UNIQUENESS AND DISJOINTNESS OF KLYACHKO MODELS

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Abstract. We show the uniqueness and disjointeness of Klyachko models for GL_n over a non-archimedean local field. This completes, in particular, the study of Klyachko models on the unitary dual. Our local results imply a global rigidity property for the discrete automorphic spectrum.

1. Introduction

In this work we show that over a local non-archimedean field, the mixed (symplectic-Whittaker) models introduced by Klyachko in [Kly84] are disjoint and that multiplicity one is satisfied. In [OS] we showed, over a p-adic field (a finite extension of $\mathbb{Q}_p$), the existence of Klyachko models for unitary representations. The up shot is then that for every irreducible, unitary representation of GL_n over a p-adic field there is a unique Klyachko model where it appears and it appears there with multiplicity one.

To formulate the main result more precisely we introduce some notation. Let F be a non-archimedean local field. For a positive integer r, denote by U_r the subgroup of upper triangular unipotent matrices in GL_r and let

$$Sp_{2k} = \{g \in GL_{2k} : {}^t g J_{2k} g = J_{2k}\}$$

where

$$J_{2k} = \begin{pmatrix} 0 & w_k \\ -w_k & 0 \end{pmatrix}$$

and $w_k \in GL_k(F)$ is the matrix with $(i,j)^{th}$ entry equal to $\delta_{i,n+1-j}$. Whenever $n = r + 2k$ we consider the subgroup $H_{r,2k}$ of GL_n defined by

$$H_{r,2k} = \{\begin{pmatrix} u & X \\ 0 & h \end{pmatrix} : u \in U_r,\, X \in M_{r\times 2k},\, h \in Sp_{2k}\}.$$

Let ψ be a non-trivial character of F. For $u = (u_{i,j}) \in U_r(F)$ we set

$$\psi_r(u) = \psi(u_{1,2} + \cdots + u_{r-1,r}). \tag{1}$$

Let $\psi_{r,2k}$ be the character of $H_{r,2k}(F)$ defined by

$$\psi_{r,2k}\begin{pmatrix} u & X \\ 0 & h \end{pmatrix} = \psi_r(u). \tag{2}$$

When $n = r + 2k$ the space

$$\mathcal{M}_{r,2k} = \mathrm{Ind}_{H_{r,2k}(F)}^{GL_n(F)}(\psi_{r,2k})$$

is referred to as a Klyachko model and we say that a representation π of $GL_n(F)$ admits the Klyachko model $\mathcal{M}_{r,2k}$ if $\mathrm{Hom}_{GL_n(F)}(\pi, \mathcal{M}_{r,2k}) \neq 0$. Here Ind denotes

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the functor of non-compact smooth induction and representations of $GL_n(F)$ are always assumed to be smooth. The main result of this paper is the following.

Theorem 1. *Let F be a non-archimedean local field and let π be an irreducible representation of $GL_n(F)$ then*

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{GL_n(F)}(\pi, \bigoplus_{k=0}^{[\frac{n}{2}]} \mathcal{M}_{n-2k,2k})) \leq 1. \tag{3}$$

Denote by

$$m_\pi = \dim_{\mathbb{C}}(\mathrm{Hom}_{GL_n(F)}(\pi, \bigoplus_{k=0}^{[\frac{n}{2}]} \mathcal{M}_{n-2k,2k}))$$

the multiplicity of π in the direct sum of the Klyachko models. When F is a finite field, it is proved in [IS91] that $m_\pi = 1$ for every irreducible representation π of $GL_n(F)$. When F is a non-archimedean local field it is shown in [HR90] that there exists an irreducible representation π of $GL_3(F)$ so that $m_\pi = 0$. Thus, we cannot expect in general for the inequality (3) to be an equality. However, in [OS] we showed that if F is a p-adic field then $m_\pi \geq 1$ for every irreducible, unitary representation π of $GL_n(F)$. We therefore have the following.

Corollary 1. *Let F be a p-adic field and let π be an irreducible, unitary representation of $GL_n(F)$ then*

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{GL_n(F)}(\pi, \bigoplus_{k=0}^{[\frac{n}{2}]} \mathcal{M}_{n-2k,2k})) = 1.$$

By Frobenius receiprocity [BZ76, §2.28] for a representation π of $GL_n(F)$ we have

$$\mathrm{Hom}_{GL_n(F)}(\pi, \mathcal{M}_{r,2k}) = \mathrm{Hom}_{H_{r,2k}(F)}(\pi, \psi_{r,2k}). \tag{4}$$

It follows that for an irreducible, unitary representation π of $GL_n(F)$ there is a unique integer $0 \leq \kappa(\pi) \leq [\frac{n}{2}]$ such that

$$\mathrm{Hom}_{H_{n-2\kappa(\pi),2\kappa(\pi)}(F)}(\pi, \psi_{n-2\kappa(\pi),2\kappa(\pi)}) \neq 0,$$

i.e. such that π is $(H_{n-2\kappa(\pi),2\kappa(\pi)}, \psi_{n-2\kappa(\pi),2\kappa(\pi)})$-*distinguished* and that the space of such functionals is one dimensional.

Remark 1. In [OS07, Theorem 1], when F is a p-adic field and n is even, we exhibited a family of irreducible, unitary representations of $GL_n(F)$ that are $Sp_n(F)$-distinguished. We promised in [OS07] that in [OS] we will show that this family exhausts all irreducible, unitary representations that are $Sp_n(F)$-distinguished. Eventually, we postponed the delivery of this statement to the current paper. It is immediate from Corollary 1.

In [OS] we also studied globally over a number field, the mixed (symplectic-Whittaker) periods on the discrete automorphic spectrum of GL_n. Let F be a number field and let ψ be a non-trivial character of $F\backslash\mathbb{A}_F$. We use (1) to view ψ_r as a character of $U_r(\mathbb{A}_F)$ and (2) to view $\psi_{r,2k}$ as a character of $H_{r,2k}(\mathbb{A}_F)$. For an automorphic form ϕ in the discrete automorphic spectrum of $GL_n(\mathbb{A}_F)$ and a decomposition $n = r + 2k$ we consider the mixed period integral

$$P_{r,2k}(\phi) = \int_{H_{r,2k}(F)\backslash H_{r,2k}(\mathbb{A}_F)} \phi(h)\, \psi_{r,2k}(h)\ dh. \tag{5}$$

We say that an irreducible, discrete spectrum automorphic representation π of $GL_n(\mathbb{A}_F)$ is $H_{r,2k}$-*distinguished* if $P_{r,2k}$ is not identically zero on the space of π. In [OS] we provided an explicit integer $0 \le \kappa(\pi) \le [\frac{n}{2}]$ such that π is $(H_{n-2\kappa(\pi),2\kappa(\pi)}, \psi_{n-2\kappa(\pi),2\kappa(\pi)})$-*distinguished*. Furthermore, we showed that this period integral is factorizable. Corollary 1 (particularly, the disjointness of Klyachko models) then shows that there is a unique such integer. Furthermore, it implies the following rigidity property of the discrete automorphic spectrum of GL_n.

Theorem 2. *Let F be a number field and let $\pi = \otimes_v \pi_v$ be an irreducible, discrete spectrum automorphic representation of $G(\mathbb{A}_F)$. Then the following are equivalent:*

1. *π is $(H_{r,2k}, \psi_r)-$distinguished;*
2. *π_v is $(H_{r,2k}, \psi_r)-$distinguished for all places v of F;*
3. *π_{v_0} is $(H_{r,2k}, \psi_r)-$distinguished for some finite place v_0 of F.*

The rest of this work is organized as follows. After setting up the notation in §2, in §3-§4 we reduce Theorem 1 to a statement about invariant distributions on orbits using the Gelfand-Kazhdan theory. This statement is made more explicit in §5 and is then proved by induction in §6.

