ON UNITARY REPRESENTATIONS OF GL_{2n} DISTINGUISHED BY THE SYMPLECTIC GROUP

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Let F be a p-adic field and let $G = GL_{2n}(F)$. We denote by

$$H_x = \{g \in G|^t gxg = x\}$$

the symplectic group associated with the skew symmetric matrix $x \in G$. We further denote by H the group $H_{\epsilon_{2n}}$ where

$$\epsilon_{2n} = \begin{pmatrix} & w_n \\ -w_n & \end{pmatrix}$$

and w_n is the $n \times n$ permutation matrix with unit anti-diagonal.

Definition 1. A representation π of G is called H-distinguished if

$$\mathrm{Hom}_H(\pi, \mathbb{C}) \neq 0.$$

In this work we show that a certain family of irreducible, unitary representations of G are distinguished by the symplectic group H. In an upcoming work ([OS]) we show, in particular, that this family exhausts the irreducible, unitary, H-distinguished representations of G.

Our interest in local symplectic periods is motivated by the work of Klyachko over finite fields [Kl84]. In [HR90], Heumos and Rallis began the study of an analogue for p-adic fields. A survey of their work, with some motivation and the relation to periods of automorphic forms, can also be found in [Heu93]. Let us briefly describe the problem at hand.

Let ψ be an additive character of F and let U_q denote the group of upper triangular unipotent matrices in $GL_q(F)$. We denote by

$$\psi_q(u) = \psi(u_{1,2} + \cdots + u_{q-1,q})$$

the associated character of U_q. We will also denote by H_{2q} the symplectic group $H_{\epsilon_{2q}}$. For $0 \leq k \leq [\frac{q}{2}]$, let $M_{q,k}$ be the subgroup of $GL_q(F)$ of matrices of the form

$$\begin{pmatrix} u & X \\ 0 & h \end{pmatrix}$$

where $u \in U_{q-2k}$, $h \in H_{2k}$ and $X \in M_{(q-2k)\times 2k}(F)$. We denote by $\psi_{q,k}$ the character

$$\psi_{q,k}\begin{pmatrix} u & X \\ 0 & h \end{pmatrix} = \psi_{q-2k}(u).$$

We refer to the spaces

$$\mathcal{M}_{q,k} = \mathrm{Ind}_{M_{q,k}}^{GL_q(F)}(\psi_{q,k})$$

as *Klyachko models*. The model $\mathcal{M}_{2n,n}$ is refered to as a symplectic model and the Klyachko models interpolate between a Whittaker model and (if q is even) a symplectic model. An irreducible representation π of $GL_q(F)$ is said to have the Klyachko model $\mathcal{M}_{q,k}$ if

$$\mathrm{Hom}_{GL_q(F)}(\pi, \mathcal{M}_{q,k}) \neq 0.$$

Note that a representation is H-distinguished if and only if it has a symplectic model. In [Kl84], Klyachko showed that each irreducible representation of GL_q over a finite field has a unique Klyachko model. In [HR90], Heumos-Rallis provide evidence that every irreducible, unitary representation of $GL_q(F)$ has a Klyachko model. In fact they prove this fact for $q \leq 4$. They also show that irreducible, unitary representations can imbed in at most one of the different Klyachko models $\mathcal{M}_{q,k}$. We refer to [Heu93, p. 143] for the local conjecture and its global analogue. The present work is a step towards proving the conjectures of [loc. cit.]. In [OS], we will show that any irreducible unitary representation has a Klyachko model. Moreover, we specify the model it has in terms of the Tadic parameter of the representation. The exact description of a Klyachko model for any unitary representation, together with the main result of the present work and the unitary disjointness of models [HR90, Theorem 3.1], imply that the representations that we consider in Theorem 1 are precisely all irreducible, unitary representations that are distinguished by H.

