RESEARCH STATEMENT

OMER OFFEN

0. The general framework

My research is in the general area of automorphic forms and representation theory. My particular interests are in the study of *periods* of automorphic forms and their connection with special values of L-functions. Jacquet's *Relative Trace Formula* (RTF) is a tool introduced by Jacquet to study period integrals of the form

$$P^H(\phi) = \int_{H(F)\backslash(H(\mathbb{A})\cap G(\mathbb{A})^1)} \phi(h)\, dh.$$

Here H is the group of fixed points of an involution θ on a reductive group G defined over a number field F with ring of adèles $\mathbb{A}$ and ϕ is an automorphic form on $G(F)\backslash G(\mathbb{A})$. If ϕ is cuspidal, this integral is known to converge. For a general automorphic form, it should be possible to regularize the period, as was carried out in [JLR99] and [LR03] in the case where θ is a Galois involution (and in [Off06a] for a specific non-Galois case).

Definition 1. A cuspidal, automorphic representation π of $G(\mathbb{A})$ is *distinguished* by H if $P^H(\phi) \neq 0$ for some ϕ in the space of π.

In many examples, the period integral is related to the value of an L-function at a special point. In general, there is a group G' associated to the pair (G, θ) and a functorial transfer (in the sense of Langlands functoriality) of automorphic forms from G' to G. It is expected that cuspidal automorphic representations on $G(\mathbb{A})$ that lie in the image of the transfer are characterized by H-distinction. In recent years, there has been much progress in the development of the RTF. Yet, at this stage, much of the progress is by means of examples rather then a general theory or even precise conjectures. One of my main interests is the understanding of the right framework and the formulation of precise conjectures concerning the scope and possible applications of the RTF. This long term goal serves me as a guideline for my research. What follows is a more detailed discussion of several specific problems I am working on, my progress to date and my plans for the future.

1. Periods of cuspidal Eisenstein series

Let G be a reductive group defined over a number field F and let H be the fixed point subgroup of an involution θ defined over F. We wish to generalize the analysis of [LR03] to the setup at hand. This includes, the definition of regularized periods of automorphic forms of G with respect to H; computing the periods of pseudo-Eisenstein series and regularized periods of a cuspidal Eisenstein series; a formula for the period of the truncated Eisenstein series generalizing the Maass-Selberg relations. The periods of truncated Eisenstein series appear in the continuous part of the spectral expansion of the RTF and are therefore crucial for the study of periods of cusp forms. This is a joint project in preparation with E. Lapid.

The first step is a careful analysis of the double coset space $P(F)\backslash G(F)/H(F)$ for a parabolic subgroup P of G. In [LR03], it was assumed that θ is a Galois involution and furthermore that the group G has a θ-stable minimal parabolic subgroup. Under these assumptions it was enough to fix such a minimal parabolic P_0 and consider only standard parabolic subgroups, i.e. those containing P_0. For our more general setting this will not suffice. The approach taken by Arthur in [Art82] appears to be most suitable for the problem at hand. That is, we fix a maximal split torus A of G, let $M_0 = Z_G(A)$ be the associated Levi subgroup and consider all parabolic subgroups that contain M_0. In addition we choose A to be θ-stable (such A always exists), thus M_0 is a θ-stable minimal Levi. Let $P = MU$ be a parabolic, with Levi subgroup M containing M_0 and with unipotent radical U. Every double coset $\mathcal{O}$ in $P(F)\backslash G(F)/H(F)$ has a representative η such that $x = \eta\theta(\eta)^{-1} \in N_G(A)$. We say that $\mathcal{O}$ is admissible, if it contains such a representative η, so that M is stable under the involution $\theta_x(y) = x\theta(y)x^{-1}$. Admissible orbits will play an important role in our study of the periods of Eisenstein series. Consider the Eisenstein series $E(g, \varphi, \lambda)$ induced from a cuspidal section φ on $U(\mathbb{A})M(F)\backslash G(\mathbb{A})$. A formal computation of the period $P^H(E(\varphi, \lambda))$ yields:

$$\int_{H(F)\backslash(H(\mathbb{A})\cap G(\mathbb{A})^1)} E(h, \varphi, \lambda)\ dh \text{``}=\text{''} \sum_{\eta} \int_{P_\eta\backslash(\eta H(\mathbb{A})\eta^{-1}\cap G(\mathbb{A})^1)} \varphi(h\eta) e^{\langle\lambda, H_P(h\eta)\rangle}\ dh$$