2. Notation

Let F be a non-archimedean local field and for any positive integer r let $G_r = GL_r(F)$. We also set $G_0 = \{1\}$. Throughout, we fix a positive integer n and let $G = G_n$. For a partition $(n_1, \dots, n_t)$ of n we denote by $P_{(n_1,\dots,n_t)}$ the standard parabolic subgroup of G of type $(n_1, \dots, n_t)$. It consists of matrices in upper triangular block form. If $P = P_{(n_1,\dots,n_t)}$ we denote by $\overline{P}$ the parabolic opposite to P. It consists of matrices in lower triangular form. When we say that $P = MU$ is the standard Levi decomposition of P we mean that U is its unipotent radical, and $M = P \cap \overline{P} = \{\mathrm{diag}(g_1, \dots, g_t) : g_i \in G_{n_i}\}$. We then denote by $\overline{U}$ the unipotent radical of $\overline{P}$. We denote by $a^{(r)}$ the r-tuple $(a, \dots, a)$, thus for example $P_{(1)^n}$ is the subgroup of upper triangular matrices in G. For any standard Levi subgroup M of G denote by W_M the Weyl group of M and let $W = W_G$. If M' is another standard Levi subgroup then any double coset in $W_M \backslash W / W_{M'}$ has a unique element of minimal length which we refer to as a left W_M and right $W_{M'}$ reduced Weyl element. We denote by ${}_M W_{M'}$ the set of all left W_M and right $W_{M'}$ reduced Weyl elements. For integers a and b we set $[a, b] = \{x \in \mathbb{Z} : a \le x \le b\}$. For any subset $A \subseteq [1, n]$ we denote by SA the permutation group in the elements of A. It will be convenient to identify W with $S[1, n]$. If $P = MU$ and $P' = M'U'$ are standard parabolic subgroups of G with their standard Levi decompositions, the Bruhat decomposition of G gives the disjoint union

$$G = \bigsqcup_{w \in {}_M W_{M'}} Pw\overline{P'}. \tag{6}$$

For any matrix X let ${}^tX$ denote the transpose matrix. For a skew-symmetric matrix $\mathcal{I} = -{}^t\mathcal{I} \in G_{2k}$ let

$$Sp(\mathcal{I}) = \{g \in G_{2k} : {}^tg\mathcal{I}g = \mathcal{I}\}$$

and let

$$J_{2k} = \begin{pmatrix} 0 & w_k \\ -w_k & 0 \end{pmatrix}$$

where $w_k \in G_k$ is the matrix with $(i,j)^{th}$ entry $\delta_{i,n+1-j}$. Denote by U_r the subgroup of upper triangular unipotent matrices and by $\overline{U}_r$ the subgroup of lower triangular unipotent matrices in G_r. For non-negative integers r and k let

$$H_{r,2k} = \{\begin{pmatrix} u & X \\ 0 & h \end{pmatrix} : u \in U_r,\ X \in M_{r\times 2k}(F),\ h \in Sp(J_{2k})\}$$

and let

$$\overline{H}_{r,2k} = \{\begin{pmatrix} u & 0 \\ X & h \end{pmatrix} : u \in \overline{U}_r,\ X \in M_{2k\times r}(F),\ h \in Sp(J_{2k})\}.$$

Note that $\overline{H}_{r,2k}$ is the image of $H_{r,2k}$ under transpose. For $g \in G$ let

$$g^\tau = {}^t g^{-1}.$$

The restriction to $H_{r,2k}$ of the involution $\tau : G \to G$ defines a group isomorphism from $H_{r,2k}$ to $\overline{H}_{r,2k}$. Let $n = r+2k = r'+2k'$ and let $\mathcal{H}^{r,r'} = \mathcal{H}^{r,r'}_n = H_{r,2k} \times \overline{H}_{r',2k'}$. Thus

$$\mathcal{H}^{r,r'} = \{(h_1, h_2^\tau) : h_1 \in H_{r,2k},\ h_2 \in H_{r',2k'}\}.$$

We denote by $e_{\mathcal{H}^{r,r'}}$ the identity element of $\mathcal{H}^{r,r'}$. It will also be useful to consider the map $\xi : \mathcal{H}^{r,r'} \to \mathcal{H}^{r',r}$ defined by

$$\xi(h_1, h_2^\tau) = (h_2, h_1^\tau).$$

The group $\mathcal{H}^{r,r'}$ acts on G by

$$h \cdot g = h_1 g\, {}^t h_2,\ h = (h_1, h_2^\tau) \in \mathcal{H}^{r,r'},\ g \in G.$$

We observe that

$$ {}^t(h \cdot g) = \xi(h) \cdot {}^t g,\ h \in \mathcal{H}^{r,r'},\ g \in G. \tag{7}$$

When $r = r'$ the map ξ is an involution of $\mathcal{H}^{r,r}$. The formula (7) allows us then to define the semi direct product

$$\widetilde{\mathcal{H}}^{r,r} = \mathcal{H}^{r,r} \rtimes \{\pm 1\}$$

with multiplication rule

$$(h,\epsilon)(h',\epsilon') = (h\,\xi_\epsilon(h'), \epsilon\epsilon') \text{ where } \xi_\epsilon(h) = \begin{cases} h & \epsilon = 1 \\ \xi(h) & \epsilon = -1. \end{cases}$$

Here $h,\ h' \in \mathcal{H}^{r,r}$, $\epsilon,\ \epsilon' \in \{\pm 1\}$. The group $\widetilde{\mathcal{H}}^{r,r}$ acts on G by

$$(h,\epsilon) \cdot g = h \cdot T_\epsilon(g) \text{ where } T_\epsilon(g) = \begin{cases} g & \epsilon = 1 \\ {}^t g & \epsilon = -1. \end{cases}$$

In order to unify notation, when $r \neq r'$ we shall set $\widetilde{\mathcal{H}}^{r,r'} = \mathcal{H}^{r,r'} \times \{1\}$.

For a non-trivial character ψ of F we define as in §1 the generic character ψ_r of U_r by (1) and the character $\psi_{r,2k}$ of $H_{r,2k}$ by (2). Let $\theta^{r,r'}$ be the character of $\mathcal{H}^{r,r'}$ defined by

$$\theta^{r,r'}(h_1, h_2^\tau) = \psi_{r,2k}(h_1)\psi_{r',2k'}(h_2).$$

We also extend $\theta^{r,r'}$ to the character $\widetilde{\theta}^{r,r'}$ of $\widetilde{\mathcal{H}}^{r,r'}$ defined by

$$\widetilde{\theta}^{r,r'}(h,\epsilon) = \epsilon\ \theta^{r,r'}(h).$$

3. Reduction to Invariant Distributions

Let $n = r + 2k = r' + 2k'$ be 2 decompositions of n. Let $\mathcal{H} = \mathcal{H}^{r,r'}$ and $\theta = \theta^{r,r'}$. The action of $\widetilde{\mathcal{H}}$ on G defines an action on $C_c^\infty(G)$ and on the space $\mathfrak{D}(G) = C_c^\infty(G)^*$ of distributions on G by

$$(h \cdot \phi)(g) = \phi(h^{-1} \cdot g) \text{ and } (h \cdot D)(\phi) = D(h^{-1} \cdot \phi)$$

for $h \in \widetilde{\mathcal{H}}$, $g \in G$, $\phi \in C_c^\infty(G)$ and $D \in \mathfrak{D}(G)$. In this section we show that Theorem 1 reduces to the following.