To state our main theorem we briefly review Tadic's classification of the unitary dual of G [Tad86]. Denote by ν the character $g \mapsto |\det g|$ on $GL_q(F)$ for any q. For representations π_i of $GL_{q_i}(F)$, $i = 1, \dots t$ and for $q = q_1 + \dots + q_t$ we denote by $\pi_1 \times \dots \times \pi_t$ the representation of $GL_q(F)$ obtained from $\pi_1 \otimes \dots \otimes \pi_t$ by normalized parabolic induction. For a representation τ of $GL_q(F)$ and $\alpha \in \mathbb{R}$ we denote $\pi(\tau, \alpha) = \nu^\alpha \tau \times \nu^{-\alpha} \tau$. For representations π_i of $GL_{q_i}(F)$ set $\pi = \pi_1 \otimes \dots \otimes \pi_t$ and let $\lambda = (\lambda_1, \dots, \lambda_t) \in \mathbb{C}^t$. We denote

$$\pi[\lambda] = \nu^{\lambda_1} \pi_1 \otimes \dots \otimes \nu^{\lambda_t} \pi_t$$

and

$$I(\pi, \lambda) = \nu^{\lambda_1} \pi_1 \times \dots \times \nu^{\lambda_t} \pi_t.$$

Let

$$\Lambda_m = (\frac{m-1}{2}, \frac{m-3}{2}, \dots, \frac{1-m}{2}) \in \mathbb{R}^m.$$

A representation of $GL_r(F)$ is called square integrable if its matrix coefficients are square integrable modulo the center. Square integrable representations are in particular unitary. For a square integrable representation δ of $GL_r(F)$, the representation $I(\delta^{\otimes m}, \Lambda_m)$ has a unique irreducible quotient which we denote by $U(\delta, m)$. Let

$$B_u = \left\{ U(\delta, m), \pi(U(\delta, m), \alpha) : \delta\text{-square integrable}, \, m \in \mathbb{N}, |\alpha| < \frac{1}{2} \right\}.$$

A representation of the form $\sigma_1 \times \cdots \times \sigma_t$ where $\sigma_i \in B_u$, is irreducible and unitary. Any irreducible, unitary representation of $GL_q(F)$ for some q has this form and is uniquely determined by the multi-set of σ_i's up to reordering. This is the classification of Tadic. Our main result is the following.

Theorem 1. *Let $\pi = \sigma_1 \times \cdots \times \sigma_t \times \tau_{t+1} \times \cdots \times \tau_s$ be a unitary representation of G, such that $\sigma_i = U(\delta_i, 2m_i) \in B_u$ and $\tau_i = \pi(U(\delta_i, 2m_i), \alpha_i) \in B_u$. Then π is H-distinguished.*

In fact we prove in Proposition 2 that π is H-distinguished for a wider family of – not necessarily unitary – representations. Theorem 1 is a generalization of a result of Heumos-Rallis. They showed in [HR90] that the representations $U(\delta, 2)$ are H-distinguished. Their argument is the following. First, they construct a non-vanishing H-invariant functional on $I(\delta \otimes \delta, \Lambda_2)$. This representation has length 2 and its unique irreducible subrepresentation has a Whittaker model. The existence of an H-invariant functional on $U(\delta, 2)$ is therefore a consequence of the fact that irreducible generic representations are not H-distinguished. This is a special case of [HR90, Theorem 3.1]. The method of proof of Heumos-Rallis does not generalize directly to the case $m > 1$. In Remark 1 we explain where the difficulties lie. Our proof of Theorem 1 is in two steps. We first use global methods to show that the building blocks $U(\delta, 2m)$ are H-distinguished. We introduce in (1) a non zero H-invariant functional j_H on $I(\delta^{\otimes 2m}, \Lambda_{2m})$ and imbed $I(\delta^{\otimes 2m}, \Lambda_{2m})$ as the local component of a certain global representation $I(\sigma^{\otimes 2m}, \Lambda_{2m})$ induced from cuspidal. The functional j_H is then the corresponding local component of a certain factorizable, period $\mathfrak{j}_H$ on $I(\sigma^{\otimes 2m}, \Lambda_{2m})$ and $U(\delta, 2m)$ is the local component of the unique irreducible quotient $L(\sigma, 2m)$ of $I(\sigma^{\otimes 2m}, \Lambda_{2m})$. We then use the results of [Off06b] to show that $\mathfrak{j}_H$ is not identically zero and factors through $L(\sigma, 2m)$. The second step consists of showing that symplectic periods on the building blocks can be induced. Our proof of this fact is rather technical. The idea, due to Heumos-Rallis, is to apply Bernstein's principle of meromorphic continuation. This requires convergence of a certain complicated integral dependent on a complex parameter in some right half-plane. We accomplish this in Lemma 2 using an integration formula of Jacquet-Rallis [JR92]. In fact we now know that the hereditary property of symplectic periods follows from a recent work of Delorme-Blanc [DB].