where $\{\eta\}$ is a set of representatives for the double cosets and $P_\eta = P \cap \eta H \eta^{-1}$. Neither side of this equation converges, but the left-hand side is defined by regularization. On the other hand, many of the terms on the right-hand side are "morally" equal to zero. This will be true for the non-admissible double cosets for cuspidality reasons. We expect an admissible double coset to contribute only if it contains a representative η as above so that P is a θ_x-split parabolic, i.e. so that $\theta_x(P)$ is the parabolic opposite to P. The fact that the remaining admissible double cosets do not contribute to $P^H(E(\varphi, \lambda))$ can be guessed in advance because the contribution of such a coset involves the integral of an exponential function over a vector space. As noted by Casselman in a related context ([Cas93]), an integral of this type should be interpreted as zero.

2. Unitary periods

Let E/F be a quadratic extension of number fields. Let $G' = GL_n$ regarded as an algebraic group defined over F and let $G = R_{E/F}(GL_n)$ be the restriction of scalars. Thus, $G'(F) = GL_n(F)$ and $G(F) = GL_n(E)$. We also denote by K the standard maximal compact subgroup of $G(\mathbb{A})$.

2.1. **Periods of cusp forms.** The functorial transfer from G' to G is quadratic base change. It has now been established by Jacquet that for cuspidal representations, distinction by unitary groups characterizes the image of this transfer. In a joint work with E. Lapid, we obtain explicit formulas for the anisotropic unitary period of certain cusp forms in terms of special values of L-functions [LO]. A special case of the result is formulated as follows. Let

E be a CM-field and F its maximal totally real subfield. Assume that E/F is unramified at all finite places. We consider the unitary group

$$H = \{g \in G | {}^t\bar{g}g = e\}$$

where $x \to \bar{x}$ denotes the Galois action. Let π' be a cuspidal automorphic representation of $G'(\mathbb{A})$ which is everywhere unramified and assume that $\pi' \not\simeq \pi' \otimes \eta$ where η is the quadratic Hecke character attached to E/F by class field theory. Let π be the lifting to $G(\mathbb{A})$ of π' under quadratic base change. Thus π is an everywhere unramified cuspidal representation of $G(\mathbb{A})$. Let ϕ_0 be the K-invariant, L^2-normalized cusp form in the space of π.

Theorem 1 (Lapid-Offen)**.** *There is a constant $c_E \neq 0$ depending only on E such that*

$$\left|\int_{H(F)\backslash H(\mathbb{A})} \phi_0(h)dh\right|^2 = c_E \frac{L(1, \pi' \times \tilde{\pi}' \otimes \eta)}{\mathrm{res}_{s=1} L(s, \pi' \times \tilde{\pi}')}. \tag{1}$$

The constant c_E is explicit and an analogue of this formula, replacing H by any anisotropic unitary group is also provided. If we allow ramification in E/F, we still obtain a similar formula for the period, explicit up to finitely many local terms at finite places. Each such term is expressed as a value of a spherical function on the p-adic space of Hermitian matrices with respect to a ramified quadratic extension. One of my plans for the near future is to compute those terms explicitly and thus obtain an explicit formula for the period in a more general context. In §5, I describe the local problem at hand.

To stress the importance of such explicit formulas and of the relation between periods and special values, let me point out a relation with a conjecture of Sarnak regarding the L^∞-norm of automorphic forms. Roughly speaking, the cusp form ϕ_0 is a function on the locally symmetric space $G(F)\backslash G(\mathbb{A})/K$. Using strong approximation, this space can be interpreted in classical terms as an arithmetic quotient of the symmetric space $(G(\mathbb{R})/H(\mathbb{R}))^d$ where $d = [F : \mathbb{Q}]$. Note that $H(\mathbb{R})$ is the standard maximal compact of $G(\mathbb{R}) = GL_n(\mathbb{C})$. In general, if ϕ is an L^2-normalized, cuspidal automorphic form on this locally symmetric space, which is an eigenfunction of the Laplacian with eigenvalue λ then

$$\|\phi\|_\infty << \lambda^{\frac{n^2-n}{4}}.$$

This bound comes from the local geometry of the symmetric space (see [Sar04]) and is valid without the assumption that ϕ is a Hecke eigenform. In this sense it should be thought of as the convexity bound. There are precise conjectures about the size of the L-functions appearing on the right hand side of (1). Since the H-period of ϕ_0 amounts to a weighted finite sum $\sum' \phi_0(x_i)$ of point evaluations over classes in the genus class of the Hermitian form defined by the identity matrix, these conjectures imply the lower bound

$$\|\phi\|_\infty >> \lambda^{\frac{n^2-n}{8}}.$$

This is an analogue of the situation with the Ramanujan conjecture. In our situation exceptional cases appear already for GL_n. We remark, however, that the exceptional forms (those with large L^∞-norm) are rare, and roughly speaking Sarnak conjectures that they are all functorial lifts from smaller groups.