Proposition 1. *If $D \in \mathfrak{D}(G)$ is such that $h \cdot D = \tilde{\theta}(h)D$ for all $h \in \widetilde{\mathcal{H}}$ then $D = 0$, i.e.*

$$\mathrm{Hom}_{\widetilde{\mathcal{H}}}(C_c^\infty(G), \tilde{\theta}) = 0. \tag{8}$$

3.1. **Proposition 1 implies Theorem 1.** Let π be an irreducible representation of G. Set $H = H_{r,2k}$, $H' = H_{r',2k'}$, $\psi = \psi_{r,2k}$ (forgive the abuse of notation) and $\psi' = \psi_{r',2k'}$. Denote by $\overline{H}$ (resp. $\overline{H'}$) the image of H (resp. H') under τ. Let $\ell \in \mathrm{Hom}_H(\pi, \psi)$ and $\ell' \in \mathrm{Hom}_{H'}(\pi, \psi')$. The representation $\pi^\tau(g) = \pi(g^\tau)$ realizes the contragradiant representation $\tilde{\pi}$ on the space V_π of π [GK75] (see also [BZ76, Theorem 7.3]). Note that $\ell' \in \mathrm{Hom}_{\overline{H'}}(\pi^\tau, (\psi')^\tau)$ defines a functional $\tilde{\ell}'$ on the space $V_{\tilde{\pi}}$ of $\tilde{\pi}$ and that $\tilde{\ell}' \in \mathrm{Hom}_{\overline{H'}}(\tilde{\pi}, (\psi')^\tau)$. Note further that $\ell \circ \pi(\phi)$ is a smooth vector in $V_{\tilde{\pi}}$. Define the distribution D on G by

$$D(\phi) = \tilde{\ell}'(\ell \circ \pi(\phi)),\ \phi \in C_c^\infty(G). \tag{9}$$

For $h \in H$ and $h' \in H'$ we have $\pi((h^{-1}, {}^t h') \cdot \phi) = \pi(h) \circ \pi(\phi) \circ \pi({}^t h')$ and therefore

$$((h, (h')^\tau) \cdot D)(\phi) = \tilde{\ell}'(\ell \circ \pi(h) \circ \pi(\phi) \circ \pi({}^t h')).$$

By our assumption on ℓ and ℓ' we have, $\ell \circ \pi(h) = \psi(h)\ell$ and $\tilde{\ell}' \circ \tilde{\pi}((h')^\tau) = \psi'(h')\tilde{\ell}'$, $h \in H$, $h' \in H'$. Also note that for any $\tilde{v} \in V_{\tilde{\pi}}$ viewed as a smooth functional on π the composition $\tilde{v} \circ \pi(g)$ is again a smooth functional on π and in fact

$$(\tilde{v} \circ \pi(g))(v) = \tilde{v}(\pi(g)v) = (\tilde{\pi}(g^{-1})\tilde{v})(v)$$

i.e.,

$$\tilde{v} \circ \pi(g) = \tilde{\pi}(g^{-1})\tilde{v}.$$

Applying this to $\tilde{v} = \ell \circ \pi(\phi)$ and $g = {}^t h'$ we get that

$$\begin{aligned} ((h, (h')^\tau) \cdot D)(\phi) &= \psi(h)\ \tilde{\ell}'((\ell \circ \pi(\phi)) \circ \pi({}^t h')) \\ &= \psi(h)\ \tilde{\ell}'(\tilde{\pi}((h')^\tau)(\ell \circ \pi(\phi))) = \theta(h, (h')^\tau)\ D(\phi). \end{aligned}$$

We see that D is $(\mathcal{H}, \theta)$-equivariant. If $r \neq r'$ it follows from Proposition 1 that $D = 0$. If we assume further that ℓ is non-zero then the vectors $\ell \circ \pi(\phi)$, $\phi \in C_c^\infty(G)$ span $V_{\tilde{\pi}}$. We conclude that $\tilde{\ell}'$ must vanish identically on $V_{\tilde{\pi}}$ and hence also $\ell' = 0$. This shows that

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{H_{r,2k}}(\pi, \psi_{r,2k})) \dim_{\mathbb{C}}(\mathrm{Hom}_{H_{r',2k'}}(\pi, \psi_{r',2k'})) = 0 \text{ whenever } r \neq r'. \tag{10}$$

Assume now that $r = r'$. Recall that $e_{\mathcal{H}}$ is the unit element of $\mathcal{H}$. Note that $(e_{\mathcal{H}}, -1) \cdot \phi = {}^t\phi$ where ${}^t\phi(g) = \phi({}^t g)$, $\phi \in C_c^\infty(G)$, $g \in G$. Note further that for every $h \in \mathcal{H}$ we have

$$(h, 1)(e_{\mathcal{H}}, -1) = (e_{\mathcal{H}}, -1)(\xi(h), 1)$$

and that $\theta(\xi(h)) = \theta(h)$. Since $D \in \mathrm{Hom}_{\mathcal{H}}(C_c^\infty(G), \theta)$, it also follows that

$$D_1 = D - (e_{\mathcal{H}}, -1) \cdot D \in \mathrm{Hom}_{\mathcal{H}}(C_c^\infty(G), \theta).$$

Furthermore, since $\tilde{\theta}(e_{\mathcal{H}}, -1) = -1$ and $(e_{\mathcal{H}}, -1) \cdot D_1 = -D_1$ we conclude that $D_1 \in \mathrm{Hom}_{\widetilde{\mathcal{H}}}(C_c^\infty(G), \tilde{\theta})$. Proposition 1 now implies that

$$D = (e_{\mathcal{H}}, -1) \cdot D. \tag{11}$$

Let $B : C_c^\infty(G) \times C_c^\infty(G) \to \mathbb{C}$ be the bilinear form defined by

$$B(\phi_1, \phi_2) = D(\phi_1 * \phi_2) \tag{12}$$

where

$$(\phi_1 * \phi_2)(g) = \int_G \phi_1(x)\phi_2(x^{-1}g)\, dx.$$

Note that

$$\pi(\phi_1 * \phi_2) = \pi(\phi_1) \circ \pi(\phi_2) \text{ and } {}^t(\phi_1 * \phi_2) = {}^t\phi_2 * {}^t\phi_1,\ \phi_1,\ \phi_2 \in C_c^\infty(G).$$

Thus, (11) implies that

$$B(\phi_1, \phi_2) = B((e_{\mathcal{H}}, -1) \cdot \phi_2, (e_{\mathcal{H}}, -1) \cdot \phi_1).$$

This implies that $R_B = (e_{\mathcal{H}}, -1) \cdot L_B$ where

$$L_B = \{\phi \in C_c^\infty(G) : B(\phi, \cdot) \equiv 0\} \text{ and } R_B = \{\phi \in C_c^\infty(G) : B(\cdot, \phi) \equiv 0\}$$

are respectively the left and right kernels of B. In other words

$$R_B = \{{}^t\phi : \phi \in L_B\}. \tag{13}$$

For a functional λ on V_π let

$$\mathfrak{K}(\lambda, \pi) = \{\phi \in C_c^\infty(G) : \lambda \circ \pi(\phi) = 0\}.$$

Note that

$$B(\phi_1, \phi_2) = (\tilde{\ell}' \circ \pi(\phi_2^\vee))(\ell \circ \pi(\phi_1))$$

where

$$\phi^\vee(g) = \phi(g^{-1})$$

and therefore

$$L_B = \mathfrak{K}(\ell, \pi) \text{ and } R_B = \{\phi^\vee : \phi \in \mathfrak{K}(\tilde{\ell}', \tilde{\pi})\}.$$

By our definitions we have

$$\mathfrak{K}(\tilde{\ell}', \tilde{\pi}) = \mathfrak{K}(\ell', \pi^\tau) = \{({}^t\phi)^\vee : \phi \in \mathfrak{K}(\ell', \pi)\}$$

and therefore

$$R_B = \{{}^t\phi : \phi \in \mathfrak{K}(\ell', \pi)\}.$$

It now follows from (13) that

$$\mathfrak{K}(\ell, \pi) = \mathfrak{K}(\ell', \pi).$$

Since π is irreducible we get that $\ker \ell = \ker \ell'$ and therefore that ℓ and ℓ' are proportional. We therefore proved that

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{H_{r,2k}}(\pi, \psi_{r,2k})) \le 1 \text{ for all } 0 \le k \le [\frac{n}{2}]. \tag{14}$$

Theorem 1 is now a straightforward consequence of (4), (10) and (14).

4. Reduction to $\mathcal{H}$-orbits

We keep the notation introduced in §3. For every $g \in G$ we denote by $\mathcal{H}_g$ the stabilizer of g in $\mathcal{H}$ and by $\widetilde{\mathcal{H}}_g$ the stabilizer of g in $\widetilde{\mathcal{H}}$. The purpose of this section is to reduce Proposition 1 to the following.

Proposition 2. *For every $g \in G$ the character $\widetilde{\theta}$ is non-trivial on $\widetilde{\mathcal{H}}_g$.*

Remark 2. The objects involved and the statement of Proposition 2 makes sense over any field F and in fact, our proof is valid in this generality. In particular, using Mackey theory, it can provide an alternative proof of the uniqueness and disjointness of Klyachko models over a finite field.