1. Symplectic period on the building blocks

Let δ be a square integrable representation of $GL_r(F)$ and let $n = mr$. We construct an explicit non-zero and H-invariant linear form l_H on the representation $U(\delta, 2m)$ of $G = GL_{2n}(F)$. For a permutation $w \in \mathfrak{S}_{2m}$ in $2m$ variables, let $M(w)$ be the standard intertwining operator

$$M(w) : I(\delta^{\otimes 2m}, \Lambda_{2m}) \to I(\delta^{\otimes 2m}, w\Lambda_{2m}).$$

Let $w' = w'_{2m}$ be the permutation defined by

$$w'(2i-1) = i,\ w'(2i) = 2m+1-i,\ i = 1, \dots, m.$$

Let M be the standard Levi of type $(r, \dots, r)$ of G and let $M_H = M \cap H$. Up to a scalar, there is a unique M_H-invariant form on $\delta^{\otimes 2m}$ which we denote by l_{M_H}. Indeed, l_{M_H} is given by the pairing of $\delta^{\otimes m}$ with its contragradiant. Let K be the standard maximal compact subgroup of G and set $K_H = K \cap H$. The linear form

$$j_H(\varphi) = \int_{K_H} l_{M_H}(M(w')\varphi(k))dk \tag{1}$$

is a non-zero H-invariant form on $I(\delta^{\otimes 2m}, \Lambda_{2m})$. Indeed, this is shown in [Off06b, §3], when δ is any irreducible, generic, unitary representation. To obtain a symplectic period on $U(\delta, 2m)$ it is therefore enough to show that the form j_H factors through the unique irreducible quotient $U(\delta, 2m)$ of $I(\delta^{\otimes 2m}, \Lambda_{2m})$.

Remark 1. If δ is supercuspidal, we can show that the representation $I(\delta^{\otimes 2m}, \Lambda_{2m})$ has a decomposition series for which no factor (except $U(\delta, 2m)$) is H-distinguished. When $m = 1$, the same is true for any square integrable δ. This was the key point in the proof of [HR90, Theorem 11.1]. For $m > 1$ and δ square integrable this is in general no longer true. The representation $I(\delta^{\otimes 2m}, \Lambda_{2m})$ may have more then one decomposition factor which is H-distinguished. For this reason the method of proof of Heumos-Rallis does not generalize directly to $U(\delta, 2m)$. To overcome this problem we use a global approach.

We imbed our local problem in a global setting. In order to construct, locally, a non-vanishing symplectic period, we construct a global, decomposable symplectic period and apply [Off06b] to show that it factors through the unique irreducible quotient.

Proposition 1. *The form j_H on $I(\delta^{\otimes 2m}, \Lambda_{2m})$ factors through $U(\delta, 2m)$, i.e. it defines a non-zero H-invariant form on $U(\delta, 2m)$.*

Proof. We start with the following lemma.

Lemma 1. *Let δ be a square integrable representation of $GL_r(F)$. There is a number field k, a place v of k so that $F = k_v$ and a cuspidal automorphic representation σ of $GL_r(\mathbb{A}_k)$ so that $\delta = \sigma_v$.*

Proof of the lemma. The lemma follows from the proof of Proposition 5.15 in [Rog83]. ☐

Let k, v and σ be as in Lemma 1. Let P be the standard parabolic subgroup of G with Levi M. For $\lambda = (\lambda_1, \dots, \lambda_{2m}) \in \mathbb{C}^{2m}$ denote

$$I(\sigma^{\otimes 2m}, \lambda) = \mathrm{Ind}_{P(\mathbb{A}_k)}^{G(\mathbb{A}_k)}(|\det|_{\mathbb{A}_k}^{\lambda_1}\sigma \otimes |\det|_{\mathbb{A}_k}^{\lambda_2}\sigma \otimes \cdots \otimes |\det|_{\mathbb{A}_k}^{\lambda_{2m}}\sigma).$$

Let $L(\sigma, 2m)$ be the unique irreducible quotient of $I(\sigma^{\otimes 2m}, \Lambda_{2m})$. Then $I(\delta^{\otimes 2m}, \Lambda_{2m})$ is the local component of $I(\sigma^{\otimes 2m}, \Lambda_{2m})$ and $U(\delta, 2m)$ is the local component of $L(\sigma, 2m)$ at v. Let

$$\mathfrak{j}_H(\varphi) = \int_{K_H} \int_{M_H \backslash M_H(\mathbb{A}_k)^1} (M_{-1}(w')\varphi)(mk)\, dm\, dk$$

where $M_{-1}(w')$ is the multi-residue at Λ_{2m} of the standard intertwining operator

$$M(w', \lambda) : I(\sigma^{\otimes 2m}, \lambda) \to I(\sigma^{\otimes 2m}, w'\lambda).$$

