by

$$\Phi(g)=\Psi({}^tg,g) \tag{42}$$

then $\Phi\xleftrightarrow{\psi}\Psi$.

Lemma 2. *For every Φ and Ψ as in (42) we have*

$$B^{st}(\Psi,\nu,\lambda)=\gamma(\nu,\lambda,\psi)B'(\Phi,\nu,\lambda).$$

The proof relies on a functional equation of Shahidi for Whittaker functions [Sha81]. It also uses the fact that the stable intertwining period in the split case satisfies

$$J^{st,({}^t g,g)}(\varphi_1' \otimes \varphi_2', \nu, \lambda) = \int_{K'} \varphi_1'(k)(\vartheta \circ M(w_n, \lambda) \circ I'(w_n g, \lambda))(\varphi_2')(k)\ dk$$

for $\varphi_1' \otimes \varphi_2' \in I(\nu) \otimes I(\nu) = I(\chi)$, from which it follows that

$$B^{st}(\Psi, \nu, \lambda) = \\ \sum_{\varphi_i', \varphi_j' \in \mathrm{ob}(I'(\nu))} ((\vartheta \circ M(w_n, \lambda) \circ I'(w_n, \lambda) \circ I'(\Phi, \lambda))\varphi_j', \overline{\varphi_i'})\ \overline{\mathcal{W}'^{\psi}(\varphi_i')}\ \overline{\mathcal{W}'^{\psi}(\varphi_j')}.$$

Here $\vartheta(\varphi)(g) = \varphi(w_n {}^t g^{-1} w_n^{-1})$.

We now turn to the case where E/F is an unramified quadratic extension of p-adic fields. The following is a generalization of [Off, Proposition 5].

Lemma 3. *Assume that ψ has conductor $\mathcal{O}_F$ and let $\Phi \in \mathcal{H}_{G'}(K')$ and $\Psi \in \mathcal{H}_X(K)$ be such that $\hat{\Phi} = \hat{\Psi} \in \mathbb{C}[Z^{\pm 1}]^W$. Then, for every unramified unitary charcter ν of T' and $\lambda \in \mathbb{C}^n$ we have*

$$B^{st}(\Psi, \nu, \lambda) = \gamma(\nu, \lambda, \psi) B'(\Phi, \nu, \lambda).$$

Proof. Let $\chi = \nu \circ \mathrm{Nm}$. It follows from the definitions that for $\varphi \in I(\chi)$ we have

$$J^{st, {}^t g x g}(\varphi, \nu, \lambda) = J^{st,x}(I(g, \lambda)\varphi, \nu, \lambda) \tag{43}$$

and since Ψ is K-invariant we then have

$$\int_X \Psi(x) J^{st,x}(\varphi, \nu, \lambda)\ dx = 0$$

whenever $(\varphi, \varphi_\chi) = 0$. It follows that

$$B^{st}(\Psi, \nu, \lambda) = \int_X \Psi(x) J^{st,x}(\varphi_\chi, \nu, \lambda)\ dx\ \overline{\mathcal{W}(\varphi_\chi, -\bar{\lambda})}.$$

We show in [Off, §3.2] that

$$J^{st,x}(\varphi, \nu, \lambda) = (1 - q^{-1})^{-n} \frac{v_n(q^{-2})}{v_n(-q^{-1})} \mathcal{L}(x^{-1}; \nu; \lambda).$$

This gives an interpretation of the stable period at the unramified section in terms of Hironaka's spherical functions. It is a straight forward computation to obtain this identity up to a constant depending only on the G-orbit of x. That the constant is independent of the G-orbits then follows from the algebraic nature of the consistent measures on the different unitary groups. The constant is then computed using an asymptotic formula for the intertwining periods. It follows that

$$B^{st}(\Psi, \nu, \lambda) = \hat{\Psi}(\nu, \lambda)\ (1 - q^{-1})^{-n} \frac{v_n(q^{-2})}{v_n(-q^{-1})} \mathcal{L}(e; \nu; \lambda)\ \overline{\mathcal{W}(\varphi_\chi, -\bar{\lambda})}.$$

On the other hand

$$B'(\Phi, \nu, \lambda) = \hat{\Phi}(\nu, \lambda)\ \mathcal{W}'(\varphi_\nu, \lambda)\ \overline{\mathcal{W}'(\varphi_\nu, -\bar{\lambda})}.$$

Thus the lemma follows from the identity

$$(1 - q^{-1})^{-n} \frac{v_n(q^{-2})}{v_n(-q^{-1})} \mathcal{L}(e; \nu; \lambda)\ \overline{\mathcal{W}(\varphi_\chi, -\bar{\lambda})} = \gamma(\nu, \lambda, \psi)\ \mathcal{W}(\varphi_\nu, \lambda)\ \overline{\mathcal{W}(\varphi_\nu, -\bar{\lambda})}.$$

Using (35) and the Shintani, Casselman-Shalika formula for the spherical whittaker functions we have

$$\begin{aligned}(1-q^{-1})^{-n}\frac{v_n(q^{-2})}{v_n(-q^{-1})}\mathcal{L}(e;\nu;\lambda)\ \overline{\mathcal{W}(\varphi_\chi,-\bar{\lambda})}\\ =\prod_{i<j}\frac{L(\nu_i\nu_j^{-1}\omega,\lambda_i-\lambda_j)}{L(\nu_i\nu_j^{-1},\lambda_i-\lambda_j+1)}L(\chi_j\chi_i^{-1},\lambda_j-\lambda_i+1)^{-1}\end{aligned}$$

whereas

$$\begin{aligned}&\gamma(\nu,\lambda,\psi)\ \mathcal{W}(\varphi_\nu,\lambda)\ \overline{\mathcal{W}(\varphi_\nu,-\bar{\lambda})}\\ &=\prod_{i<j}\frac{L(\nu_i\nu_j^{-1}\omega,\lambda_i-\lambda_j)}{L(\nu_j\nu_i^{-1}\omega,\lambda_j-\lambda_i+1)}L(\nu_i\nu_j^{-1},\lambda_i-\lambda_j+1)^{-1}L(\nu_j\nu_i^{-1},\lambda_j-\lambda_i+1)^{-1}.\end{aligned}$$

Since $L(\mu,s)L(\mu\omega,s)=L(\mu\circ\mathrm{Nm},s)$ for every character μ of $F^\times$ the identity follows. ☐

Note that the fundamental lemma of Jacquet in the form of Theorem 6 is valid even if we replace the transfer factor $\gamma(a)$ by the transfer factor $\omega(\det a)\gamma(a)$ in the definition of matching, since the functions involved are supported in the kernel of $\omega\circ\det$. This is the transfer factor dichotomy. In [Off, Theorem 2 and Theorem 3] we show that there exists $\delta=\delta(n)\in\{0,1\}$ depending only on n, so that the Bessel identities of Theorem 11 and Theorem 12 hold for functions Φ and Ψ matching with respect to the transfer factor $\omega(\det a)^\delta\gamma(a)$ (we shall then write $\Phi\overset{\delta}{\longleftrightarrow}\Psi$). The matching (15) for odd k, however, only holds with the transfer factor $\gamma(a)$. This fact, together with Lemma 3 and Lemma 1 solves the transfer factor dichotomy. Indeed, it shows that if there exists δ for which the Bessel identities hold for $\Phi\overset{\delta}{\longleftrightarrow}\Psi$ (i.e. if we have [Off, Theorem 2 and Theorem 3]) then $\delta=0$ (i.e. we have the more refined Theorem 11 and Theorem 12). Furthermore, Lemma 3 can be viewed as a spectral version of the fundamental lemma. Motivated by it I suggested