4.1. **Proposition 2 implies Proposition 1.** Assume now that Proposition 2 holds. We deduce that Proposition 1 also holds. Let $\mathbf{1}_{\widetilde{\mathcal{H}}_g}$ denote the trivial character of $\widetilde{\mathcal{H}}_g$. Note that $h \cdot g \mapsto \widetilde{\mathcal{H}}_g\, h^{-1}$ is a homeomorphism of $\widetilde{\mathcal{H}}$-spaces $\widetilde{\mathcal{H}} \cdot g \simeq \widetilde{\mathcal{H}}_g \backslash \widetilde{\mathcal{H}}$ that induces an $\widetilde{\mathcal{H}}$-isomorphism

$$C_c^{\infty}(\widetilde{\mathcal{H}} \cdot g) \simeq \operatorname{ind}_{\widetilde{\mathcal{H}}_g}^{\widetilde{\mathcal{H}}}(\mathbf{1}_{\widetilde{\mathcal{H}}_g}) \tag{15}$$

where ind denotes smooth induction with compact support. Therefore, by Frobenius reciprocity [BZ76, §2.29]

$$\operatorname{Hom}_{\widetilde{\mathcal{H}}}(C_c^{\infty}(\widetilde{\mathcal{H}} \cdot g), \tilde{\theta}) = \operatorname{Hom}_{\widetilde{\mathcal{H}}_g}(\delta_{\widetilde{\mathcal{H}}_g}, \theta_{|\widetilde{\mathcal{H}}_g}) \tag{16}$$

where $\delta_{\widetilde{\mathcal{H}}_g}$ is the modulus function of $\widetilde{\mathcal{H}}_g$. Since the image of $\tilde{\theta}$ lies in the unit circle (in fact, the image of θ lies in the group of p-powered roots of unity where p is the residual characteristic of F) and since $\delta_{\widetilde{\mathcal{H}}_g}$ is positive, we get that whenever $\tilde{\theta}_{|\widetilde{\mathcal{H}}_g}$ is non-trivial we also have

$$\tilde{\theta}_{|\widetilde{\mathcal{H}}_g} \neq \delta_{\widetilde{\mathcal{H}}_g}. \tag{17}$$

It follows from Proposition 2 that (17) holds for every $g \in G$ and therefore by (15) that

$$\operatorname{Hom}_{\widetilde{\mathcal{H}}}(C_c^{\infty}(\widetilde{\mathcal{H}} \cdot g), \tilde{\theta}) = 0,\ g \in G. \tag{18}$$

Proposition 1 follows from (18) using the theory of Gelfand-Kazhdan [GK75]. Indeed, we apply [BZ76, Theorem 6.9] to the following setting. We view $C_c^{\infty}(G)$ as a module over itself by convolution. By [BZ76, Proposition 1.14] it uniquely defines a sheaf $\mathcal{F}$ over the l-space G. We let $\widetilde{\mathcal{H}}$ act on $C_c^{\infty}(G)$ by

$$h \cdot_{\tilde{\theta}} \phi = \tilde{\theta}(h) h \cdot \phi.$$

This defines an action of $\widetilde{\mathcal{H}}$ on the sheaf $\mathcal{F}$. The space of $\widetilde{\mathcal{H}}$-invariant distributions on $\mathcal{F}$ is then precisely $\operatorname{Hom}_{\widetilde{\mathcal{H}}}(C_c^{\infty}(G), \tilde{\theta})$. The action of $\widetilde{\mathcal{H}}$ on G is constructible by [BZ76, §6.15, Theorem A]. The second assumption of [BZ76, Theorem 6.9] is precisely (18). It follows that there are no $\widetilde{\mathcal{H}}$-invariant distributions on the sheaf $\mathcal{F}$, i.e. that (8) holds.

5. The property of $\mathcal{H}$-orbits made explicit

In order to prove Proposition 2 it will be convenient to reformulate it, by describing more explicitly the property of the $\widetilde{\mathcal{H}}$-orbits that we wish to prove. We begin with this reformulation.

5.1. **The property $\mathcal{P}(g,r,r')$.** For $g\in G$ let $\mathcal{P}(g,r,r')=\mathcal{P}_n(g,r,r')$ be the following property: either

$$\text{there exists } y\in H_{r,2k} \text{ such that } g^{-1}yg\in \overline{H}_{r',2k'} \text{ and } \theta^{r,r'}(y,g^{-1}yg)\neq 1 \tag{19}$$

or $r=r'$ and

$$\text{there exists } y\in H_{r,2k} \text{ such that } g^{-1}y\ {}^tg\in \overline{H}_{r,2k} \text{ and } \theta^{r,r}(y,g^{-1}y\ {}^tg)=1. \tag{20}$$

Lemma 1. *For every $g\in G$, $\widetilde{\theta}^{r,r'}$ is non-trivial on $\widetilde{\mathcal{H}}_g^{r,r'}$ if and only if $\mathcal{P}(g,r,r')$.*

Proof. Note that

$$\mathcal{H}_g^{r,r'}=\{(y,g^{-1}yg):y\in H_{r,2k}\cap g\overline{H}_{r',2k'}g^{-1}\}$$

and therefore (19) holds if and only if $\theta^{r,r'}$ is not trivial on $\mathcal{H}_g^{r,r'}$. If $r\neq r'$ this proves the lemma. If $r=r'$ it remains to show that when $\theta^{r,r}$ is trivial on $\mathcal{H}_g^{r,r}$ then $\widetilde{\theta}^{r,r}$ is not trivial on $\widetilde{\mathcal{H}}^{r,r}$ if and only if we have (20). Note that

$$\{h\in\mathcal{H}^{r,r}:(h,-1)\in\widetilde{\mathcal{H}}_g^{r,r}\}=\{(y,g^{-1}y\ {}^tg):y\in H_{r,2k}\cap g\overline{H}_{r',2k'}g^{\tau}\}.$$

If $y\in H_{r,2k}\cap g\overline{H}_{r',2k'}g^{\tau}$ then for $h=(y,g^{-1}y\ {}^tg)\in\mathcal{H}^{r,r}$ we have $h\cdot{}^tg=g$ and therefore by (7) we get that $h\,\xi(h)\in\mathcal{H}_g^{r,r}$ so that $\theta^{r,r}(h\,\xi(h))=1$. Since $\theta^{r,r}=\theta^{r,r}\circ\xi$ we have $\theta^{r,r}(h)\in\{\pm1\}$. The remaining of the lemma follows. □

We make here another simple observation that will help to shorten some of the arguments in the proof of Proposition 2.

Lemma 2. *If $\mathcal{P}(g,r,r')$ then $\mathcal{P}(h\cdot g,r,r')$ for all $h\in\widetilde{\mathcal{H}}$ and $\mathcal{P}({}^tg,r',r)$.*

Proof. Note that $\widetilde{\mathcal{H}}_{h\cdot g}=h\widetilde{\mathcal{H}}_gh^{-1}$ and that $\widetilde{\theta}$ is a character. Thus, the first statement is immediate from Lemma 1. If $r=r'$ this argument with $h=(e_{\mathcal{H}},-1)$ also contains the second statement. If $r\neq r'$ the second statement follows from the fact that $\mathcal{H}_{{}^tg}^{r',r}=\xi(\mathcal{H}_g^{r,r'})$ (that follows from (7)) and the fact that $\theta\circ\xi=\theta$. □

In light of Lemma 1 in order to show Proposition 2 we need to show that for every r, $r'\leq n$ such that $n-r\equiv n-r'\equiv 0(\mod 2)$ and for every $g\in G$ we have $\mathcal{P}(g,r,r')$. This will occupy the rest of this paper.

5.2. **Two cases where $\mathcal{P}(g,r,r')$ is already known.** There are two extremes that are already known. The first is a well known fact concerned with the double coset space $U_n\backslash G/\overline{U}_n$. It can be found in the proof of [GK75, Lemma 4.3.8] (it is essentially the steps (a)-(d) verifying condition 4 of [GK75, Theorem 4.2.10]) and it is applied in order to prove the uniqueness of Whittaker models.