The proof of Theorem 1 is based on a comparison of the RTF for G with a Kuznetsov trace formula (KTF) for G'. The main ingredients in the proof are

(1) The fundamental lemma of Jacquet [Jac05, Jac04].
(2) The fine spectral expansion of the RTF [Lap06].
(3) Local identities of Bessel distributions for principle series [Off].
(4) Explicit formulas for Hironaka's spherical functions [Hir99].

In the notation of the theorem, (1) and (2) allow us to compare the contribution of π to the RTF with the contribution of π' to the KTF. The latter, factorizes into a product of local distributions, and this comparison already provides a formula for $\left|P^H(\phi_0)\right|^2$ explicit up to finitely many terms expressed in terms of local distributions on G'. The local results I obtain in (3) relate those distributions to local distributions on G which in turn can be expressed in terms of Hironaka's spherical functions. The formula is then made explicit by the results of (4), which are explained here in §5 in more detail.

2.2. **Periods of Eisenstein series and Bessel identities.** An irreducible cuspidal automorphic representation of $G(\mathbb{A})$ that lies in the image of quadratic base change is (essentially) the image of a unique cuspidal automorphic representation of $G'(\mathbb{A})$. Accordingly, the unitary period integral is factorizable. An Eisenstein automorphic representation is the base change of several automorphic representations. Accordingly, the regularized unitary period of Eisenstein series should be expressed as a finite sum of factorizable linear forms.

For the most continuous part of the spectrum, i.e. for Eisenstein series induced from principal series representations, this was established first for $n = 3$ by Lapid and Rogawsky in [LR00] and then for a general n in [Off]. In fact, the period of the Eisenstein series over any unitary group is expressed as a finite sum of quotients of a product of Dirichlet L-functions, explicit up to local data at finitely many places [Off, Corollary 1]. As in the case of periods of cusp forms the formula is based on Hironaka's computations. The project suggested in §5 will make the formula explicit for the K-invariant Eisenstein series. For an anisotropic unitary group, as in the cuspidal case, the period can be interpreted as a finite sum of point evaluations of the Eisenstein series over classes in the genus class of the Hermitian form. The period then contains arithmetic information about certain representation numbers on flag varaieties. In a joint work with G. Chinta ([CO]), we study furhter these representation numbers in the special case where $E/\mathbb{Q}$ is purely imaginary and of class number one.

Denote by H_ξ the unitary group associated to the hermitian matrix ξ, B the standard Borel subgroup of G, B' that of G' and by $\mathcal{S}$ the space of hermitian matrices in G. Let $E(\varphi, \lambda)$ be an Eisenstein series induced from a section φ in a principle series representation $I(\chi, \lambda)$ induced from B with a unitary character χ and an exponent λ. Assume further that χ is a base change, i.e $\chi = \nu \circ \mathrm{Nm}$ for some character ν on B'. Denote by $\mathcal{B}(\chi)$ the set of 2^n characters that base change to χ. It follows from [LR03, Theorem 9.1.1] that the unitary period can be expressed as

$$P^{H_\xi}(E(\varphi, \lambda)) = \sum_a J_\xi^a(\varphi, \lambda)$$

where the sum is over $a \in (F^\times / \mathrm{Nm}\, E^\times)^n$ and $J_\xi^a(\cdot, \lambda)$ is a certain H_ξ-invariant linear functional. The right hand side is an infinite sum and the summands are not factorizable. The formula for the period of the Eisenstein series mentioned above, is obtained by stabilization of these linear functionals. In [Off] we define the stable functionals $J_\xi^{st}(\cdot,\nu,\lambda)$ parameterized by $\nu \in \mathcal{B}(\chi)$. We show that

$$P^{H_\xi}(E(\varphi,\lambda)) = \sum_{\nu\in\mathcal{B}(\chi)} J_\xi^{st}(\varphi,\nu,\lambda)$$