Conjecture 1 ([Off06]). *Let $\Phi\in\mathcal{H}_{G'}(K')$ and $\Psi\in\mathcal{H}_X(K)$ be such that $\hat{\Phi}=\hat{\Psi}$ then $\Phi\longleftrightarrow\Psi$.*

The Hecke algebra $\mathcal{H}_{G'}(K')$ is a free $\mathcal{H}_G(K)$-module of rank 2^n via the Satake transform. Similarly, convolution defines an $\mathcal{H}_G(K)$-module structure on $\mathcal{H}_X(K)$. The respective spherical Fourier transforms of Satake and Hironaka identify between the two $\mathcal{H}_G(K)$-modules. Theorem 6 proves the conjecture for pairs $\Phi\longleftrightarrow\Psi$ that lie respectively in the rank one submodules $\mathcal{H}_G(K)\cdot\Phi^{(0)}$ and $\mathcal{H}_G(K)\cdot\Psi^{(0)}$. The more general matching (15) proves the conjecture for other $\mathcal{H}_G(K)$-orbits, but even when $n=2$ not for all orbits. For $n=2$, the conjecture is proved in [Off06]. From now on, we explain the proof of [Off, Theorem 2 and Theorem 3].

Let us go back to a global setting. Lapid and Rogawski generalized in [LR00, Lemma 4] the argument of Langlands in [**?**, §11], of linear independence of characters, to suit the setting of the RTF. From the fundamental Lemma of Jacquet and the fine spectral expansion of Lapid for the RTF it can then be shown that

$$\sum_{\nu\in\mathcal{B}(\chi)}B^{st}(\Psi,\nu,\lambda)=\sum_{\nu\in\mathcal{B}(\chi)}B'(\Phi,\nu,\lambda)\tag{44}$$

for every unitary Hecke character χ of $T_{\mathbb{A}}$ that lies in the image of base change and matching functions $\Phi \overset{\psi}{\longleftrightarrow} \Psi$. Indeed, as in the proof of [LR00, Proposition 6] the identity (44) can be proved up to a combinatorial constant $\mathfrak{c}$ that depends only on n (the constant $\mathfrak{c}(T,\chi)$ in (22) does not depend on χ and not even on the quadratic field extension E/F). Let

$$\alpha(\chi,\lambda)=\prod_v \alpha_v(\chi_v,\lambda)=\prod_{i<j} L(\chi_j\chi_i^{-1},\lambda_j-\lambda_i+1).$$

From the unramified local computations it follows that at least when $\operatorname{Re}\lambda$ is sufficiently large each summand of (44) is a factorizable distribution, indeed we can write for $\operatorname{Re}\lambda$ sufficiently large

$$B^{st}(\Psi,\nu,\lambda)=\alpha(\chi,\lambda)^{-1}\prod_v \alpha_v(\chi_v,\lambda)\tilde{B}^{st}(\Psi_v,\nu_v,\lambda)$$

and

$$B'(\Phi,\nu,\lambda)=\alpha(\chi,\lambda)^{-1}\prod_v \gamma(\nu_v,\lambda,\psi_v)\alpha_v(\chi_v,\lambda)\tilde{B}'(\Phi_v,\nu_v,\lambda).$$

Choosing an extension E/F which is split at all archimedean places and unramified at all finite places it follows from Lemma 2 and Lemma 3 that $\mathfrak{c}=1$. In [Off] we used the fact that $\mathfrak{c}=1$ without justification. From the global identity (44) we can deduce the following identity on the summands.

Lemma 4 (Corollary 3 [Off])**.** *Let E/F be a quadratic extension of number fields which is split at all real places of F, let $\delta\in\{0,1\}$ and let χ be a unitary Hecke character of $T_{\mathbb{A}}$ that is a base change from $T'_{\mathbb{A}}$. There exists a permutation $\tau=\tau_{\delta,\chi}$ on $\mathcal{B}(\chi)$ so that*

$$B^{st}(\Psi,\nu,\lambda)=B'(\Phi,\tau(\nu),\lambda)$$

whenever $\Phi \overset{\delta}{\longleftrightarrow} \Psi$.

To deduce Lemma 4 from (44) we proceed as in [LR00]. Using the localization principle of Gelfand-Kazdhan [GK75] Jacquet observed that, for an inert p-adic place v of F, the local Bessel distribution on G'_v depends only on the orbital integrals of the functions and may therefore be viewed as a distribution on X_v via matching. Since this is also the case at the split places, under the assumption of Lemma 4 on E/F, (44) may be viewed as an identity between distributions on $X_{\mathbb{A}}$. The key for deducing the termwise identity from (44) is a lemma in linear algebra [LR00, Lemma 5]. We now recall its content. Let V_1, V_2 and V_3 be vector spaces. Let $\{x_i^j\}_{i=1}^m$ and $\{y_i^j\}_{i=1}^m$ be two sets of m vectors in V_i so that the vectors $\{x_i^j\}_{i=1}^m$ are linearly independent for $i=1,2,3$ and such that

$$\sum_{i=1}^m x_i^1\otimes x_i^2\otimes x_i^3=\sum_{i=1}^m y_i^1\otimes y_i^2\otimes y_i^3.$$

Then there exists a permutation τ of $\{1,2,\dots,m\}$ such that

$$y_i^1\otimes y_i^2\otimes y_i^3=x_{\tau(i)}^1\otimes x_{\tau(i)}^2\otimes x_{\tau(i)}^3$$

for all $i=1,\dots,m$. In [Off, Lemma 8], we show that for an inert p-adic place v of F the local relative Bessel distributions $\{B^{st}(\cdot,\nu_v,\lambda)\}_{\nu_v\in\mathcal{B}(\chi_v)}$ on X_v are linearly independent. This allows us to apply [LR00, Lemma 5] using two auxiliary p-adic inert places of F and obtain Lemma 4 from (44).

Using this global identity we first prove the local Bessel identity in the p-adic case. The idea is as follows. Fix a quadratic extension E^0/F^0 of p-adic fields. We embed the local p-adic setting in a favorable global setting. There exists a quadratic extension of number fields E/F so that the set S^0 of all places v of F so that $E_v/F_v \simeq E^0/F^0$ is not empty and so that for all $v \notin S^0$ the extension E_v/F_v is either split or an unramified quadratic extension of p-adic fields for p odd. For $\delta \in \{0,1\}$ and a unitary Hecke character χ of $T_{\mathbb{A}}$ which is a base change from $T'_{\mathbb{A}}$ let $\tau_{\delta,\chi}$ be the permutation on $\mathcal{B}(\chi)$ given by Lemma 4. Since $B'(\omega \circ \det\ \Phi, \nu, \lambda) = B'(\Phi, \omega\nu, \lambda)$ and $\Phi \overset{\delta}{\longleftrightarrow} \Psi$ if and only if $\omega \circ \det\ \Phi \overset{1-\delta}{\longleftrightarrow} \Psi$, it follows from the linear independence of the local p-adic relative Bessel distributions that

$$\tau_{1-\delta,\chi}(\nu) = \omega\tau_{\delta,\chi}(\nu) \tag{45}$$

for $\nu \in \mathcal{B}(\chi)$. We now pick an odd prime p relatively prime to the residual characteristic of F^0 and so that denoting by S^p the set of all places of F with residual characteristic p, the character ψ_v has conductor $\mathcal{O}_v$ for all $v \in S^p$. For a finite set of finite places S of F we denote by $\mathcal{U}_S$ the set of unramified unitary characters of T'_S.