Lemma 3. *For every $g\in G$ we have $\mathcal{P}_n(g,n,n)$.*

The second extreme is with respect to the symplectic group. It was proved by Heumus and Rallis [HR90, Proposition 2.3.1] based on results of Klyachko [Kly84, Corollary 5.6]. Recently, Goldstein and Guralnick essentially provided an independent proof over any field [GG07, Proposition 3.1].

Lemma 4. *When n is even for every $g\in G$ we have $\mathcal{P}_n(g,0,0)$.*

Proof. We show that when $r = r' = 0$ (20) holds for every $g \in G$. That is, we show that for every $g \in G$ we have ${}^t g \in Sp(J_n)gSp(J_n)$. As observed in the proof of Lemma 2, it is enough to prove that there exists $y \in Sp(J_n)gSp(J_n)$ such that ${}^t y \in Sp(J_n)gSp(J_n)$. Let $n = 2k$ and let

$$J'_n = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} = {}^t\sigma J_n \sigma \text{ where } \sigma = \begin{pmatrix} w_k & 0 \\ 0 & I_k \end{pmatrix}.$$

Thus,

$$Sp(J'_n) = \sigma^{-1} Sp(J_n)\sigma.$$

It follows from [GG07, Proposition 3.1] that there exists $g' \in G_k$ such that $\mathrm{diag}(I_k, g') \in Sp(J'_n)\sigma^{-1} g\sigma Sp(J'_n)$, i.e that $y = \sigma\,\mathrm{diag}(I_k, g')\sigma^{-1} \in Sp(J_n)gSp(J_n)$. Since every matrix in G_k is conjugate to its transpose and since $\mathrm{diag}(x, {}^t x) \in Sp(J'_n)$ for every $x \in G_k$ we see that $\mathrm{diag}(I_k, {}^t g') \in Sp(J'_n)\sigma g\sigma Sp(J'_n)$, i.e. that ${}^t y = \sigma\,\mathrm{diag}(I_k, {}^t g')\sigma^{-1} \in Sp(J_n)gSp(J_n)$. □

6. Proof by induction of $\mathcal{P}_n(g, r, r')$

Fix 2 decompositions $n = r + 2k = r' + 2k'$. We prove by induction on n that for every $g \in G$ we have $\mathcal{P}_n(g, r, r')$. If $r = r' = 0$ then this is Lemma 4. We assume from now on that $r + r' > 0$. By the induction hypothesis we may also assume that for all $n_1 < n$, all r_1, $r'_1 \le n_1$ such that $n_1 - r_1 \equiv n_1 - r'_1 \equiv 0(\mod 2)$ and all $g' \in G_{n_1}$ we have $\mathcal{P}_{n_1}(g', r_1, r'_1)$. Set $H = H_{r,2k}$, $H' = H_{r',2k'}$, $\mathcal{H} = H \times \overline{H}'$ and $\theta = \theta^{r,r'}$. Let $P = P_{(1^{(r)},2k)}$ and $P' = P_{(1^{(r')},2k')}$. For $w \in W$ viewed as a permutation in $S[1, n]$ let

$$I_w = \{i \in [1, r] : w^{-1}(i) \in [1, r']\}.$$

6.1. A simple proof for most Bruhat cells.

Lemma 5. *Let $w \in {}_M W_{M'}$ be such that I_w is not empty then $\mathcal{P}(g, r, r')$ holds for every $g \in Pw\overline{P'}$.*

Proof. Note that $U \times \overline{U}' \subseteq \mathcal{H}$ and therefore that every $\mathcal{H}$-orbit in $Pw\overline{P'}$ contains an element of MwM'. In light of Lemma 2 we may assume without loss of generality that $g \in MwM'$.

Assume first that there exists an integer i such that $1 \le i \le \min\{r, r'\}$ and $I_w = w^{-1}(I_w) = [1, i]$. We can then write $w = \mathrm{diag}(w_1, w_2)$ for some $w_1 \in S[1, i]$ and $w_2 \in S[i + 1, n]$. Thus for $g \in MwM'$ there exists g_1, $g_2 \in G_{n-i}$, and $a = \mathrm{diag}(a_1, \dots, a_i) \in G_i$ such that $g = \mathrm{diag}(I_i, g_1)w\,\mathrm{diag}(a, g_2) = \mathrm{diag}(w_1 a, g')$ for $g' = g_1 w_2 g_2 \in G_{n-i}$. Let $(u_1, u_2^\tau, \epsilon) \in (\widetilde{\mathcal{H}}_i^{i,i})_{w_1 a}$ be such that $\widetilde{\theta}^{i,i}(u_1, u_2^\tau, \epsilon) \ne 1$ and let $(h_1, h_2^\tau, \epsilon') \in (\widetilde{\mathcal{H}}_{n-i}^{r-i,r'-i})_{g'}$ be such that $\widetilde{\theta}^{r-i,r'-i}(h_1, h_2^\tau, \epsilon') \ne 1$. The first exists by Lemma 3. For the second we apply the induction hypothesis to have $\mathcal{P}_{n-i}(g', r - i, r' - i)$. If $\epsilon = 1$ then

$$h = (\mathrm{diag}(u_1, I_{n-i}), \mathrm{diag}(u_2, I_{n-i})^\tau, 1) \in \widetilde{\mathcal{H}}_g \text{ and } \widetilde{\theta}(h) = \widetilde{\theta}^{i,i}(u_1, u_2^\tau, 1) \ne 1.$$

Similarly, if $\epsilon' = 1$ then

$$h = (\mathrm{diag}(I_i, h_1), \mathrm{diag}(I_i, h_2)^\tau, 1) \in \widetilde{\mathcal{H}}_g \text{ and } \widetilde{\theta}(h) = \widetilde{\theta}^{r-i,r'-i}(h_1, h_2^\tau, 1) \ne 1.$$

If on the other hand $\epsilon = \epsilon' = -1$ then

$$h = (\mathrm{diag}(u_1, h_1), \mathrm{diag}(u_2, h_2)^\tau, -1) \in \widetilde{\mathcal{H}}_g \text{ and } \widetilde{\theta}(h) = -1.$$

We are now left with the case that either I_w or $w^{-1}(I_w)$ is not of the form $[1,i]$ as above. Note that if $g \in Pw\overline{P'}$ then ${}^tg \in P'w^{-1}\overline{P}$ and that $w^{-1} \in {}_{M'}W_M$. It follows from Lemma 2 that it is enough to prove our lemma either for g or for ${}^tg$. We may therefore assume, without loss of generality, that I_w is not of the form $[1,i]$ for any $1 \le i \le \min\{r,r'\}$. Since we assume that $g \in MwM'$ there exist $g_1 \in G_{2k}$, $g_2 \in G_{2k'}$ and $a = \operatorname{diag}(a_1,\dots,a_{r'})$ a diagonal matrix in $G_{r'}$ such that $g = \operatorname{diag}(I_r, g_1) w \operatorname{diag}(a, g_2)$. By our assumption on w we have that $[1,r] \setminus I_w$ is not empty. Let $\ell = \min([1,r] \setminus I_w)$. Since $[1,\ell-1]$ is contained but does not equal I_w the set $[\ell+1,r] \cap I_w$ is not empty. Let $q = \min([\ell+1,r] \cap I_w)$. Then $q-1 \notin I_w$ and $q \in I_w$. In particular, $w^{-1}(q-1) > r'$ and $w^{-1}(q) \le r'$. Let $E_{i,j} \in M_{n\times n}(F)$ be the matrix with $(b,c)^{th}$ entry equal to $\delta_{(i,j),(b,c)}$ and let $u_{i,j}(s) = I_n + s\,E_{i,j}$, $s \in F$. Note that $u_{q-1,q}(s) \in U \subseteq H_{r,2k}$ and that $\psi_{r,2k}(u_{q-1,q}(s)) = \psi(s)$. Thus, there exists $s \in F$ such that $\psi_{r,2k}(u_{q-1,q}(s)) \ne 1$. On the other hand,

$$g^{-1}u_{q-1,q}(s)g = \begin{pmatrix} a^{-1} & 0 \\ 0 & g_2^{-1} \end{pmatrix} u_{w^{-1}(q-1),w^{-1}(q)}(s) \begin{pmatrix} a & 0 \\ 0 & g_2 \end{pmatrix} = \begin{pmatrix} I_{r'} & 0 \\ * & I_{2k'} \end{pmatrix} \in \overline{H}_{r',2k'}$$

and $\psi_{r',2k'}(g^{-1}u_{q-1,q}(s)g) = 1$. It follows that $h_s = (u_{q-1,q}(s), g^{-1}u_{q-1,q}(s)g) \in \mathcal{H}_g$ and if s is such that $\psi_{r,2k}(u_{q-1,q}(s)) \ne 1$ then $\theta(h_s) \ne 1$. □