and that the stable periods are factorizable and we compute the local factors at almost all places. This stabilization, also provides both locally and globally, identities between certain distributions on G and on G'. If (π,V) is a unitary representation of $G(\mathbb{A})$ and l_1, l_2 are linear functionals on V, we may define a distribution on the space of compactly supported, smooth functions on $G(\mathbb{A})$ by the formula

$$\mathcal{B}^{l_1,l_2}(f) = \sum_{\{\phi\}} L_1(\pi(f)\phi)\overline{L_2(\phi)}$$

where the sum is over an orthonormal basis of V. The distributions occurring in the spectral expansion of the KTF for G' and of the RTF for G are all of this type. They are referred to as Bessel distributions and relative Bessel distributions respectively. We study the contributions of the most continuous part of the spectrum. On $G'(\mathbb{A})$ the corresponding Bessel distribution is of the form $B'(f',\nu,\lambda) = \mathcal{B}^{\mathcal{W}',\mathcal{W}'}(f')$ where $\mathcal{W}'$ is the Whittaker functional on the principle series representation $I'(\nu,\lambda)$. We also define the stable relative Bessel distribution on $\mathcal{S}(\mathbb{A})$ by $B^{st}(\Phi,\nu,\lambda) = \sum_\xi \mathcal{B}^{J_\xi^{st}(\cdot,\nu,\lambda),\mathcal{W}}(f_\xi)$ where the sum is over representatives of the $G(F)$-orbits in $\mathcal{S}(F)$ and $\Phi(g\cdot\xi) = \int_{H_\xi(\mathbb{A})} f_\xi(gh)dh$. We then have

Theorem 2 (Offen)**.** *There exists $\delta = \delta(n) \in \{0,1\}$ so that for δ-matching functions $f' \leftrightarrow \Phi$ we have,*

$$B'(f',\nu,\lambda) = B^{st}(\Phi,\nu,\lambda).$$

The distributions on both sides are factorizable, i.e. if $\Phi = \otimes'_v \Phi_v$, the product being over all places of F, then there are distributions $\tilde{B}_v^{st}(\Phi_v,\nu_v,\lambda)$ on $G(F_v)$ defined by the local analogues of their global counterpart, such that

$$B^{st}(\Phi,\nu,\lambda) = \prod_v B_v^{st}(\Phi_v,\nu_v,\lambda)$$

and similarly for $B'(f',\nu,\lambda)$. Passing to a local setting and omitting the index v from our notation we have,

Theorem 3 (Offen)**.** *Let E/F be a quadratic extension of local fields of characteristic zero. There is a root of unity $d_{E/F}$ depending only on the extension E/F so that for any $\delta(n)$-matching $\Phi \leftrightarrow f'$ we have*

$$B^{st}(\Phi,\nu,\lambda) = d_{E/F}\gamma(\nu,\lambda,\psi)B'(f',\nu,\lambda).$$

Moreover, if E/F is unramified then, $d_{E/F} = 1$.

The term $\gamma(\nu,\lambda,\psi)$ is a product of Tate gamma factors defined in [Off]. The notation $\Phi \leftrightarrow f'$ both in the local and in the global case refers to functions with matching orbital integrals. There is a "δ-ambiguity" concerning the definition of the transfer factor. This is the reason for the appearance of δ in the above statements. We discuss this ambiguity further and suggest how it can be resolved in §3.1.

3. The fundamental lemma of Jacquet

Roughly speaking, the fundamental lemma is an explicit matching of orbital integrals between Hecke functions. It is a local problem that became a major obstacle for establishing trace formula comparisons. In the context of the RTF we refer to it as the FL of Jacquet. The geometric expansion of the RTF on $\mathcal{S}$ consists of distributions that can be expressed as factorizable orbital integrals and similarly for the KTF on G'. Thus a comparison between the two trace formulas is based on the FL of Jacquet.