Lemma 5 (Lemma 11 [Off]). *There is a non empty open set $U^p \subset \mathcal{U}_{S^p}$ so that if ν is a unitary Hecke character of $T'_{\mathbb{A}}$ such that $\nu_{S^p} \in U^p$ then $\tau_{\delta,\chi}(\nu) \in \{\nu, \omega\nu\}$.*

The lemma is proved by writing explicitly the identity of Lemma 4 for a choice of decomposable matching functions $\Phi \overset{\delta}{\longleftrightarrow} \Psi$ so that $\Phi_v = \mathbf{1}_{K'_v}$ and $\Psi_v = \mathbf{1}_{X_v \cap K_v}$ for almost all v. Using the fundamental Lemma of Jacquet and the explicit Bessel identities in the split and unramified cases we obtain an identity between two Dirichlet series. Comparing the p part of the Dirichlet series we show that $\tau_{\delta,\chi}(\nu) \notin \{\nu, \omega\nu\}$ provides a non trivial closed condition on ν_{S^p}.

Let U^p be the set of characters given by Lemma 5. It follows from (45) that for every ν such that $\nu_{S^p} \in U^p$ there exists $\delta = \delta(\nu) \in \{0,1\}$ such that $\tau_{\delta,\chi}(\nu) = \nu$. In order to apply Lemma 5, we use a Lemma of Lapid-Rogawski on the distributions of Hecke characters.

Lemma 6 (Corollary 2 [LR00]). *Let S_f be a finite set of finite places of F and let $S = S_\infty \cup S_f$. Given a place $w \notin S$, a unitary character $\eta = (\eta_v)_{v \in S_f}$ of T'_{S_f} and an open set $U \subset \mathcal{U}_{S_f}$, there exists a Hecke character ϱ of $T'_{\mathbb{A}}$ which is unramified outside $S \cup \{w\}$ such that $\varrho_{S_f}^{-1}\eta \in U$.*

We apply Lemma 6 with $S_f = S^0 \cup S^p$, $\eta = (\eta_v)_{v \in S_f}$ with $\eta_v = \mathbf{1}_{T'_v}$ for $v \in S^p$ and $\eta_v = \mu$ for a fixed character μ of $(F^0)^\times$ for all $v \in S^0$, $U = U^0 \times U^p$ where U^0 will soon be specified and w is a place of F which is split in E. Thus, there exists a unitary Hecke character ν of $T'_{\mathbb{A}}$ such that $\nu_{S_f} = \eta$. In particular we have $\tau_{\delta(\nu),\chi}(\nu) = \nu$. It follows from the global identity of Bessel distributions that there exists constants $\kappa_v(\nu_v, \lambda)$ such that

$$B^{st}(\Psi_v, \nu_v, \lambda) = \kappa_v(\nu_v, \lambda)\gamma(\nu_v, \lambda_v, \psi)B'(\Phi_v, \nu_v, \lambda)$$

whenever $\Phi_v \overset{\delta(\nu)}{\longleftrightarrow} \Psi_v$. From the split and unramified explicit identities we get that $\kappa_v(\nu_v, \lambda) = 1$ for $v \notin S^0$ and therefore the global identity implies that

$$\prod_{v \in S^0} \kappa_v(\nu_v, \lambda) = 1. \tag{46}$$

Note that for $v \in S^0$ there exists $\lambda_v \in i\mathbb{R}^n$ such that $nu_v = \mu\left|\cdot\right|^{\lambda_v}$ and the function $\kappa_v(\nu_v,\lambda) = \kappa_v(\mu,\lambda{+}\lambda_v)$ is a rational function in $q_{F^0}^{\lambda}$. Assume now that the function $\kappa_v(\mu,\lambda)$ is not constant in λ. Then (46) is a closed condition on ν_{S^0} and we may choose $U^0 \subset \mathcal{U}_{S^0}$ for which (46) does not hold. This will contradict Lemma 6. Thus we see that $\kappa_v(\mu,\lambda) = \kappa_v(\mu)$ is a $\left|S^0\right|$-root of unity independent of λ. Repeating the argument, but changing η_v to be a different character μ_1 of $(F^0)^\times$ only at a single place $v \in S^0$ we will get that $\kappa(\mu)^{\left|S^0\right|-1}\kappa(\mu_1) = 1$ and therefore that $\kappa(\mu)$ is independent of μ. We therefore obtain the desired p-adic Bessel identity for $\delta(\mu)$-matching functions. To see that $\delta(\mu)$ is independent of μ, given any two quadratic extensions of p-adic fields E_i/F_i and characters ν_i for $i = 1, 2$ there is a quadratic extension of number fields E/F, places v_i of F and a Hecke character ν of $T'_{\mathbb{A}}$ such that $E_{v_i}/F_{v_i} \simeq E_i/F_i$ and $\nu_{v_i} = \nu_i$. This shows that $\delta(\nu_1) = \delta(\nu) = \delta(\nu_2)$. This proves the local Bessel identity [Off, Theorem 3] in the p-adic case. The archimedean case now follows from (44) applied to $E/F = \mathbb{Q}[\sqrt{-1}]/\mathbb{Q}$, using the linear independence of the Bessel distributions. Similarly, [Off, Theorem 3] and (44) now prove the global Bessel identity [Off, Theorem 2].

6. A formula for an anisotropic unitary period of certain cusp forms

We now explain how the results of §2-§5 are applied to obtain Theorem 2. Fix a CM-extension E/F, an everywhere unramified cuspidal automorphic representation π of $G^1_{\mathbb{A}}$ which is a base change from $(G'_{\mathbb{A}})^1$ and a cuspidal automorphic representation π' such that $\pi = \mathrm{bc}(\pi')$. Then $\pi' \not\simeq \pi' \otimes \omega$ and we also have $\pi = \mathrm{bc}(\pi' \otimes \omega)$. We denote by ϕ_0 the everywhere unramified L^2-normalized cusp form in the space of π. Let $\alpha \in X$ be such that $\alpha_v = \pm{}^t\bar{\theta}_v\theta_v$ for some $\theta_v \in G_v$, is either positive or negative definite for every real place v of F and let $\mathbf{H} = \mathbf{H}^\alpha$ be the associated unitary group. For a finite place v of F let $\theta_v = e$ and let $\theta = (\theta_v) \in G_{\mathbb{A}}$. For a cuspidal automorphic form ϕ on $G_{\mathbb{A}}$ we denote by $P^H(\phi)$ the anisotropic unitary period

$$P^H(\phi) = \int_{H\backslash H_{\mathbb{A}}} \phi(h)\ dh.$$

Our goal is to compute $\left|P^H(\pi(\theta)\phi_0)\right|^2$.