It is shown in [Off06b], that the form $\mathfrak{j}_H$ is a non-zero $H(\mathbb{A}_{k,f})$-invariant form on $I(\sigma^{\otimes 2m}, \Lambda_{2m})$, where $\mathbb{A}_{k,f}$ is the ring of finite adèles of k. It is decomposable into local factors $\mathfrak{j}_H = \otimes_w j_{H,w}$ and $j_{H,v}$ is proportional to j_H given by (1). Let E_{-1} denote the intertwining operator that projects $I(\sigma^{\otimes 2m}, \Lambda_{2m}) \to L(\sigma, 2m)$. It is also decomposable. In [Off06a] it is shown that $\mathfrak{j}_H$ factors through $L(\sigma, 2m)$, i.e. there is a linear form $\mathfrak{l}_H$ on $L(\sigma, 2m)$ that makes the following diagram commute

$$(2) \qquad \begin{array}{ccc} I(\sigma^{\otimes 2m}, \Lambda_{2m}) & \xrightarrow{E_{-1}} & L(\sigma, 2m) \\ \mathfrak{j}_H \searrow & & \swarrow \mathfrak{l}_H \\ & \mathbb{C} & \end{array}.$$

Fix a decomposable element $\varphi_0 = \otimes_w \varphi_{0,w} \in I(\sigma^{\otimes 2m}, \Lambda_{2m})$ such that $\mathfrak{j}_H(\varphi_0) \neq 0$. For each $\varphi_v \in I(\delta^{\otimes 2m}, \Lambda_{2m})$ denote $\varphi = \otimes_{w \neq v} \varphi_{0,w} \otimes \varphi_v$. If φ_v is in the kernel of the projection $I(\delta^{\otimes 2m}, \Lambda_{2m}) \to U(\delta, 2m)$ then φ is in the kernel of E_{-1} and therefore $\mathfrak{j}_H(\varphi) = 0 = \mathfrak{j}_{H,v}(\varphi_v)\prod_{w\neq v}\mathfrak{j}_{H,w}(\varphi_{0,w})$. Thus $\mathfrak{j}_{H,v}(\varphi_v) = 0$. This shows that $\mathfrak{j}_{H,v}$ factors through $U(\delta, 2m)$. The proposition follows. $\square$

We only needed to introduce global notation for the proof of Proposition 1. For the remainder of this work we remain strictly in a local setting. Recall that j_H is the H-invariant form on $I(\delta^{\otimes 2m}, \Lambda_{2m})$ defined by (1). It follows from Proposition 1 that there exists an H-invariant form l_H on $U(\delta, 2m)$ such that $j_H = l_H \circ M(w_{2n})$.

2. Induction of the symplectic period

In this section we fix irreducible, square integrable representations δ_i of $GL_{r_i}(F)$, $i = 1, \dots, t$. We also fix $\alpha_1, \dots, \alpha_t \in \mathbb{R}$ and positive integers $m_1, \dots, m_t$.

Proposition 2. *The representation*

$$J = \nu^{\alpha_1} U(\delta_1, 2m_1) \times \cdots \times \nu^{\alpha_t} U(\delta_t, 2m_t) \tag{3}$$

is distinguished by H.

The rest of this work is devoted to the proof of Proposition 2. Let $k_i = m_i r_i$ and let $\mathbf{k} = (2k_1, \dots, 2k_t)$ be a partition of $2n$. Let $Q = LV$ be the standard parabolic subgroup of G of type $\mathbf{k}$, and let x be the skew symmetric matrix

$$x = \mathrm{diag}(\epsilon_{2k_1}, \dots, \epsilon_{2k_t}).$$

We denote $Q_x = Q \cap H_x$ and let $P = MU$ be the standard parabolic of G of type

$$(\overbrace{r_1, \dots, r_1}^{2m_1}, \dots, \overbrace{r_t, \dots, r_t}^{2m_t}).$$