3.1. **Quadratic base change.** We use the notation introduced in §2 and remain in a local p-adic setting, assuming in addition that E/F is an unramified quadratic extension. Denote by $\mathcal{H}_G$ the Hecke algebra of bi K-invariant functions on G, by $\mathcal{H}_{G'}$ the Hecke algebra on G' and by $\mathcal{H}_{\mathcal{S}}$ the space of K-invariant functions on the space $\mathcal{S}$ of hermitian matrices in G. Let Φ_0 be the characteristic function of $K \cap \mathcal{S}$. There is an action of $\mathcal{H}_G$ on $\mathcal{H}_{\mathcal{S}}$ by convolution. The spherical Fourier transform defines an embedding of $\mathcal{H}_G$ in $\mathcal{H}_{G'}$ and Jacquet showed that if $f \in \mathcal{H}_G$ and $f' \in \mathcal{H}_{G'}$ are such that $\hat{f}(\lambda) = \hat{f}'(\lambda)$ then $f * \Phi_0 \leftrightarrow f'$. As mentioned in §2.2 there are two ways to define the transfer factor for this matching. Since Jacquet's results hold for both of them, they do not determine which is the correct transfer factor. In [Off06c], I conjecture a generalization of the FL of Jacquet that will, in particular, determine the transfer factor uniquely. Based on Hironaka's theory of spherical functions, there is a spherical Fourier transform on $\mathcal{H}_{\mathcal{S}}$ that defines an isomorphism $\mathcal{H}_{\mathcal{S}} \simeq \mathcal{H}_{G'}$. I conjecture that (for one of the two possible ways to define the transfer factor) if $\Phi \in \mathcal{H}_{\mathcal{S}}$ and $f' \in \mathcal{H}_{G'}$ are such that $\hat{\Phi}(\lambda) = \hat{f}'(\lambda)$ then $\Phi \leftrightarrow f'$. I prove the conjecture when $n = 2$. One of my plans for the near future is to work on this conjecture. It is my hope that some variation of the beautiful machinary developed by Jacquet for his FL can also be used to prove the above suggested generalization. Furthermore, I hope that this project will help me acquire the necessary techniques in order to tackle the FL in a more difficult setting that I now explain.

3.2. **Metaplectic Correspondence.** One of the mysteries of Langlands functoriality conjectures is that even for the study of automorphic forms on GL_n one has to leave the framework of algebraic groups. Let $G = GL_n$. Distinction of cuspidal representations on $G(\mathbb{A})$ by orthogonal groups is expected to be characterized by the image of the metaplectic correspondence. This is a transfer of automorphic forms on the metaplectic double cover $G' = \widetilde{GL_n}$ of G to automorphic forms on G. The relevant symmetric space $\mathcal{S}$ for this setting is the space of symmetric matrices in G. As already explained, to obtain this result we wish to establish an identity of the RTF on $\mathcal{S}$ with the KTF on G'. A crucial step is then the FL of Jacquet. To prove the FL in the case of quadratic base change, Jacquet developed a new machinery. First he linearized the problem by considering the space of all Hermitien forms rather then the invertible ones. Next, he introduced a Jacquet-Kloosterman transform on the space of orbital integrals and expressed the orbital integral of the Fourier transform of a function in terms of the Jacquet-Kloosterman transform. He then proved an inversion formula for the Jacquet-Kloosterman transform and obtained smooth matching. The fundammental lemma required in addition some "uncertainty principle" for the Jacquet-Kloosterman transform, that was reduced by Jacquet to the usual uncertainty principle (for a Schwarts function and its Fourier transform). It seems that the method used to prove the uncertainty principle can be adjusted for the case at hand. Thus, a cruisal step is the inversion formula for the Jacquet-Kloosterman transform both for $\mathcal{S}$ and for G'. In [Off05], I obtain the inversion formula for the space of symmetric matrices (the linearized version of $\mathcal{S}$). Let ψ be a non-additive character of the p-adic field F. For a smooth function of compact support Φ on the space of $n \times n$ symmetric matrices and a diagonal matrix $a = \mathrm{diag}(a_1, \dots, a_n)$, the orbital integral is defined by,

$$\Omega[\Phi, \psi; a] = \int_U \Phi({}^t uau) \psi_U(u)^2 \, du.$$

Here U is the group of upper-triangular unipotent matrices and $\psi_U(u) = \psi(u_{1,2} + \cdots + u_{n-1,n})$. The Jacquet-Kloosterman transform of a function $\Omega(a)$ in the space of orbital integrals is

$$K_\psi \Omega(a_1, \dots, a_n) =$$

$$\int \Omega(p_1, \dots, p_n) \, \psi \left[-\sum_{i=1}^{n} p_i \, a_{n+1-i} + \sum_{i=1}^{n-1} \frac{1}{p_i \, a_{n-i}} \right] \left(\prod_{i=1}^{n-1} \prod_{j=1}^{n-i} [a_i, p_j] \right) dp_n \, dp_{n-1} \cdots dp_1 \tag{2}$$