The RTF of Jacquet is taking all unitary periods into consideration. Since we focus on a single unitary group $H_{\mathbb{A}}$, we may simplify the RTF and consider it as a distribution on $G_{\mathbb{A}}$ as follows. We choose a set of representatives $\{\xi\}$ for the G-orbits in X so that α is the representative of its orbit. For every $\xi \neq \alpha$ we set $f^\xi = 0$ and we let $f = f^\alpha \in C_c^\infty(G_{\mathbb{A}})$. If we express the RTF as a sum over the orbits as in (20), we obtain

$$RTF(\Psi) = RTF_\alpha(f)$$

where Ψ is supported on the $G_{\mathbb{A}}$-orbit of α and $\Psi({}^t\bar{g}\alpha g) = \int_{H_{\mathbb{A}}} f(hg)\ dh$ for $g \in G_{\mathbb{A}}$. For a functions $f' \in C_c^\infty(G'_{\mathbb{A}})$, we shall also say that f and f' have matching orbital integrals for ψ and write

$$f' \overset{\psi}{\longleftrightarrow} f \text{ whenever } f' \overset{\psi}{\longleftrightarrow} \Psi.$$

From the fundamental lemma of Jacquet and the linear independence of characters of Lapid-Rogawski we obtain the identity

$$\tilde{B}_\pi(f) = B_{\pi'}(f') + B_{\pi'\otimes\omega}(f')$$

whenever $f' \xleftrightarrow{\psi} f$. Note that in order to apply [LR00, Lemma 4] we make full use of the matching in Theorem 6. The matching of the Hecke unit element is not enough! If in addition we assume that *supp*$(f') \subset \ker(\omega \circ \det)$ then $B_{\pi'}(f') = B_{\pi' \otimes \omega}(f')$ and we obtain the identity of Bessel distributions

$$\tilde{B}_\pi(f) = 2B_{\pi'}(f'). \tag{47}$$

Fix $g \in G_{\mathbb{A}}$ such that $W^\psi(\phi_0, g) \neq 0$. Let S be a finite set of places of F containing all real places, all even places all inert ramified places and such that for all $v \notin S$ the character ψ_v has conductor $\mathcal{O}_v$ and g_v, $\alpha_v \in K_v$. Consider a function $f \in C_c^\infty(G_{\mathbb{A}})$ of the form

$$f = \prod_{v \in S} f_v \prod_{v \notin S} \mathbf{1}_{K_v}$$

where f_v is a bi-K_v-invariant function for all $v \in S$. Let $f_\theta^g(x) = f(\theta x g)$, $x \in G_{\mathbb{A}}$. Based on Jacquet's smooth matching at the finite places and the fundamental lemma for the Hecke unit element it is not hard to choose a function $f' \in C_c^\infty(G'_{\mathbb{A}})$ of the form

$$f' = \prod_{v \in S} f'_v \prod_{v \notin S} \mathbf{1}_{K'_v}$$

with support contained in $\ker(\omega \circ \det)$ and such that

$$f' \xleftrightarrow{\psi} f_\theta^g \tag{48}$$

(see [LO, §4] for the choice of f'_v when v is real). Choosing a basis $\mathrm{ob}(\pi)$ containing $\pi(g)\phi_0$ and taking the K-invariance of f into consideration we get that

$$\tilde{B}_\pi(f) = \hat{f}_S(\pi_S) P^H(\pi(\theta)\phi_0) \overline{W^\psi(\phi_0, g)} \tag{49}$$

where

$$\hat{f}_S(\pi_S) = \prod_{v \in S} \hat{f}_v(\pi_v)$$

is the spherical Fourier transform of f. Applying (47) to the pair of matching functions (48), we obtain from (49) that

$$\left|P^H(\pi(\theta)\phi_0)\right|^2 = 4 \left|\hat{f}_S(\pi_S) W^\psi(\phi_0, g)\right|^{-2} \left|B_{\pi'}(f')\right|^2. \tag{50}$$

6.1. **Application of a formula of Jacquet for the inner product of cusp forms.** Jacquet showed in [Jac01] that the Bessel distribution $B_{\pi'}(f')$ is decomposable. This observation is based on a formula he obtained for the inner product of cusp forms on $G'_{\mathbb{A}}$ that, in turn, is based on a Rankin-Selberg integral that expresses the inner product in terms of Whittaker functions [JS81]. We recall the formula here and refer to [LO, §2.2] for details.

In the local setting, for an irreducible, generic, unitary representation π' of G' let $\mathcal{W}^\psi(\pi')$ be the Whittaker model of π'. An inner product on $\mathcal{W}^\psi(\pi')$ was given by Bernstein in the non archimedean case [Ber84] and by Baruch in the archimedean case [Bar03] by the formula

$$[W_1, W_2] = \mathfrak{d}_F^{1-n} L(n, \mathbf{1}_{F^\times}) \int_{U'_{n-1} \backslash G'_{n-1}} W_1(\mathrm{diag}(g,1)) \overline{W_2(\mathrm{diag}(g,1))}\, dg. \tag{51}$$

Note the normalization factor outside the integral, which appears for our convenience. We define the local Bessel distribution

$$B_{\pi'}^{\psi'}(f') = \mathfrak{B}^{\delta_{w_n}, \delta_e, [\cdot,\cdot]}_{\mathcal{W}^{\psi'}(\pi'), \mathcal{W}^{\psi'}(\pi')}(f')$$

where $\delta_g(W) = W(g)$.

Globally, for a cusp form ϕ in the space of $\pi' = \otimes_v \pi'_v$ which is a pure tensor, we may write $W^{\psi'}(\phi, g) = \prod_v W_v(g_v)$ with $W_v \in \mathcal{W}^{\psi'_v}(\pi_v)$ and $W_v(e) = 1$ almost everywhere. Let S be a finite set of places containing the archimedean places, so that for $v \notin S$, π_v is unramified, ψ'_v has conductor $\mathcal{O}_v$, W_v is spherical and $W_v(e) = 1$. Then

$$(\phi, \phi)_{G' \backslash (G'_{\mathbb{A}})^1} = \operatorname{Res}_{s=1} L^S(s, \pi' \times \tilde{\pi}') \prod_{v \in S} [W_v, W_v].$$

We get that

$$\left| W'^{\psi}(g, \phi) \right|^2 = \frac{(\phi, \phi)_{G' \backslash (G'_{\mathbb{A}})^1}}{\operatorname{Res}_{s=1} L^S(s, \pi' \times \tilde{\pi}')} \prod_{v \in S} \left| W^{\psi'}_{1,v}(g_v) \right|^2 \tag{52}$$

where $W^{\psi'}_{1,v}$ is a spherical Whittaker function of π' normalized so that $[W^{\psi'}_1, W^{\psi'}_1] = 1$. The inner product formula also gives rise to the decomposition of the Bessel distribution. Let $f' = \otimes_{v \in S} f'_v \otimes_{v \notin S} \mathbf{1}_{K'_v}$ then

$$B^{\psi'}_{\pi'}(f') = \frac{1}{\operatorname{Res}_{s=1} L^S(s, \pi' \times \tilde{\pi}')} \prod_{v \in S} B^{\psi'_v}_{\pi'_v}(f'_v). \tag{53}$$

In order to evaluate $B^{\psi'}_{\pi'}(f')$ in the setting of (50), we will apply the local Bessel identities. Note that locally, if $\pi' = I'(\nu, \lambda)$ is a principal series representation then the distributions $B^{\psi'}_{\pi'}(\cdot)$ and $B^{\psi'}(\cdot, \nu, \lambda)$ are normalized differently. The normalizing factor was computed in [LO, Proposition 1] and the remark following Proposition 1. If $\lambda \in \mathbb{C}^n$ is such that $|\operatorname{Re} \lambda_i| < \frac{1}{2}$ (which is the case, in particular, when π' is unitary) then the integral in (51) converges and defines a G'-invariant pairing between $\mathcal{W}^{\psi'}(\pi')$ and $\mathcal{W}^{\psi'}((\pi')^*)$ where $(\pi')^*$ denotes the conjugate contragredient of π'. We showed that for such λ and for φ'_1, $\varphi'_2 \in I'(\nu, \lambda)$ we have

$$(\varphi'_1, \varphi'_2) = \frac{[W^{\psi}(\varphi'_1, \lambda), W^{\psi}(\varphi'_2, -\bar{\lambda})]}{L(1, \mathbf{1}_{F^\times})^n}. \tag{54}$$