Its Levi component is $M = M_1 \times \cdots \times M_t$ where M_i is the standard Levi of GL_{2k_i} of type $(r_i, \dots, r_i)$. We denote by $M_{i,H}$ the intersection of M_i with the symplectic group H_{2k_i} and by $K_{i,H}$ the intersection of H_{2k_i} with the standard maximal compact subgroup of $GL_{2k_i}(F)$. In [Off06b] we provided an H-filtration of induced representations and a useful description of their composition factors, using the geometric lemma of Bernstein-Zelevinsky. The filtration of J is parameterized by $Q\backslash G/H$. Let l_i be the symplectic period on $U(\delta_i, 2m_i)$ introduced in §1. It gives rise to a period on the first composition factor coming from the open double coset. Let $\eta \in G$ be such that $x = \eta\epsilon_{2n}{}^t\eta$. Then $\eta H \eta^{-1} = H_x$ and $Q\eta H$ is the open double coset. It is a consequence of Frobenious reciprocity that on the subspace of J, of functions supported on $Q\eta H$ we obtain a non-zero H-invariant functional defined by the formula

$$l_H(\varphi) = \int_{(H\cap\eta^{-1}Q\eta)\backslash H} (l_1 \otimes \cdots \otimes l_t)(\varphi(\eta h))\, dh =$$

$$\int_{Q_x\backslash H_x} (l_1 \otimes \cdots \otimes l_t)(\varphi(h\eta))\, dh. \tag{4}$$

However, this integral need not converge on the fully induced space J. We follow the ideas of [HR90] to bypass this obstacle. We let

$$J_s = \mathrm{Ind}_Q^G(\delta_Q^s \otimes (\nu^{\alpha_1}U(\delta_1, 2m_1) \otimes \cdots \otimes \nu^{\alpha_t}U(\delta_t, 2m_t)))$$

where δ_Q is the modulus function of Q. Denote by $l_{s,H}$ the linear form on J_s defined by the right hand side of (4). We show that for Res large enough and for $\varphi \in J_s$, the integral defining $l_{s,H}(\varphi)$ is absolutely convergent. It will then follow from the uniqueness of symplectic periods [HR90, Theorem 2.4.2], and from Bernstein's principle of meromorphic continuation as used in [HR90, p. 277-278], that J_s has a non-zero symplecic period, which is a rational function of q^s, where q is the cardinality of the residual field of F. This will provide a non-zero symplectic period on $J = J_0$. Indeed, there will be an integer m so that $s^m l_{s,H}$ is holomorphic and non-zero at $s = 0$. We therefore only need to show that for Re$s >> 0$ and for $\varphi \in J_s$, the integral on the right hand side of (4) is absolutely convergent. Let

$$I'_s = \mathrm{Ind}_P^G(\delta_Q^s|_P \otimes (\nu^{\alpha_1}\delta_1^{\otimes 2m_1}[\Lambda_{2m_1}] \otimes \cdots \otimes \nu^{\alpha_t}\delta_t^{\otimes 2m_t}[\Lambda_{2m_t}])).$$

Let E_i denote the projection from $I(\delta_i^{\otimes 2m_i}, \Lambda_{2m_i})$ to $U(\delta_i, \Lambda_{2m_i})$. The projection $E = E_1 \otimes \cdots \otimes E_t$ gives rise to a projection $\tilde{E}_s : I'_s \to J_s$ given by

$$(\tilde{E}_s(f))(g) = E(f(g)).$$

It is easy to see that if $\varphi = \tilde{E}_s(f)$, $f \in I'_s$ then

$$l_{s,H}(\varphi) = \int_{Q_x\backslash H_x} (j'_1 \otimes \cdots \otimes j'_t)(f(h\eta))\, dh \tag{5}$$

where $j'_i = l_i \circ E_i$ is the non-zero symplectic period on $I(\delta_i^{\otimes 2m_i}, \Lambda_{2m_i})$ introduced in (1). We let $j'_{s,H}$ be the linear form on I'_s defined by the right hand side of (5). Let

$$I_s = \mathrm{Ind}_P^G(\delta_Q^s|_P \otimes (\nu^{\alpha_1}\delta_1^{\otimes 2m_1}[w'_{2m_1}\Lambda_{2m_1}] \otimes \cdots \otimes \nu^{\alpha_t}\delta_t^{\otimes 2m_t}[w'_{2m_t}\Lambda_{2m_t}]))$$

and let $w' = \mathrm{diag}(w'_{2m_1}, \ldots, w'_{2m_t})$. Then $M(w')$ is the standard intertwining operator from I'_s to I_s. Making the j'_i's explicit we observe that

$$j'_{s,H} = j_{s,H} \circ M(w'),$$

where $j_{s,H}$ is the linear form on I_s given by

$$j_{s,H}(\varphi) = \tag{6}$$

$$\int_{Q_x \backslash H_x} \int_{K_{1,H} \times \cdots \times K_{t,H}} (l_{M_{1,H}} \otimes \cdots \otimes l_{M_{t,H}})(f(\mathrm{diag}(k_1, \ldots, k_t)h\eta))d(k_1, \ldots, k_t)\, dh$$

and $l_{M_{i,H}}$ is the $M_{i,H}$-invariant form on $\delta_i^{\otimes 2m_i}$. It is left to prove the following.

