Stochastic Resonance

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Dipl. Math. Ilya Pavlyukevich

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Präsident der Humboldt-Universität zu Berlin
Prof. Dr. Jürgen Mlynek

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät II
Prof. Dr. Elmar Kulke

Gutachter

1. Prof. Dr. Peter Imkeller	Humboldt-Universität zu Berlin
2. Prof. Dr. Ludwig Arnold	Universität Bremen
3. Priv.-Doz. Dr. Anton Bovier	WIAS Berlin
4. Prof. N. Sri Namachchivaya	University of Illinois at Urbana-Champaign (USA)

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Abstract

In this thesis we investigate the phenomenon of *stochastic resonance*, i.e. the optimal noise-induced increase of a dynamical system's sensitivity and ability to amplify small periodic signals.

We consider bistable stochastic systems fed with a small periodic signal — two-state *Markov chains* with discrete and continuous time and *diffusions* in double-well potentials. In the Markov chain case, one-step or infinitesimal probabilities are periodically modulated. In the diffusion case, depths of the potential wells are periodically changed in time. We introduce several measures of goodness for tuning, the most important one of which is the coefficient of the *spectral power amplification* (SPA), which describes the spectral energy carried by the averaged random output corresponding to the frequency of a small deterministic periodic perturbation.

For the Markov chains, this coefficient is studied as a function of noise, and the coordinate of its local maximum, the resonance point, is determined asymptotically in the small noise limit. Six more measures of goodness are investigated, including the SPA-to-noise ratio, the energy, the out-of-phase measure, the relative entropy, and the entropy of the invariant measure. Optimal tuning rates are obtained for all these measures.

To investigate optimal tuning for diffusions we study adapted Markov chains whose dynamical properties retain the essentials of the diffusions' hopping properties between the two metastable states. Surprisingly, due to the influence of many small oscillations near the potential valley bottoms, the tuning properties of a diffusion's SPA coefficient do not match those of the adapted Markov chain's coefficient. However, if the fluctuations of the diffusion near the wells' minima are cut off, the modified SPA coefficient has a local maximum with approximately the same coordinates as the adapted Markov chain.

Our methods are based on the study of the equilibrium measures of the considered random processes. To determine the invariant densities of diffusions, we use spectral analysis of the respective infinitesimal generators which describes them in Fourier type series expansions. The analysis of optimal tuning and the comparison to the adapted Markov chains become possible by a precise study of the lowest order eigenvalues and eigenfunctions and the discovery of a spectral gap between the first and second eigenvalue.

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Chapter 1

Introduction

Usually when we speak about *noise* we mean something that worsens the operation of a system. The dictionary [1] says that noise is *a disturbance, especially a random and persistent disturbance, that obscures or reduces the clarity of a signal*. Indeed, we all suffer from the background noise in TV and radio receivers, and engineers want to screen it out. The sensitivity of measuring devices like tele- and microscopes is also limited due to the presence of natural and technical noise. The fact is that fluctuation phenomena are part of our world and cannot be neglected or eliminated.

However, in nonlinear systems the presence of noise may play a constructive role. This work is devoted to the study of systems displaying *stochastic resonance*, in which the essentially non-zero level of noise enhances the systems' sensitivity and ability to amplify small periodic deterministic signals.

1.1 Historical development of the concept

1.1.1 Ice Ages and the origin of stochastic resonance

In about 1980 R. Benzi et al. [3, 4, 5] and C. Nicolis [49] introduced a mathematical approach to qualitative explanation of the phenomenon of glacial cycles. The modern methods of acquiring and interpreting climate records indicate at least seven major climate changes in the last 700,000 years. These changes occurred with the periodicity of about 100,000 years and are characterised by a substantial variation of the average Earth's temperature of about $10K$.

The effect can be explained with the help of a simple energy balance model (for an extended review on the subject see [32]). The Earth is considered as a point in space, and its temporally and spatially averaged temperature $X(t)$ satisfies the equation

$$c\dot{X}(t) = E_{in} - E_{out}. \tag{1.1}$$

In other words, the instant change of the Earth temperature is determined by the difference between the incoming (E_{in}) and outgoing (E_{out}) radiative energy. Here c is a positive constant signifying the Earth's heat capacity.