This implies that

$$B^{\psi'}(f', \nu, \lambda) = L(1, \mathbf{1}_{F^*})^n \cdot \mathfrak{B}^{\delta_{w_n}, \delta_e, [\cdot,\cdot]}_{\mathcal{W}^{\psi'}(\pi'), \mathcal{W}^{\psi'}((\pi')^*)}(f').$$

In particular, if $I'(\nu, \lambda)$ is unitary then

$$B^{\psi'}(f', \nu, \lambda) = L(1, \mathbf{1}_{F^\times})^n B^{\psi}_{I(\nu,\lambda)}(f). \tag{55}$$

We also note that for the unramified section (54) gives

$$\left| W^{\psi'}(\varphi'_\nu, -\bar{\lambda}, g) W^{\psi'}(\varphi'_\nu, \lambda, g) \right| = L(1, \mathbf{1}_{F^*})^n \left| W^{\psi'}_1(g) \right|^2. \tag{56}$$

6.2. **Application of the local Bessel identities.** Here E/F is a quadratic extension of local fields. We make as explicit as we can the Bessel distribution in the following setting. Let $\pi = I(\lambda)$ be a unitary, unramified principal series representation of G and let $\nu \in \mathcal{B}(\mathbf{1}_T)$. Thus, $\pi' = I'(\nu, \lambda)$ is such that $\pi = b(\pi')$. Once again, we focus on a single unitary group and we therefore consider the relative Bessel distribution on G defined by

$$B^{st,\alpha}(f, \nu, \lambda) = \mathfrak{B}^{J^{st,\alpha}(\cdot,\nu,\lambda), \mathcal{W}(\cdot,-\bar{\lambda}), (\cdot,\cdot)}_{\pi,\pi^*}.$$

Note that if Ψ is supported on the G-orbit of α and $\Psi({}^t g\alpha g) = \int_H f(hg)\,dh$ then

$$B^{st}(\Psi,\nu,\lambda) = B^{st,\alpha}(f,\nu,\lambda).$$

For θ, $g \in G$ and a Hecke function $f \in \mathcal{H}_K(G)$ let $f^g_\theta(y) = f(\theta y g)$, $y \in G$ and let $f' \in C^\infty_c(G')$ be such that $f' \xleftrightarrow{\psi} f^g_\theta$. Combining (55) with Theorem 11 we get that

$$B^\psi_{I'(\nu,\lambda)}(f') = (L(1,\mathbf{1}_{F^\times})^n \kappa_{E/F}\gamma(\nu,\lambda,\psi))^{-1}\tilde{B}^{st,\alpha}(f^g_\theta,\nu,\lambda).$$

Since f is bi-K-invariant, choosing an orthonormal basis containing $\pi(g)\varphi_{\mathbf{1}_T}$ we see that

$$\begin{aligned}(57)\quad B^{st,\alpha}(f^g_\theta,\nu,\lambda) &= \hat{f}(\lambda)J^{st,\alpha}(I(\theta,\lambda)\varphi_{\mathbf{1}_T},\nu,\lambda)\overline{W^\psi(g,\varphi_{\mathbf{1}_T},-\bar{\lambda})}\\ &= \hat{f}(\lambda)J^{st,e}(\varphi_{\mathbf{1}_T},\nu,\lambda)\overline{W^\psi(g,\varphi_{\mathbf{1}_T},-\bar{\lambda})}.\end{aligned}$$

The latter equality follows from (43). Since $I'(\nu,\lambda)$ is unitary it is isomorphic to $I'(\nu,-\bar{\lambda})$ and therefore $B^\psi_{I'(\nu,\lambda)}(f')$ is invariant under $\lambda \mapsto -\bar{\lambda}$. It follows that

$$B^\psi_{I'(\nu,\lambda)}(f') = (\kappa_{E/F}L(1,\mathbf{1}_{F^\times})^n\gamma(\nu,\lambda,\psi))^{-1}\overline{\hat{f}(\lambda)}J^{st,e}(\varphi_{\mathbf{1}_T},\nu,\lambda)W^\psi(g,\varphi_{\mathbf{1}_T},-\bar{\lambda})$$

and

$$\overline{B^\psi_{I'(\nu,\lambda)}(f')} = \kappa_{E/F}(L(1,\mathbf{1}_{F^\times})^n\gamma(\nu,-\lambda,\bar{\psi}))^{-1}\overline{\hat{f}(\lambda)}J^{st,e}(\varphi_{\mathbf{1}_T},\nu,-\lambda)W^\psi(g,\varphi_{\mathbf{1}_T},\lambda).$$

Observe that since

$$\varepsilon(s,\omega,\psi)\varepsilon(-s,\omega,\overline{\psi}) = (\frac{\mathfrak{d}_F}{\mathfrak{d}_E})^2$$

we also have

$$\gamma(\nu,\lambda,\psi)\gamma(\nu,-\lambda,\bar{\psi}) = (\frac{\mathfrak{d}_F}{\mathfrak{d}_E})^{(n-1)n}\prod_{i,\,j=1}^n \frac{L(\lambda_i-\lambda_j,\omega\nu_i\nu_j)}{L(\lambda_i-\lambda_j+1,\omega\nu_i\nu_j)}$$

and therefore

$$\frac{L(1,\mathbf{1}_{E^*})^n}{L(1,\mathbf{1}_{F^*})^{2n}}\frac{1}{\gamma(\nu,\lambda,\psi)\gamma(\nu,-\lambda,\bar{\psi})} = \frac{L(0,\omega)^n}{L(0,\pi'\times\tilde{\pi}'\otimes\omega)}\frac{L(1,\pi'\times\tilde{\pi}'\otimes\omega)}{L(1,\mathbf{1}_{F^*})^n}.$$

Applying (56) to the group G we get that

$$\begin{aligned}(58)\quad \left|B_{I'(\nu,\lambda)}(f')\right|^2 = &\frac{L(0,\omega)^n}{L(0,\pi'\times\tilde{\pi}'\otimes\omega)}\frac{L(1,\pi'\times\tilde{\pi}'\otimes\omega)}{L(1,\mathbf{1}_{F^*})^n}\\ &\left|\hat{f}(\lambda)v\right|^2(\frac{\mathfrak{d}_F}{\mathfrak{d}_E})^{(n-1)n}J^{st,e}(\varphi_{\mathbf{1}_T},\nu,\lambda)J^{st,e}(\varphi_{\mathbf{1}_T},\nu,-\lambda)\left|W^\psi_1(g)\right|^2.\end{aligned}$$