6.2. **The closed Bruhat cell.** We are now left with the case that I_w is empty. Since this means that w^{-1} maps $[1,r]$ into $[r'+1,n]$ we must have, in particular, $n \ge r + r'$. It is not difficult to see that there is then a unique such element in ${}_MW_{M'}$, namely,

$$w = w^{r,r'} = \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix}.$$

Note then that $Pw\overline{P'}$, is the closed Bruhat cell. We remark further that this contains the case that either r or r' is 0. Let $g \in MwM'$. Note that there exist $g_1 \in G_{2k}$ and $g_2 \in G_{2k'}$ such that

$$g = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix}.$$

Indeed, for $t \in G_r$, $t' \in G_{r'}$ (and in particular when t and t' are diagonal) if $g_1' \in G_{2k}$ and $g_2' \in G_{2k'}$ we have

$$\begin{pmatrix} t & \\ & g_1' \end{pmatrix} w \begin{pmatrix} t' & \\ & g_2' \end{pmatrix}$$
$$= \begin{pmatrix} I_r & \\ & g_1' \end{pmatrix} \begin{pmatrix} 0 & t & 0 \\ t' & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} \begin{pmatrix} I_r' & \\ & g_2' \end{pmatrix} = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix}$$

where $g_1 = g_1' \operatorname{diag}(t', I_{2k-r'})$ and $g_2 = \operatorname{diag}(t, I_{2k'-r})g_2'$.

In order to show $\mathcal{P}(g,r,r')$ we distinguish between 2 cases. We denote by $\langle v_1,\dots,v_i\rangle$ the subspace of a vector space V spanned by $v_1,\dots,v_i \in V$. Let V be a subspace of the vector space $M_{\ell\times 1}(F)$ for some positive integer ℓ. We say that a skew symmetric matrix $\mathcal{I} \in M_{\ell\times\ell}(F)$ is totally isotropic on V if ${}^tv\mathcal{I}v' = 0$ for all

$v,\, v' \in V$. Denote by e_i the column vector with 1 in the i^{th} row and 0 in each other row. Thus $e_i \in M_{\ell\times 1}(F)$ for an integer ℓ which is implicit in our notation. Let

$$\mathcal{I}_1 = {}^t g_1 J_{2k} g_1 \text{ and } \mathcal{I}_2 = g_2^\tau J_{2k'} g_2^{-1}.$$

We say that g belongs to the totally isotropic case if both $\mathcal{I}_1^{-1}$ is totally isotropic on $\langle e_1, \dots, e_{r'}\rangle$ and $\mathcal{I}_2$ is totally isotropic on $\langle e_1, \dots, e_r\rangle$. Otherwise we say that g does not belong to the totally isotropic case. It is easy to verify that this property indeed depends only on g and not on g_1 and g_2. We now prove $\mathcal{P}(g, r, r')$ separately in each of the 2 cases.

6.2.1. *When g does not belong to the totally isotropic case.* In this case we prove that g satisfies (19). It will be convenient to make this property more explicit. We say that the 2 skew-symmetric forms $\mathcal{I}_1,\, \mathcal{I}_2 \in G$ satisfy the property $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$ if there exist $u \in U_r$ and $u' \in U_{r'}$ such that $\psi_r(u) \neq \psi_{r'}(u')$ and for some $X \in M_{r\times 2k'-r}(F)$, $Y \in M_{r'\times 2k-r'}(F)$ and $D \in G_{n-(r+r')}$ we have

$$\begin{pmatrix} u & X \\ 0 & D \end{pmatrix} \in Sp(\mathcal{I}_2) \text{ and } \begin{pmatrix} {}^t u' & 0 \\ Y & D \end{pmatrix} \in Sp(\mathcal{I}_1).$$

Lemma 6. *Let*

$$g = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix} \in MwM'$$

and let

$$\mathcal{I}_1 = {}^t g_1 J_{2k} g_1 \text{ and } \mathcal{I}_2 = g_2^\tau J_{2k'} g_2^{-1}.$$

Then g satisfies (19) *if and only if* $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$.

Proof. Let

$$y = \begin{pmatrix} u & Z \\ & h \end{pmatrix} \in H$$

with $u \in U_r$, $h \in Sp(J_{2k})$ and $Z \in M_{r\times 2k}(F)$. To explicate condition (19) we compute $g^{-1}yg$. First note that we have

$$\begin{pmatrix} I_r & \\ & g_1^{-1} \end{pmatrix} \begin{pmatrix} u & Z \\ & h \end{pmatrix} \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} = \begin{pmatrix} u & Zg_1 \\ & g_1^{-1} h g_1 \end{pmatrix}.$$

We write

$$g_1^{-1} h g_1 = \begin{pmatrix} {}^t u' & B \\ Y & D \end{pmatrix} \text{ and } Zg_1 = (Z_1, Z_2)$$

with $u' \in M_{r'\times r'}(F)$, $D \in M_{2k-r'\times 2k-r'}(F)$, $Z_1 \in M_{r\times r'}(F)$ and $Z_2 \in M_{r\times 2k-r'}(F)$. We then have

$$\begin{pmatrix} 0 & I_{r'} & 0 \\ I_r & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} \begin{pmatrix} u & Z_1 & Z_2 \\ 0 & {}^t u' & B \\ 0 & Y & D \end{pmatrix} \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} = \begin{pmatrix} {}^t u' & 0 & B \\ Z_1 & u & Z_2 \\ Y & 0 & D \end{pmatrix}.$$

Therefore,

$$g^{-1} y g = \begin{pmatrix} {}^t u' & (0, B) g_2 \\ g_2^{-1} \begin{pmatrix} Z_1 \\ Y \end{pmatrix} & g_2^{-1} \begin{pmatrix} u & Z_2 \\ 0 & D \end{pmatrix} g_2 \end{pmatrix}.$$

We see that $g^{-1} y g \in \overline{H'}$ if and only if $u' \in U_{r'}$, $B = 0$ and

$$g_2^{-1} \begin{pmatrix} u & Z_2 \\ 0 & D \end{pmatrix} g_2 \in Sp(J_{2k'}).$$

Recall also that

$$\begin{pmatrix} {}^t u' & 0 \\ Y & D \end{pmatrix} \in g_1^{-1} Sp(J_{2k}) g_1.$$

With this notation, when $g^{-1}yg \in \overline{H'}$ we have

$$\theta(y, g^{-1}yg) = \psi_r(u)\psi_{r'}((u')^{-1}).$$

Since

$$g_1^{-1} Sp(J_{2k}) g_1 = Sp(\mathcal{I}_1) \text{ and } g_2 Sp(J_{2k}) g_2^{-1} = Sp(\mathcal{I}_2),$$

the lemma is now immediate. □

In order to proceed we need the following Lemma of Klyachko [Kly84, §1.3, p. 368, Step 3].