Lemma 2. *For Res >> 0 and $f \in I_s$, the integral (6) is absolutely convergent.*

Proof. It will be convenient to use the integration formula of Jacquet-Rallis [JR92] or rather its generalization to $Q_x \backslash H_x$ given in [Off06a]. We will need to introduce some new notation. We will try to minimize the notation and details and focus only on the information we need for our proof of convergence. More details regarding the integration formula can be found in ([Off06a], §5). For $Y = (y_1, \ldots, y_m) \in F^m$ let

$$\|Y\| = \max_{i=1}^{m}(|y_i|)$$

and let

$$\lambda(Y) = \max(\|Y\|, 1).$$

For $X = (X_1, \ldots, X_{t-1})$ where $X_i \in M_{2(k_{i+1}+\cdots+k_t)\times k_i}(F)$, we define a unipotent matrix $\sigma_{\mathbf{k}}(X) \in G$ by recursion on t as follows. Let $\mathbf{k}'$ be the partition of $2n$ defined by $\mathbf{k}' = (2k_1, \ldots, 2k_{t-2}, 2k_{t-1} + 2k_t)$. Define

$$\sigma_{\mathbf{k}}(X) = \begin{pmatrix} 1_{2(k_1+\cdots+k_{t-2})} & & & \\ & 1_{k_{t-1}} & & \\ & & 1_{k_{t-1}} & \\ & & X_{t-1} & 1_{k_t} \end{pmatrix} \sigma_{\mathbf{k}'}(X_1, \ldots, X_{t-2}).$$

For our purpose, it is enough to give the integration formula for the H_x-invariant measure on $Q_x \backslash H_x$, for functions ϕ on G which are left U-invariant. There is a function $\gamma(X)$ such that for functions ϕ as above we have

$$\int_{Q_x \backslash H_x} \phi(h)dh = \int_{K_{H_x}} \int \gamma(X)\phi(\sigma_{\mathbf{k}}(X)k)dX dk$$

where $K_{H_x} = K \cap H_x$. On the factor $\gamma(X)$ all we need to know is that there are constant c and m such that

$$\gamma(X) \leq c \left(\prod_{i=1}^{t-1} \lambda(X_i) \right)^m .$$

For $f \in I_s$ we therefore have

$$j_{s,H}(I_s(\eta^{-1})f) = \int_{K_{H_x}} \int \gamma(X) \int_{K_{1,H}\times\cdots\times K_{t,H}}$$

$$(l_{M_{1,H}} \otimes \cdots \otimes l_{M_{t,H}})(f(\mathrm{diag}(k_1,\dots,k_t)\sigma_{\mathbf{k}}(X)k))d(k_1,\dots,k_t)dXdk.$$

Since f is K-finite, fixing a basis $\{f_j\}$ of $I_s(K)f$, there are smooth functions a_j on K such that $I_s(k)(f) = \sum_j a_j(k)f_j$. It follows that $j_{s,H}(I_s(\eta^{-1})f)$ is the finite sum over j of $\int_{K_{H_x}} a_j(k)dk$ times

$$\int \gamma(X) \int_{K_{1,H}\times\cdots\times K_{t,H}} (l_{M_{1,H}}\otimes\cdots\otimes l_{M_{t,H}})(f_j(\mathrm{diag}(k_1,\dots,k_t)\sigma_{\mathbf{k}}(X)))d(k_1,\dots,k_t)dX.$$