6.3. **Proof of Theorem 2.** Since π is everywhere unramified, each component of $\pi' = \otimes_v\pi'_v$ is a unitary, principal series representation $\pi'_v = I'(\nu_v,\lambda_v)$ with $\nu_v \in \mathcal{B}(\mathbf{1}_{T_v})$. Thus π'_v is unramified for every place $v \notin S$ and we may then assume that $\nu_v = \mathbf{1}_{T'_v}$. In order to obtain the formula for $\left|P^H(\pi(\theta)\phi_0)\right|^2$ we now only need to collect together the results we have already recalled. We apply (52) to the group G, the factorization (53) of the Bessel distribution and (58). Plugging these results into (50) we get that

$$\left|P^H(\pi(\theta)\phi_0)\right|^2 = 4\frac{\operatorname{Res}_{s=1}L^S(s,\pi\times\tilde{\pi})}{\operatorname{Res}_{s=1}L^S(s,\pi'\times\tilde{\pi}')^2}(*...*)\prod_{v\in S}J^{st,e}_v(\varphi_{\mathbf{1}_T},\nu_v,\lambda_v)J^{st,e}_v(\varphi_{\mathbf{1}_T},\nu_v,-\lambda_v).$$

7. A generalization of Jacquet's explicit Kloosterman identities

In this section E^0/F^0 is an unramified quadratic extension of p-adic fields. Based on Hironaka's spherical Fourier transform, we show that the fundamental lemma of Jacquet can be generalized to an explicit matching of orbital integrals for every element of the Hecke algebra $\mathcal{H}_{G'}(K')$.

Theorem 13. *Let* $\Phi \in \mathcal{H}_{G'}(K')$ *and* $\Psi \in \mathcal{H}_X(K)$ *be such that* $\hat{\Phi} = \hat{\Psi}$. *If* ψ *has conductor* $\mathcal{O}_{F^0}$ *then* $\Phi \overset{\psi}{\longleftrightarrow} \Psi$.

Remark 11. In the theorem, there is no restriction on the characteristic of the residual field of F^0. In particular, this proves the fundamental lemma in the case of even residual characteristic. The proof is global using the RTF and it relies on the main local result of [Jac05].

Proof. A function $\Phi \in C_c^\infty(G'(F^0))$ is called supercuspidal if for all g_1, $g_2 \in G'(F^0)$ and V the unipotent radical of a parabolic subgroup of G' we have

$$\int_V \Phi(g_1 v g_2)\ dv = 0.$$

Denote by $C_{sc}(G')$ the space of supercuspidal functions on $G'(F^0)$. We have a direct sum decomposition $C_c^\infty(G'(F^0)) = C_{sc}(G') \oplus C_{sc}(G')^\perp$. Where the orthogonal complement is taken with respect to the standard inner product

$$\langle \Phi, \Phi' \rangle_{G'} = \int_{G'} \Phi(g)\bar{\Phi}'(g)\ dg.$$

Lemma 7. *Let* (π', V) *be a supercuspidal representation of* $G'(F^0)$ *then for all* $\Phi \in C_{sc}(G')^\perp$ *and* $v \in V$ *we have* $\pi'(\Phi)v = 0$.

Proof. Let $v \in V$ and $\tilde{v} \in \tilde{V}$. Then

$$\langle \pi'(\Phi)v, \tilde{v} \rangle = \int_{G'} \Phi(g) f_{v,\tilde{v}}(g)\ dg$$

where

$$f_{v,\tilde{v}}(g) = \langle \pi'(g)v, \tilde{v} \rangle$$

is the associated matrix coefficient. Since π' is supercuspidal and since $C_{sc}(G')$ is stable under complex conjugation there exists $\Phi' \in C_{sc}(G')$ such that

$$f_{v,\tilde{v}}(g) = \int_{Z_{G'}} \bar{\Phi}'(zg)\omega_{\pi'}^{-1}(z)\ dz.$$

Note also that $C_{sc}(G')$ is invariant under translations and therefore

$$\langle \pi'(\Phi)v, \tilde{v} \rangle = \int_{Z_{G'}} \langle \Phi, \Phi'_z \rangle_{G'}\, \omega_{\pi'}(z)^{-1}\ dz = 0$$

where Φ'_z is the translate of Φ' by z. □

Lemma 8. *There exists* $\Phi \in C_{sc}(G'(F^0))$ *and* $a \in T'$ *such that* $\Omega[\Phi, \psi; a] \neq 0$.

Proof. For the proof we will use the global KTF. Let F be a number field and v_0 a place of F such that $F^0 \simeq F_{v_0}$. Let π' be a cuspidal automorphic representation of $G'_{\mathbb{A}}$ such that π'_{v_0} is a supercuspidal representation. Since π'_{v_0} has a Whittaker model it follows that the local Bessel distribution $B_{\pi'_{v_0}}$ is not identically zero. From Lemma 7 it then follows that there exists $\Phi_{v_0} \in C_{sc}(G'_{v_0})$ such that $B_{\pi'_{v_0}}(\Phi_{v_0}) \neq 0$.

If $\Phi = \otimes_v \Phi_v \in C_c^\infty(G'_{\mathbb{A}})$ is any decomposable test function with a supercuspidal factor at some finite place then

$$KTF(\Phi) = \sum_\pi B_\pi(\Phi)$$

where the sum is over all cuspidal representations of $G'_{\mathbb{A}}$. Let S be a finite set of places of F containing v_0 and all the archimedean places and such that ψ_v has conductor $\mathcal{O}_v$ and π'_v is unramified for $v \notin S$. For every $v \in S \setminus \{v_0\}$ let $\Phi_v \in C_c^\infty(G')$ be such that $B_{\pi'_v}(\Phi_v) \neq 0$ and if $v|\infty$ then Φ_v is supported in the open Bruhat cell $\overline{B}'_v w_n U'_v$ (see the lemma of Shalika for the infinite places...). For $v \notin S$ we let $\Phi_v \in \mathcal{H}_{G'}(K')$ be arbitrary. As in Jacquet's paper, there are only finitely many irreducible, cuspidal, representations π of $G'_{\mathbb{A}}$ such that $\pi_\infty = \pi'_\infty$ and such that $B_\pi(\Phi) \neq 0$. Let them be $\pi_1, \dots, \pi_l$. It then follows from the strong multiplicity one that there are places $v_1, \dots, v_l$ not in S such that $(\pi_i)_{v_i} \not\simeq \pi'_{v_i}$ and therefore there are Hecke functions $\Phi_{v_i} \in \mathcal{H}_{G'_{v_i}}(K'_{v_i})$ such that $\hat{\Phi}_{v_i}((\pi_i)_{v_i}) = 0$ and $\hat{\Phi}_{v_i}(\pi'_{v_i}) \neq 0$. For such Φ we claim that $RTF(\Phi) \neq 0$. Indeed, by our choice of Φ, summing up only over cuspidal representation with infinite component π'_∞ we have

$$\sum_{\pi_\infty = \pi'_\infty} B_\pi(\Phi) = B_{\pi'}(\Phi) \neq 0$$

and therefore by the infinite linear independence lemma of Jacquet ([Jac05, Lemma 1]) we have

$$RTF(\Phi) = \sum_\pi B_\pi(\Phi) \neq 0$$

where the sum is over all cuspidal representations. On the other hand, by our choice of Φ_v at the infinite places we have

$$RTF(\Phi) = \sum_{a \in T'} \Omega[\Phi; \psi; a].$$

The lemma follows. □

In order to prove the local matching we embed the setting once more into a global setting and apply the RTF identity. We will also need the following global lemma.