Lemma 7. *Let $\mathcal{I} = -{}^t\mathcal{I} \in G_{2k}$ and let $r \le 2k$ be such that $\mathcal{I}$ is not totally isotropic on $\langle e_1, \dots, e_r \rangle$ then there exists $u \in U_r$ with $\psi_r(u) \ne 1$ and $X \in M_{r \times 2k-r}(F)$ such that*

$$\begin{pmatrix} u & X \\ 0 & I_{2k-r} \end{pmatrix} \in Sp(\mathcal{I}). \tag{21}$$

Proof. Let $i \in [1, r-1]$ be maximal so that $\mathcal{I}$ is totally isotropic on $\langle e_1, \dots, e_i \rangle$. There is therefore $v_0 \in \langle e_1, \dots, e_i \rangle$ such that ${}^t v_0 \mathcal{I} e_{i+1} \ne 0$. We may further assume that $v_0 \in e_i + \langle e_1, \dots, e_{i-1} \rangle$ since if ${}^t e_i \mathcal{I} e_{i+1} \ne 0$ then we may take $v_0 = e_i$ and otherwise, we may replace v_0 by its sum with any scalar multiple of e_i. Let $V = M_{2k \times 1}(F)$ and for every $s \in F$ define $\lambda_s \in \mathrm{Hom}_F(V, F)$ by $\lambda_s(v) = s\, {}^t v_0 \mathcal{I} v$. Note that the map $s \mapsto \lambda_s(e_{i+1})$, $s \in F$ is onto F. Identify $GL(V)$ with G_{2k} via the standard basis $\{e_1, \dots, e_{2k}\}$ and define an element $h_s \in G_{2k}$ by

$$h_s(v) = v + \lambda_s(v)\, v_0.$$

Thus, $h_s \in Sp(\mathcal{I})$ is of the form (21) with $\psi_r(u) = \psi(\lambda_s(e_{i+1}))$. □

Lemma 8. *Let*

$$g = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix} \in MwM'$$

not belong to the totally isotropic case and let

$$\mathcal{I}_1 = {}^t g_1 J_{2k} g_1 \text{ and } \mathcal{I}_2 = g_2^{\tau} J_{2k'} g_2^{-1}.$$

Then we have $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$.

Proof. If $\mathcal{I}_2$ is not totally isotropic on $\langle e_1, \dots, e_r \rangle$ then by Lemma 7 there exist $u \in U_r$ and $X \in M_{r \times 2k'-r}$ such that $\psi_r(u) \ne 1$ and

$$\begin{pmatrix} u & X \\ 0 & I_{2k'-r} \end{pmatrix} \in Sp(\mathcal{I}_2).$$

Then $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$ is satisfies with $Y = 0$, $u' = I_{r'}$ and $D = I_{n-(r+r')}$. Note further that $Sp(\mathcal{I}_1^{-1}) = \{{}^t g : g \in Sp(\mathcal{I}_1)\}$. Thus, if $\mathcal{I}_1^{-1}$ is not totally isotropic on $\langle e_1, \dots, e_{r'} \rangle$ then by Lemma 7 applied to $\mathcal{I}_1^{-1}$ there exist $u' \in U_{r'}$ and $Y \in M_{2k-r' \times r'}$ such that $\psi_{r'}(u') \ne 1$ and

$$\begin{pmatrix} {}^t u' & 0 \\ Y & I_{2k-r'} \end{pmatrix} \in Sp(\mathcal{I}_1).$$

Thus, $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$ is satisfies with $X = 0$, $u = I_r$ and $D = I_{n-(r+r')}$. □

6.2.2. *When g belongs to the totally isotropic case.* Assume from now on that both $\mathcal{I}_2$ is totally isotropic on $\langle e_1, \dots, e_r \rangle$ and $\mathcal{I}_1^{-1}$ is totally isotropic on $\langle e_1, \dots, e_{r'} \rangle$. In the case at hand $\mathcal{H} \cdot g$ contains an element of a rather simple form that will allow us the inductive argument. In order to bring g to this simpler form we need the following lemma.

Lemma 9. *Let $\ell \le m$ and $Q = P_{(\ell, 2m-\ell)}$. Then*

$$Sp(J_{2m})\, Q = \{g \in G_{2m} : {}^t g J_{2m} g \text{ is totally isotropic on } \langle e_1, \dots, e_\ell \rangle\}.$$

Proof. If $h \in Sp(J_{2m})$ and $q \in Q$ then ${}^t(hq) J_{2m} hq = {}^t q J_{2m} q$. Since q preserves the space $\langle e_1, \dots, e_\ell \rangle$ and since J_{2m} is totally isotropic on $\langle e_1, \dots, e_\ell \rangle$ we get that ${}^t q J_{2m} q$ is also totally isotropic on $\langle e_1, \dots, e_\ell \rangle$. To prove the other direction let $g \in G_{2m}$ be such that ${}^t g J_{2m} g$ is totally isotropic on $\langle e_1, \dots, e_\ell \rangle$. Then

$$x = g^t J_{2m} g = \begin{pmatrix} 0_\ell & A \\ -{}^t A & D \end{pmatrix} \in G_{2m}$$

for some $D = -{}^t D \in M_{2m-\ell \times 2m-\ell}(F)$. We must show that there exists $q \in Q$ such that ${}^t q x q = J_{2m}$. Since x is invertible and $\ell \le 2m - \ell$ the matrix A is of rank ℓ. Performing elementary operations, there exists $\alpha \in G_\ell$ and $\gamma \in G_{2m-\ell}$ such that ${}^t \alpha A \gamma = (0_{\ell \times 2(m-\ell)}, w_\ell)$. It follows that for $q = \mathrm{diag}(\alpha, \gamma) \in Q$, ${}^t q x q$ has the form

$$\begin{pmatrix} 0 & 0 & w_\ell \\ 0 & a & b \\ -w_\ell & -{}^t b & d \end{pmatrix}$$

where $a = -{}^t a \in G_{2(m-\ell)}$ and $d = -{}^t d \in M_{\ell \times \ell}(F)$. Write $\beta = (\beta_1, \beta_2)$ with $\beta_1 \in M_{\ell \times 2(m-\ell)}(F)$ and $\beta_2 \in M_{\ell \times \ell}(F)$. Note that

$$\begin{pmatrix} I_\ell & 0 & 0 \\ {}^t\beta_1 & I_{2(m-\ell)} & 0 \\ {}^t\beta_2 & 0 & I_\ell \end{pmatrix} \begin{pmatrix} 0 & 0 & w_\ell \\ 0 & a & b \\ -w_\ell & -{}^t b & d \end{pmatrix} \begin{pmatrix} I_\ell & \beta_1 & \beta_2 \\ 0 & I_{2(m-\ell)} & 0 \\ 0 & 0 & I_\ell \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & w_\ell \\ 0 & a & b + {}^t\beta_1 w_\ell \\ -w_\ell & -{}^t b - w_\ell \beta_1 & d + {}^t\beta_2 w_\ell - w_\ell \beta_2 \end{pmatrix}.$$

We may now take $\beta_1 = -w_\ell {}^t b$. Any skew symmetric matrix in $M_{\ell \times \ell}(F)$ can be written as a difference $X - {}^t X$ for some $X \in M_{\ell \times \ell}(F)$. Thus, there also exists β_2 such that ${}^t\beta_2 w_\ell - w_\ell \beta_2 = -d$. We get that there exists $q \in Q$ such that

$${}^t q x q = \begin{pmatrix} 0 & 0 & w_\ell \\ 0 & a & 0 \\ -w_\ell & 0 & 0 \end{pmatrix}.$$

Let $y \in G_{2(m-\ell)}$ be such that ${}^t y a y = J_{2(m-\ell)}$. Thus $q' = q\, \mathrm{diag}(I_\ell, y, I_\ell) \in Q$ and ${}^t q' x q' = J_{2m}$. ∎

For $x \in G_\ell$ let

$$\tilde{x} = w_\ell \, x^\tau \, w_\ell.$$

The following property of the group $Sp(J_{2m})$ will be used several times in the proof of $\mathcal{P}(g,r,r')$. Assume that $\ell \le m$.

(22) For all $x \in G_\ell$, $s \in Sp(J_{2(m-\ell)})$ and y there exists y^* uniquely determined by x, s and y and dependent linearly on y and there exists z such that

$$\begin{pmatrix} x & y^* & z \\ 0 & s & y \\ 0 & 0 & \tilde{x} \end{pmatrix} \text{ (resp. } \begin{pmatrix} x & 0 & 0 \\ y^* & s & 0 \\ z & y & \tilde{x} \end{pmatrix}\text{) lies in } Sp(J_{2m}).$$

We now choose a convenient representative for g.