To prove the proposition it is therefore enough to show that the integral

$$(7) \quad \int \left(\prod_{i=1}^{t-1} \lambda(X_i)\right)^m \int_{K_{1,H}\times\cdots\times K_{t,H}}$$
$$\left|(l_{M_{1,H}} \otimes \cdots \otimes l_{M_{t,H}})(f(\mathrm{diag}(k_1,\dots,k_t)\sigma_{\mathbf{k}}(X)))\right| d(k_1,\dots,k_t)dX$$

is convergent. For any matrix g we will denote by $\|\epsilon_i g\|$ the maximum of the absolute values of the $i\times i$ minors in the lower i rows of g. For each $j \in [1,t]$, $i \in [1,2m_j]$, let $R_{i,j} = ir_j + 2\sum_{q=j+1}^t k_q$. We write the coordinates of each Λ_{2m_j} as $\Lambda_{2m_j} = (\mu_1^j,\dots,\mu_{2m_j}^j)$ (in fact the convergence is proved for μ_i^j arbitrary). Let $\mu = (\Lambda_{2m_1},\dots,\Lambda_{2m_t}) \in \mathbb{R}^{2(m_1+\cdots+m_t)}$. For $p \in P$ with diagonal blocks $p_i^j \in GL_{r_j}(F)$, $j \in [1,t]$, $i \in [1,2m_j]$ we denote $p^\mu = \prod_{i,j} |\det p_i^j|^{\mu_i^j}$. If we write an Iwasawa decomposition of $g \in G$ with respect to P as $g = p(g)\kappa(g)$ then we have

$$f(g) = \delta_Q^s(p(g))p(g)^{\mu+\rho_P}(\bigotimes_{j=1}^t \delta_j^{\otimes 2m_i})(p(g))f(\kappa(g))$$

where ρ_P is half the sum of positive roots with respect to the parabolic P of G. If $g = pk$ where $p \in P$ has diagonal blocks denoted as before we may write

$$|\det p_i^j| = \frac{\|\epsilon_{R_{2m_j+1-i,j}}g\|}{\|\epsilon_R g\|}$$

where $R = R_{2m_j-i,j}$ if $i < 2m_j$ and $R = R_{1,j+1}$ otherwise. In other words we may find $\lambda \in \mathbb{R}^{2(m_1+\cdots+m_t)}$ dependent only on μ so that

$$f(g) = \delta_Q^s(p(g))\prod_{i,j} \|\epsilon_{R_{i,j}}g\|^{\lambda_i^j}(\bigotimes_{j=1}^t \delta_j^{\otimes 2m_i})(p(g))f(\kappa(g)).$$

The integral (7) then becomes

(8)
$$\int \left(\prod_{i=1}^{t-1} \lambda(X_i)\right)^m \int_{K_{1,H}\times\cdots\times K_{t,H}} \delta_Q^s(p(\sigma_{\mathbf{k}}(X))) \prod_{i,j} \|\epsilon_{R_{i,j}} \mathrm{diag}(k_1,\dots,k_t)\sigma_{\mathbf{k}}(X)\|^{\lambda_i^j}$$
$$\big|(l_{M_{1,H}} \otimes \cdots \otimes l_{M_{t,H}})((\otimes_{j=1}^t \delta_j^{\otimes 2m_i})(p(\mathrm{diag}(k_1,\dots,k_t)\sigma_{\mathbf{k}}(X)))$$
$$f(\kappa(\mathrm{diag}(k_1,\dots,k_t)\sigma_{\mathbf{k}}(X))))\big|\, d(k_1,\dots,k_t)\, dX.$$

We first claim that the expression in the absolute value is now bounded, independently of $k_1,\dots,k_t$ and X. Indeed, f being smooth, obtains only finitely many values on K and therefore it is enough to bound

$$(l_{M_{1,H}} \otimes \cdots \otimes l_{M_{t,H}})((\otimes_{j=1}^t \delta_j^{\otimes 2m_i})(p(\mathrm{diag}(k_1,\dots,k_t)\sigma_{\mathbf{k}}(X)))v$$

for any v in the space of $\otimes_{j=1}^t \delta_j^{\otimes 2m_j}$. We may further assume that v decomposes as $v = v_{1,1} \otimes v_{1,2} \otimes \cdots \otimes v_{t,1} \otimes v_{t,2}$ where $v_{i,j}$ lies in the space of $\delta_i^{\otimes m_i}$ for $j = 1, 2$. For $p \in M$ we denote by $p_i^j \in GL_{r_i}(F)$ its diagonal blocks as before. The map

$$p \mapsto (l_{M_{1,H}} \otimes \cdots \otimes l_{M_{t,H}})((\bigotimes_{j=1}^t \delta_j^{\otimes 2m_i})(p)v)$$

is a matrix coefficient of the unitary representation $\bigotimes_{j=1}^t \delta_j^{\otimes m_j}$ evaluated at

$$\mathrm{diag}((\tilde{p}^1_{2m_1})^{-1}p^1_1,\dots,(\tilde{p}^1_{m_1+1})^{-1}p^1_{m_1},(\tilde{p}^2_{2m_2})^{-1}p^2_1,\dots,(\tilde{p}^t_{m_t+1})^{-1}p^t_{m_t}).$$