where $[\cdot, \cdot]$ is the quadratic Hilbert symbol. The integral is only iterated. To formulate the result, it is convenient to normalize the orbital integrals by certain transfer factors. We set

$$\widetilde{\Omega}[\Phi, \psi; a] = \gamma(a_1, \psi)^{n-1} \gamma(a_2, \psi)^{n-2} \cdots \gamma(a_{n-1}, \psi) \, |a_1^{n-1} \, a_2^{n-2} \cdots a_{n-1}|_F^{\frac{1}{2}} \, \omega[\Phi, \psi; a_1, \dots, a_n],$$

where $\gamma(a, \psi)$ is the Weil constant.

Theorem 4 (Offen). *The integral (2) is a convergent iterated integral. Moreover,*

$$\left(K_\psi \widetilde{\Omega}[\Phi, \psi] \right)(a) = |2|^{n(n-1)/2} \, \gamma(1, \bar{\psi})^{n(n-1)/2} \, \widetilde{\Omega}[\widehat{\Phi}, \bar{\psi}; a]. \tag{3}$$

Here $\widehat{\Phi}$ is the Fourier transform of Φ. The inversion formula then follows.

Corollary 1 (Offen).

$$K_{\bar{\psi}, n} \circ K_{\psi, n} = |2|^{n(n-1)} \, Id.$$

As already explained,in order to carry out the ideas of Jacquet, we need a similar inversion formula for (a linearized version of) G'. In the case at hand, it is not even clear how to formalize the linearized problem. So far, I had some progress with the case of $\widetilde{GL_2}$ and I plan to address the problem in the future.

4. Symplectic-Whittaker periods local and global

Let F be either a number field or a p-adic field. Let G_r denote the group GL_r and let U_r be its subgroup of upper triangular unipotent matrices. Fix n and set $G = G_n$ and if $n = r + 2k$ let

$$H_{r,2k} = \{ \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} \in G : u \in U_r, \, X \in M_{r \times 2k} \text{ and } h \in Sp(2k) \}.$$

Let ψ be a non trivial character of F in the local case (resp. of $F \backslash \mathbb{A}$ in the global case). We associate to ψ the character ψ_r of $U_r(F)$ (resp. of $U_r(F) \backslash U_r(\mathbb{A})$) defined by

$$\psi_r(u) = \psi(u_{1,2} + \cdots + u_{r-1,r}).$$

By abuse of notation we will also denote by ψ_r the character of $H_{r,2k}$ defined by

$$\psi_r \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} = \psi_r(u).$$

In a recent joint work with E. Sayag [OSb], we obtained

Theorem 5 (Offen-Sayag)**.** *Let π be an irreducible, discrete spectrum automorphic representation of $GL_n(\mathbb{A})$. Then, there exists $k = k(\pi)$, $0 \le k \le [\frac{n}{2}]$ such that π is $(H_{n-2k,2k}, \psi_{n-2k})$-distinguished.*

We also obtain a local p-adic analogue of this result.

Definition 2. Let G be a local group, H a subgroup of G and χ a character of H. We say that a representation π of G is (H,χ)-distinguished if $\mathrm{Hom}_H(\pi,\chi) \neq 0$.

Theorem 6 (Offen-Sayag)**.** *Let π be an irreducible, unitary representation of $GL_n(F)$. There exists $k = k(\pi)$, $0 \le k \le [\frac{n}{2}]$ such that π is $(H_{n-2k,2k}, \psi_{n-2k})$-distinguished.*

In fact, the integer $k(\pi)$ is explicit, in the global case in terms of the classification of Mœglin and Waldspurger of the discrete spectrum of $G(\mathbb{A})$ and in the local case in terms of the classification of Tadic of the unitary dual of $G(F)$.

Note that $H_{r,2k} \simeq (U_r \times Sp(2k))V$ where V is the unipotent radical of the standard maximal parabolic Q of G of type $(r, 2k)$. It follows that for an automorphic form ϕ on $G(\mathbb{A})$ we have

$$\int_{H_{r,2k}(F)\backslash H_{r,2k}(\mathbb{A})} \phi(h)\psi_r(h)dh = (l_r^\psi \otimes l_{Sp(2k)})(\phi_Q)$$

where ϕ_Q is the constant term of ϕ along Q, l_r^ψ is the Whittaker functional on $G_r(\mathbb{A})$ and $l_{Sp(2k)}$ is the symplectic period. The study of these mixed (Whittaker-symplectic) periods can then be reduced to that of Whittaker functionals and purely symplectic periods.