Lemma 10. *Let*

$$g = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix} \in MwM'$$

belong to the totally isotropic case. Then there exists $\gamma \in G_{n-(r+r')}$ such that

$$\begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \in \mathcal{H} \cdot g.$$

Proof. Since $-\mathcal{I}_1^{-1} = g_1^{-1} J_{2k} g_1^\tau$ is totally isotropic on $\langle e_1, \dots, e_{r'} \rangle$ and $\mathcal{I}_2 = g_2^\tau J_{2k} g_2^{-1}$ is totally isotropic on $\langle e_1, \dots, e_r \rangle$, it follows from Lemma 9 that

$$g_1 \in Sp(J_{2k}) \begin{pmatrix} \alpha_1 & 0 \\ \beta_1' & \gamma_1 \end{pmatrix} \text{ and } g_2 \in \begin{pmatrix} \alpha_2 & \beta_2' \\ 0 & \gamma_2 \end{pmatrix} Sp(J_{2k'})$$

for some $\alpha_1 \in G_{r'}$, $\gamma_1 \in G_{2k-r'}$, $\alpha_2 \in G_r$, $\gamma_2 \in G_{2k'-r}$ and β_1' and β_2' of the appropriate size. Therefore,

$$\begin{pmatrix} 0 & \alpha_2 & \beta_2' \\ \alpha_1 & 0 & 0 \\ \beta_1' & 0 & \gamma_1\gamma_2 \end{pmatrix} = \begin{pmatrix} I_r & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & \beta_1' & \gamma_1 \end{pmatrix} \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} \begin{pmatrix} I_{r'} & 0 & 0 \\ 0 & \alpha_2 & \beta_2' \\ 0 & 0 & \gamma_2 \end{pmatrix} \in \mathcal{H} \cdot g.$$

Note that $\mathrm{diag}(\alpha_1, I_{2(k-r')}, \tilde{\alpha}_1) \in Sp(J_{2k})$ and $\mathrm{diag}(\alpha_2, I_{2(k'-r)}, \tilde{\alpha}_2) \in Sp(J_{2k'})$ and therefore that

$$h = \mathrm{diag}(I_r, \alpha_1^{-1}, I_{2(k-r')}, \tilde{\alpha}_1^{-1}) \in H \text{ and } h' = \mathrm{diag}(I_{r'}, \alpha_2^{-1}, I_{2(k'-r)}, \tilde{\alpha}_2^{-1}) \in \overline{H'}.$$

Thus,

$$h \begin{pmatrix} 0 & \alpha_2 & \beta_2' \\ \alpha_1 & 0 & 0 \\ \beta_1' & 0 & \gamma_1\gamma_2 \end{pmatrix} h' = \begin{pmatrix} 0 & I_r & \beta_2 \\ I_{r'} & 0 & 0 \\ \beta_1 & 0 & \gamma \end{pmatrix} \in \mathcal{H} \cdot g$$

for some $\gamma \in G_{n-(r+r')}$, β_1 and β_2. Now note that

$$\begin{pmatrix} I_r & \beta_2\gamma^{-1}\beta_1 & -\beta_2\gamma^{-1} \\ 0 & I_{r'} & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} \begin{pmatrix} 0 & I_r & \beta_2 \\ I_{r'} & 0 & 0 \\ \beta_1 & 0 & \gamma \end{pmatrix} \begin{pmatrix} I_{r'} & 0 & 0 \\ 0 & I_r & 0 \\ -\gamma^{-1}\beta_1 & 0 & I_{n-(r+r')} \end{pmatrix} = \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \in \mathcal{H} \cdot g.$$

□

Lemma 11. *Let* $\gamma \in G_{n-(r+r')}$ *and let*

$$g = \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

then $\mathcal{P}(g, r, r')$.

Proof. Recall that $r + r' > 0$. Let

$$\sigma_1 = \begin{pmatrix} & I_{2(k-r')} \\ w_{r'} & \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} & I_{2(k'-r)} \\ w_r & \end{pmatrix}.$$

For $x = \sigma_1^{-1}\gamma\sigma_2$ we have by the induction hypothesis $\mathcal{P}_{n-(r+r')}(x, r, r')$. Fix $y \in H_{r,2(k'-r)}$ such that either

$$x^{-1}yx \in \overline{H}_{r',2(k-r')} \text{ and } \theta(y, x^{-1}yx) \neq 1 \tag{23}$$

or

$$r = r',\ x^{-1}y\ {}^tx \in \overline{H}_{r',2(k-r')} \text{ and } \theta(y, x^{-1}y\ {}^tx) = 1. \tag{24}$$

For every invertible matrix z denote by $z^\star$ the matrix z if y satisfies (23) and the matrix ${}^tz$ otherwise. Note that if (24) holds then $\sigma_1 = \sigma_2$ and therefore in either case we have

$$x^\star = \sigma_1^{-1}\gamma^\star\sigma_2.$$

There exist $s' \in Sp(J_{2(k-r')})$, $u' \in U_{r'}$ and $\varrho' \in M_{r'\times 2(k-r')}(F)$ such that

$$\sigma_1 y \sigma_1^{-1} = \begin{pmatrix} s' & \\ \varrho' & {}^t(\tilde{u'}) \end{pmatrix}$$

and there exist $s \in Sp(J_{2(k'-r)})$, $u \in U_r$ and $\varrho \in M_{2(k'-r)\times r}(F)$ such that

$$\gamma^{-1}\sigma_1 y \sigma_1^{-1}\gamma^\star = \sigma_2 x^{-1} y x^\star \sigma_2^{-1} = \begin{pmatrix} s & \varrho \\ & \tilde{u} \end{pmatrix}.$$

Note then that

$$\theta(y, x^{-1}yx^\star) = \psi_r(u)\psi_{r'}(u')^{-1}. \tag{25}$$

By (22) there exist $(\varrho')^* \in M_{2(k-r')\times r'}(F)$, $\varrho^* \in M_{r\times 2(k'-r)}(F)$, z' and z such that

$$h = \begin{pmatrix} {}^tu' & 0 & 0 \\ (\varrho')^* & s' & 0 \\ z' & \varrho' & {}^t\tilde{u}' \end{pmatrix} \in Sp(J_{2k}) \text{ and } h' = \begin{pmatrix} u & \varrho^* & z \\ 0 & s & \varrho \\ 0 & 0 & \tilde{u} \end{pmatrix} \in Sp(J_{2k'}).$$

Note that

$$g^\star = \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & \gamma^\star \end{pmatrix}.$$

Let

$$\zeta_1 = (\varrho^*, z)(\gamma^\star)^{-1} \text{ and } \zeta = (0_{r\times r'}, \zeta_1)$$

then

$$Y = \begin{pmatrix} u & \zeta \\ 0 & h \end{pmatrix} \in H,\ g^{-1}Yg^\star = \begin{pmatrix} {}^tu' & 0 \\ \zeta' & h' \end{pmatrix} \in \overline{H'} \text{ where } \zeta' = \gamma^{-1}\begin{pmatrix} (\varrho')^* \\ z' \end{pmatrix}$$

and $\theta(Y, g^{-1}Yg^\star) = \psi_r(u)\psi_{r'}(u')^{-1}$. The property $\mathcal{P}_n(g, r, r')$ therefore follows from (25) and the fact that either (23) holds or (24) holds. □

6.3. **Conclusion.** For $g \in G$, by (6) there exists $w \in {}_M W_{M'}$ such that $g \in Pw\overline{P'}$. If I_w is not empty then $\mathcal{P}(g, r, r')$ is proved in Lemma 5. If I_w is empty then we separated in §6.2 the statement $\mathcal{P}(g, r, r')$ into 2 cases. If g belongs to the totally isotropic case then $\mathcal{P}(g, r, r')$ follows from Lemma 2, Lemma 10 and Lemma 11. Otherwise $\mathcal{P}(g, r, r')$ follows from Lemma 6 and Lemma 8. It follows that for every $g \in G$ we have $\mathcal{P}(g, r, r')$. Proposition 2 now follows from Lemma 1. Therefore, Proposition 1 follows from §4.1 and Theorem 1 follows from §3.1.

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