Lemma 9. *Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $G_{\mathbb{A}}$ and let $\Psi = \Psi_v \otimes \Psi^v$ be such that $\Psi_v \in \mathcal{H}_{X_v}(K_v)$ then if $\tilde{B}_\pi(\Psi) \neq 0$ then π_v is unramified.*

Proof. A straight forward manipulation shows that

$$\tilde{B}_\pi(\Psi) = \sum_{\phi \in \mathrm{ob}(\pi)} \int_{G \backslash G_{\mathbb{A}}} \left[\sum_{x \in X} \Psi({}^t\bar{g} x g) \right] \phi(g)\, dg.$$

In particular, the summand associated to ϕ is the same as the summand associated to the function

$$g \mapsto \int_{K_v} \phi(gk)\, dk.$$

If π_v is not unramified then the latter function is zero for all Φ in the space of *pi*. □

Let E/F be a quadratic extension of number fields that is split at all infinite places and such that there exist a place v_{rel} of F such that $E_{v_{\mathrm{rel}}}/F_{v_{\mathrm{rel}}} \simeq E^0/F^0$. Let ψ be a character of $F\backslash\mathbb{A}$ such that $\psi_{v_{\mathrm{rel}}}$ has conductor $\mathcal{O}_{v_{\mathrm{rel}}}$. Let $\Phi^\circ_{v_{\mathrm{rel}}} \in \mathcal{H}_{G'_{v_{\mathrm{rel}}}}(K'_{v_{\mathrm{rel}}})$ and $\Psi^\circ_{v_{\mathrm{rel}}} \in \mathcal{H}_{X_{v_{\mathrm{rel}}}}(K)$ be such that $\hat{\Phi}^\circ_{v_{\mathrm{rel}}} = \hat{\Psi}^\circ_{v_{\mathrm{rel}}}$ and let $\alpha \in T'_{v_{\mathrm{rel}}}$. Our goal is to prove the identity

$$\Omega[\Psi^\circ_{v_{\mathrm{rel}}};\psi_{v_{\mathrm{rel}}};\alpha] = \gamma(\alpha)\Omega[\Phi^\circ_{v_{\mathrm{rel}}};\psi_{v_{\mathrm{rel}}};\alpha]. \tag{59}$$

For every place v of F let $\mathfrak{D}_v$ be the kernel of the character $\omega_v \circ \det$ on G'_v. By linearity, it is enough to prove (59) for $\Phi^\circ_{v_{\mathrm{rel}}}$ with support contained either in $\mathfrak{D}_{v_{\mathrm{rel}}}$ or in $G'_{v_{\mathrm{rel}}} \setminus \mathfrak{D}_{v_{\mathrm{rel}}}$. Let $D \in \{\mathfrak{D}_{v_{\mathrm{rel}}}, G'_{v_{\mathrm{rel}}} \setminus \mathfrak{D}_{v_{\mathrm{rel}}}\}$ and assume from now on that the support of $\Phi^\circ_{v_{\mathrm{rel}}}$ is contained in D. Let

$$\epsilon = \begin{cases} 1 & D = \mathfrak{D}_{v_{\mathrm{rel}}} \\ -1 & D = G'_{v_{\mathrm{rel}}} \setminus \mathfrak{D}_{v_{\mathrm{rel}}}. \end{cases}$$

Let v_{cusp} be a finite place of F which is split in E and let $\Phi_{v_{\mathrm{cusp}}} \overset{\psi_{v_{\mathrm{cusp}}}}{\longleftrightarrow} \Psi_{v_{\mathrm{cusp}}}$ be supercuspidal matching functions and $\beta \in T'_{v_{\mathrm{cusp}}}$ such that

$$\Omega[\Psi_{v_{\mathrm{cusp}}},\psi_{v_{\mathrm{cusp}}};\beta] = \Omega[\Phi_{v_{\mathrm{cusp}}},\psi_{v_{\mathrm{cusp}}};\beta] \neq 0.$$

Since the local orbital integrals are smooth in $a \in T'_v$, by the weak approximation there exists $a \in T'$ so that under the diagonal embedding of T' into $T'_{v_{\mathrm{rel}}} \times T'_{v_{\mathrm{cusp}}}$ it is close enough to (α,β) so that

$$\Omega[\Phi^\circ_{v_{\mathrm{rel}}};\psi_{v_{\mathrm{rel}}};\alpha] = \Omega[\Phi^\circ_{v_{\mathrm{rel}}};\psi_{v_{\mathrm{rel}}};a],\ \Omega[\Phi_{v_{\mathrm{cusp}}},\psi_{v_{\mathrm{cusp}}};\beta] = \Omega[\Phi_{v_{\mathrm{cusp}}},\psi_{v_{\mathrm{cusp}}};a]$$

and therefore $\Psi^\circ_{v_{\mathrm{rel}}}$ and $\Psi_{v_{\mathrm{cusp}}}$ also satisfy the analog identities. In particular, it is enough to prove (59) with α replaced by a.

Let S be a finite set of places of F containing all infinite places all even places and the places v_{rel}, v_{cusp} such that $a \in K_v$ and ψ_v has conductor $\mathcal{O}_v$ for $v \notin S$. For every $v \in S \setminus \{v_{\mathrm{cusp}}\}$ let $\Phi_v \overset{\psi_v}{\longleftrightarrow} \Psi_v$ be such that $\Omega[\Phi_v,\psi_v;a] \neq 0$. For every $v \in S$ let C_v be a compact subset of G'_v such that the support of Φ_v is contained in C_v and such that the support of $\Phi^\circ_{v_{\mathrm{rel}}}$ is also contained in $C_{v_{\mathrm{rel}}}$ and let $C = \prod_{v\in S} C_v \times \prod_{v\notin S} M_n(\mathcal{O}_v)$. For $v \notin S$ we choose at this point arbitrarily matching functions $\Phi_v \overset{\psi_v}{\longleftrightarrow} \Psi_v$ in such a way that $\Omega[\Phi_v,\psi_v;a] \neq 0$ and $\Phi_v = \mathbf{1}_{K'_v}$ and $\Psi_v = \mathbf{1}_{K_v\cap X_v}$ for almost all v and such that the support of $\Phi = \otimes_v \Phi_v$ is contained in C. We then have $\Phi \overset{\psi}{\longleftrightarrow} \Psi = \otimes_v \Psi_v$, $\Omega[\Psi;\psi;a] = \Omega[\Phi;\psi;a] \neq 0$ and furthermore that

$$RTF(\Psi) = \sum_{b\in\Sigma} \Omega[\Psi;\psi;b] = \sum_{b\in\Sigma} \Omega[\Phi;\psi;b] = KTF(\Phi)$$

for a finite set Σ in T' dependent only on C. Next we choose an auxiliary place $v_{\mathrm{aux}} \notin S$ and replace $\Psi_{v_{\mathrm{aux}}}$ (resp. $\Phi_{v_{\mathrm{aux}}}$) by its product with the characteristic function of all $g \in X_{v_{\mathrm{aux}}}$ (resp. $g \in G'_{v_{\mathrm{aux}}}$) such that

$$d_i(g) \in d_i(a) + \mathfrak{p}^{\ell}_{v_{\mathrm{aux}}},\ i = 1\dots,n \tag{60}$$

for some integer ℓ. Since the respective characteristic functions are constant on the unipotent orbits we still have $\Phi_{v_{\mathrm{aux}}} \overset{\psi_{v_{\mathrm{aux}}}}{\longleftrightarrow} \Psi_{v_{\mathrm{aux}}}$. We choose ℓ large enough so that a is the unique element $g \in \Sigma$ for which (60) holds and furthermore so that for $g \in G'_{v_{\mathrm{aux}}}$ the conditions (60) imply that $g \in \overline{U'}_{v_{\mathrm{aux}}} T'_{v_{\mathrm{aux}}} U'_{v_{\mathrm{aux}}} = \{g \in G'_{v_{\mathrm{aux}}} :$