The study of symplectic periods was initiated by Jacquet and Rallis in [JR92]. I continued their study in [Off06a] and [Off06b] and characterized precisely, which discrete spectrum automorphic representations of $GL_{2n}(\mathbb{A})$ are distinguished by $Sp(2n)$. This is a first example of general rank where distinction is completely analyzed for the entire discrete spectrum. The analysis is based on the fact that discrete spectrum representations are spanned by multi residues of Eisenstein series. In [Off06a], I obtain a formula for the symplectic period of a multi residue of an Eisenstein series. The classification of Mœglin and Waldspureger says that the discrete spectrum of G is the direct sum of $L(\sigma,t)$ for all pairs (σ,t) such that $n = tr$ for some r and σ is an irreducible, cuspidal representation of $G_r(\mathbb{A})$. The representation $L(\sigma,t)$ is the unique irreducible quotient of the representation, parabolically induced from

$$|\det|^{\frac{t-1}{2}}\sigma \otimes \cdots \otimes |\det|^{\frac{1-t}{2}}\sigma.$$

In [Off06b], I apply the formula I obtained for the residue of an Eisenstein series and show that it is not identically zero on $L(\sigma,t)$ if and only if t is even. In an unramified situation, an explicit formula is obtained for the symplectic period of a cusp form on $GL_{2n}(\mathbb{A})$. Denote by ϕ_0 the K-invariant automorphic form in $L(\sigma,2m)$ normalized so that $\|\phi_0\|_2 = 1$. With an appropriate normalization of Haar measures (independent of σ) I obtain the formula

$$\left|\int_{Sp(2n,F)\backslash Sp(2n,\mathbb{A})} \phi_0(h)dh\right|^2 = \frac{L_\sigma(2)L_\sigma(4)\cdots L_\sigma(2m)}{\mathrm{res}_{s=1}L_\sigma(s)L_\sigma(3)\cdots L_\sigma(2m-1)}. \tag{4}$$

Here $L_\sigma(s) = L(s, \sigma \times \tilde{\sigma})$ is the Rankin-Selberg L-function. We prove Theorem 5 in [OSb] using the results of [Off06b] on symplectic periods and the well known fact that cuspidal representations admit a Whittaker functional. For unramified data, a formula similar to (4) is also obtained for the mixed period.

As in the global case, the existence of the mixed periods in the local case can be reduced to that of Whittaker and purely symplectic functionals. Since we consider all unitary representations, the combinatorics of the local problem becomes more complicated.

Let $U(\delta, t)$ denote the (generalized) Speh representation associated to a (unitary) discrete series representation δ of $G_r(F)$ (here $n = tr$ and $U(\delta, t)$ is the local analogue of $L(\sigma, t)$). In [OSa], we show that any representation, parabolically induced from

$$|\det|^{s_1} U(\delta_1, 2m_1) \otimes \cdots \otimes |\det|^{s_q} U(\delta_q, 2m_q)$$

is distinguished by the symplectic group. The main difficulty here is to show distinction for the building blocks. In order to show that $U(\delta, 2m)$ is always distinguished by the symplectic group we use global methods and the results of [Off06b]. Theorem 6 is based on this analysis and the hereditary property of Whittaker functionals.

Using Frobenius reciprocity we see that an irreducible, admissible representation of $G(F)$ is $(H_{r,2k}, \psi_r)$-distinguished if and only if it embeds into the, so called, mixed model

$$\mathcal{M}_{r,2k} = \mathrm{Ind}_{H_{r,2k}(F)}^{G(F)}(\psi_r).$$

These models were first introduced by Klyachko when F is a finite field. In this case, if

$$m_\pi = \dim(\mathrm{Hom}_{G(F)}(\pi, \oplus_{k=1}^{[\frac{n}{2}]} \mathcal{M}_{n-2k,2k})$$

then, it was suggested by Klyachko and later proved by Inglis and Saxl that $m_\pi = 1$ for every irreducible representation of $G(F)$. In fact, Klyachko claimed to have proved the result, but his prove is incomplete. The study of mixed models when F is a p-adic field was initiated by Heumos and Rallis in [HR90]. They showed that already when $n = 3$ there exists an irreducible, admissible representation π of $G(F)$ so that $m_\pi = 0$ but for $n \leq 4$ if π is unitary then $m_\pi \geq 1$. Our result shows this for any n.