Here $\tilde{g} = w_q {}^t g^{-1} w_q$ for $g \in GL_q(F)$. Matrix coefficients of unitary representations are bounded. It is therefore enough to show that for $\mathrm{Re} s$ large enough the expression

(9)
$$\int \left(\prod_{i=1}^{t-1} \lambda(X_i)\right)^m \int_{K_{1,H}\times\cdots\times K_{t,H}} \delta_Q^s(p(\sigma_{\mathbf{k}}(X)))$$
$$\prod_{i,j} \|\epsilon_{R_{i,j}} \mathrm{diag}(k_1,\dots,k_t)\sigma_{\mathbf{k}}(X)\|^{\lambda_i^j} \times d(k_1,\dots,k_t)dX$$

converges. In order to bound the integrand in (9), we will use the following two claims.

Claim 1: There exist an N such that

$$1 \le \|\epsilon_{R_{i,j}} \mathrm{diag}(k_1,\dots,k_t)\sigma_{\mathbf{k}}(X)\| \le \left(\prod_{i=1}^{t-1} \lambda(X_i)\right)^N$$

and

Claim 2:

$$\delta_Q^s(p(\sigma_{\mathbf{k}}(X))) \leq \left(\prod_{i=1}^{t-1} \lambda(X_i)\right)^{-s}.$$

The upper bound in **Claim 1** is obvious. We show the lower bound. To avoid ambiguity of notation let us denote by $k_{i,H}$ the elements of $K_{i,H}$. Note that the lower $R_{1,j}$ rows of $\mathrm{diag}(k_{1,H},\dots,k_{t,H})\sigma_{\mathbf{k}}(X)$ have the form

$$\begin{pmatrix} * & k_{j,H} & 0 \\ * & * & \mathrm{diag}(k_{j+1,H},\dots,k_{t,H})\sigma_{(2k_{j+1},\dots,2k_t)}(X_{j+1},\dots,X_{t-1}) \end{pmatrix}$$

where we put $*$ in each block that will play no role for us. For each $i \in [1, 2m_j]$ there is a $ir_j \times ir_j$ minor A in the lower ir_j rows of $k_{j,H}$ of absolute value 1. Together with the lower right $2(k_{j+1}+\dots k_t) \times 2(k_{j+1}+\dots k_t)$-block of $\mathrm{diag}(k_{1,H},\dots,k_{t,H})\sigma_{\mathbf{k}}(X)$ we get that

$$\begin{pmatrix} A & 0 \\ * & \mathrm{diag}(k_{j+1,H},\dots,k_{t,H})\sigma_{(2k_{j+1},\dots,2k_t)}(X_{j+1},\dots,X_{t-1}) \end{pmatrix}$$

is a $R_{i,j} \times R_{i,j}$ minor of absolute value 1 in the lower $R_{i,j}$ rows of the matrix $\mathrm{diag}(k_{1,H},\dots,k_{t,H})\sigma_{\mathbf{k}}(X)$. This shows **Claim 1**. To show **Claim 2**, we note that if $|\det g| = 1$ then

$$\delta_Q(p(g)) = \prod_{j=1}^{t-1} \|\epsilon_{R_{1,j}} g\|^{-2k_{t+1-j}-2k_{t-j}}.$$

It can be proved as in [Off06a, Lemma 5.5], that

$$\|\epsilon_{R_{1,j}} p(\sigma_{\mathbf{k}}(X))\| \geq \lambda(X_{t-j}).$$

Claim 2 readily follows. Using the two claims, we bound the integral (9) by replacing each term $\|\epsilon_{R_{i,j}}\mathrm{diag}(k_1,\dots,k_t)\sigma_{\mathbf{k}}(X)\|^{\lambda_i^j}$ by 1 if $\lambda_i^j \leq 0$ and by a certain fixed and large enough power of $\left(\prod_{i=1}^{t-1}\lambda(X_i)\right)$ otherwise. It is shown in [JR92] that for any q the integral

$$\int_{F^q} \lambda(Y)^{-s} dY$$

is convergent for $s >> 0$. The lemma therefore follows from the two claims. $\square$

This concludes the proof of Proposition 2 and in particular also of Theorem 1.

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