$d_i(g) \neq 0$, $i = 1, \ldots, n\}$. Thus, $\Omega[\Psi_{v_{\rm aux}}; \psi_{v_{\rm aux}}; b] = \Omega[\Phi_{v_{\rm aux}}; \psi_{v_{\rm aux}}; b] = 0$ for all $b \in \Sigma \setminus \{a\}$ and furthermore $\Omega[\Psi_{v_{\rm aux}}; \psi_{v_{\rm aux}}; w_{M'} b] = \Omega[\Phi_{v_{\rm aux}}; \psi_{v_{\rm aux}}; w_{M'} b] = 0$ whenever $M' \neq T'$ is a standard Levy subgroup of G' and $b \in A_{M'}$. We now get for all $\Phi \overset{\psi}{\longleftrightarrow} \Psi$ of the form chosen above that

$$RTF(\Psi) = \Omega[\Psi; \psi; a] = \Omega[\Phi; \psi; a] = KTF(\Phi) \neq 0$$

and furthermore that

$$\Omega[\Psi^\circ; \psi; b] = \Omega[\Phi^\circ; \psi; b] = 0$$

for all $b \in T' \setminus \{a\}$, where $\Psi^\circ = \Psi^\circ_{v_{\rm rel}} \otimes (\otimes_{v \neq v_{\rm rel}} \Psi_v)$ and $\Phi^\circ = \Phi^\circ_{v_{\rm rel}} \otimes (\otimes_{v \neq v_{\rm rel}} \Phi_v)$. We then get that

$$RTF(\Psi^\circ) = \Omega[\Psi^\circ; \psi; a] \text{ and } KTF(\Phi^\circ) = \Omega[\Phi^\circ; \psi; a]. \tag{61}$$

To complete the prove of the theorem we therefore need to show that $RTF(\Psi^\circ) = KTF(\Phi^\circ)$. We turn to the respective spectral expansions. We have

$$RTF(\Psi^\circ) = \sum_{\pi} \tilde{B}_\pi(\Psi^\circ) \text{ and } KTF(\Phi^\circ) = \sum_{\pi'} B_{\pi'}(\Phi^\circ)$$

where each sum is over cuspidal representations. Recall that for every pair $\Phi' \overset{\psi}{\longleftrightarrow} \Psi'$ and every cuspidal representation π' of $G'_{\mathbb{A}}$ the linear independence of characters implies that

$$\tilde{B}_{{\rm bc}(\pi')}(\Psi') = B_{\pi'}(\Phi') + B_{\pi' \otimes \omega}(\Phi'). \tag{62}$$

In order to complete the proof of the theorem it is now enough to show that for every cuspidal representation π' of $G'_{\mathbb{A}}$ we have

$$\tilde{B}_{{\rm bc}(\pi')}(\Psi^\circ) = B_{\pi'}(\Phi^\circ) + B_{\pi' \otimes \omega}(\Phi^\circ). \tag{63}$$

If $\pi' \simeq \pi' \otimes \omega$ then ${\rm bc}(\pi')$ is not cuspidal and therefore $\tilde{B}_{{\rm bc}(\pi')}$ is the zero distribution. Since in this case $B_{\pi'} = B_{\pi' \otimes \omega}$ it follows from (62) that each of the Bessel distributions is zero. Thus, in this case both sides of (63) equal zero. Assume now that $\pi' \not\simeq \pi' \otimes \omega$ and let $\pi = {\rm bc}(\pi') = {\rm bc}(\pi' \otimes \omega)$. As we already explained, the Bessel distribution $B_{\pi'}$ is factorizable. In particular, for any function of the form $\Phi' = \Phi'_v \otimes (\Phi')^v$ there is a distribution $B_{(\pi')^v}$ on $(G')^v$ such that $B_{\pi'}(\Phi') = B_{\pi'_v}(\Phi'_v) B_{(\pi')^v}((\Phi')^v)$. Essentially, this is also the case for $\tilde{B}_\pi$. For any function $\Psi' = \Psi'_{v_{\rm rel}} \otimes (\Psi')^{v_{\rm rel}}$ such that the support of $\Psi'_{v_{\rm rel}}$ is contained in $\{x \in X : \omega(\det x) = \epsilon\}$ we can find a matching function $\Phi'_{v_{\rm rel}}$ with support contained in D. For $\Phi' \overset{\psi}{\longleftrightarrow} \Psi'$ as above we then have

$$\tilde{B}_\pi(\Psi') = B_{\pi'_{v_{\rm rel}}}(\Phi'_{v_{\rm rel}})(B_{(\pi')^v} + \epsilon B_{(\pi' \otimes \omega)^v})((\Phi')^v). \tag{64}$$

From the factorization of $B_{\pi'}$ it is clear that $B_{\pi'}(\Phi^\circ) = 0$ unless $\pi'_{v_{\rm rel}}$ is unramified. It follows from Lemma 9 that $B_\pi(\Psi^\circ) = 0$ unless $\pi_{v_{\rm rel}}$ is unramified. We may therefore assume from now on that $\lambda_{v_{\rm rel}} \in \mathbb{C}_n$ is such that $\pi'_{v_{\rm rel}} = I'(\lambda_{v_{\rm rel}})$ and $\pi_{v_{\rm rel}} = I(\lambda_{v_{\rm rel}})$. For a finite place v of F, from the localization principal of Kazdhan...? the Bessel distribution $B_{\pi'_v}$ may be considered as a distribution on X_v, i.e. there is a distribution B_{π_v} on X_v so that $B_{\pi_v}(\Psi'_v) = B_{\pi'_v}(\Phi'_v)$ whenever $\Phi'_v \overset{\psi_v}{\longleftrightarrow} \Psi'_v$. For every function $\Psi'_{v_{\rm rel}}$ supported in $\{x \in X_{v_{\rm rel}} : \omega_{v_{\rm rel}}(\det x) = \epsilon\}$ the equality (64) for matching functions now becomes

$$\tilde{B}_\pi(\Psi') = B_{\pi_{v_{\rm rel}}}(\Psi'_{v_{\rm rel}})(B_{(\pi')^v} + \epsilon B_{(\pi' \otimes \omega)^v})((\Phi')^v). \tag{65}$$

The local Bessel identities (Theorem 11) imply that there is an explicit factor $c(\pi_v)$ such that $B_{\pi_{v_{\mathrm{rel}}}} = c(\pi_v) B^{st}(\cdot, \mathbf{1}_{T_v}, \lambda_v)$ and furtheremore we get from Lemma 3 that

$$B_{\pi'_{v_{\mathrm{rel}}}}(\Phi^\circ_{v_{\mathrm{rel}}}) = c(\pi_v) B^{st}(\Psi^\circ_{v_{\mathrm{rel}}}, \mathbf{1}_{T_v}, \lambda_v) = B_{\pi_{v_{\mathrm{rel}}}}(\Psi^\circ_{v_{\mathrm{rel}}}). \tag{66}$$

We now plug in the functions Φ° and Ψ° into (65). Applying (66) we get (63) as desired.

□

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