In their paper, Heumos and Rallis also claim to obtain disjointness of models for unitary representations. Unfortunately, their prove is based on a false statement made by Klyachko and is therefore incomplete. We believe, however, that the methods of Klyachko properly adjusted can be used to show both disjointness and uniqueness for the mixed models, i.e. that $m_\pi \leq 1$ for every irreducible, admissible representation π of $G(F)$ (and in particular that $m_\pi = 1$ if in addition π is unitary). We currently work on this local problem. We also hope to extend the results on the global mixed periods to the continuous spectrum.

Finally, we remark that our existence result, together with unitary disjointness will imply the following

Conjecture 1 (Offen-Sayag)**.** *Let $\pi = \otimes_v \pi_v$ be an irreducible, discrete spectrum automorphic representation of $GL_n(\mathbb{A})$. The following are equivalent.*

(1) π *is* $(H_{r,2k}, \psi_r)$*-distinguished;*
(2) π_v *is* $(H_{r,2k}, \psi_r)$*-distinguished for all places* v *of* F*;*
(3) π_v *is* $(H_{r,2k}, \psi_r)$*-distinguished for some finite place* v *of* F*.*

5. Spherical functions on p-adic symmetric spaces

Let G be a reductive p-adic group and let θ be an involution on G. We consider the symmetric space

$$\mathcal{S} = \{s \in G | s\theta(s) = 1\}.$$

The group G acts on $\mathcal{S}$ by twisted conjugation $g \cdot s = gs\theta(g)^{-1}$. This induces an action of the Hecke algebra $\mathcal{H}(G, K)$ of G with respect to a maximal compact K of G, on the space $C^\infty(K\backslash\mathcal{S})$ of K-invariant functions on $\mathcal{S}$, by the convolution

$$f * \varphi(s) = \int_G f(g)\varphi(g^{-1} \cdot s)\ dg.$$

Definition 3. A *relative spherical function* on $\mathcal{S}$ is a K-invariant function on $\mathcal{S}$ which is an $\mathcal{H}(G, K)$-eigenfunction.

To date, there is no general theory of spherical functions in this context. The case where the group in mind is $G \times G$ and the involution is $(x, y) \mapsto (y, x)$ is referred to as the group case. In this case, Macdonald obtained explicit formulas for the spherical functions (which are then bi-K-invariant functions on G) and an explicit Plancherel inversion formula. Later on Casselman simplified the computations of Macdonald using principle series representations [Cas80] and together with Shalika they also obtained with similar methods, explicit formulas for the Whittaker spherical functions [CS80]. The method of Casselman-Shalika, was further used to obtain explicit results in several other cases. In my thesis, I provide in 3 cases, the K-orbit decomposition of the symmetric space $\mathcal{S}$, explicit formulas for all relative spherical functions parameterized by a complex variable and an explicit Plancherel inversion formula [Off04]. In all 3 cases $\mathcal{S}$ is a unique G-orbit, and can therefore be identified with G/H where H is the group of θ-fixed points. Let F be a p-adic field of odd residual characteristic and let E/F be an unramified quadratic extension. The 3 cases are

(1) $G = GL_{2n}(F)$ and $H = GL_n(F) \times GL_n(F)$;
(2) $G = GL_m(E)$ and $H = GL_m(F)$;
(3) $G = GL_{2n}(F)$ and $H = GL_n(E)$.

The analogue of these results were first obtained by Hironaka and Sato for the space of skew symmetric matrices [HS88], and by Hironaka for the space of Hermitian matrices with respect to an unramified quadratic extension of p-adic fields [Hir99]. As explained in §2, for global applications, we also need explicit formulas for Hironaka's spherical functions on the space of Hermitian matrices with respect to a ramified quadratic extension. One of the main difficulties in obtaining explicit formulas for the spherical functions is obtaining the functional equations satisfied between them. Some partial results were already obtained by Hironaka in [Hir88] by relating the spherical functions to local densities and by complicated computations. Her results contain minor inaccuracies that should carefully be corrected before applied. I am currently working on this problem and hope that the methods will also apply for the more difficult case of the space of symmetric matrices. The values of spherical functions in this case should be related to orthogonal periods of automorphic forms on GL_n and to the problem described in §3.2.

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