The Earth receives energy from the Sun. On the one hand, this energy is proportional to the so-called solar constant $Q(t)$ which shows how much solar energy reaches the Earth at time t. It is known that due to the influence of Jupiter the eccentricity of the Earth's orbit oscillates with the period about 10^5 years, and this causes the solar constant to vary with the same period. The amplitude of this variation is estimated to be about 0.1%, and we may assume that $Q(t) = Q(1 + A\sin(\frac{2\pi t}{2T}))$, where the period $2T \approx 10^5$ years, $A = 0.001$.

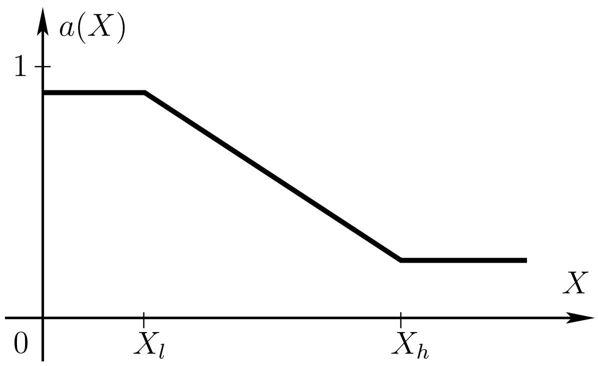

Fig. 1.1: The Earth's albedo in Budyko–Sellers model as a function of temperature X.

On the other hand, E_{in} is proportional to the absorption coefficient of the surface, which is equal to $1 - a$, where $a(X)$ is the average albedo, i.e., the proportion of the reflected solar radiation. The model suggested by Budyko [10] and Sellers [55] was used to describe the albedo. It is supposed that there are two characteristic temperature levels $X_l < X_h$. Below X_l most of the surface water is supposed to have turned into ice, the planet is bright, and consequently, the albedo is high. Conversely, if $X > X_h$, it is very warm, there is a lot of water and vegetation on the surface, the planet is green and brown, and the albedo is low. For temperatures between X_l and X_h we interpolate a linearly (it can also be made smooth), see Fig. 1.1.

To describe E_{out} we assume that the Earth is a black body, so that due to the Stefan–Boltzmann law its radiative energy is proportional to X^4. Finally, the difference of the energies takes the form

$$E_{in} - E_{out} = Q(1 + A\sin\left(\tfrac{2\pi t}{2T}\right))\{1 - a(X(t))\} - \gamma X(t)^4, \tag{1.2}$$

where γ is a positive constant. For appropriate parameter values, the graph of $E_{in} - E_{out}$ has three time-dependent zeroes, say $X_i(t)$, $X_m(t)$ and $X_w(t)$. As we are in the one-dimensional situation, $E_{in} - E_{out}$ can be represented as a gradient

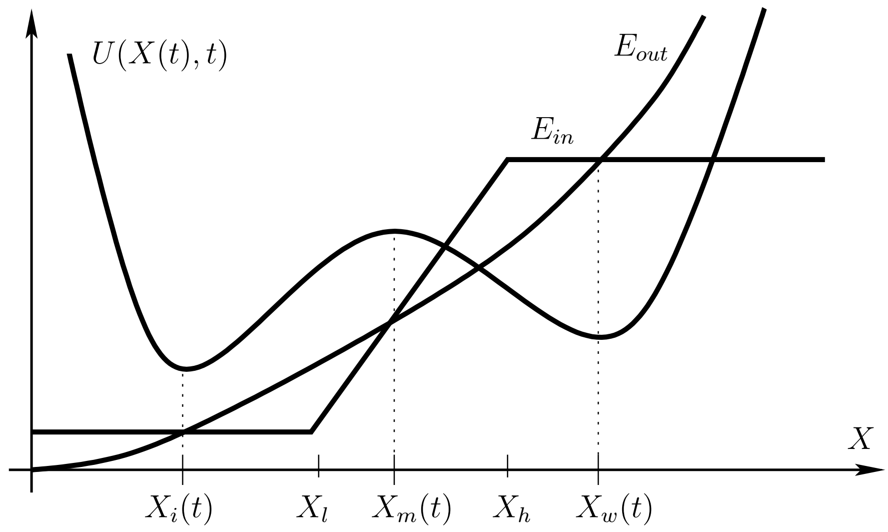

Fig. 1.2: The energies E_{in} and E_{out}, and the corresponding potential $U(x,t)$.

of some *potential* $U(x,t)$, which has two wells with minima in $X_i(t)$ and $X_w(t)$ and a saddle point at $X_m(t)$. Without loss of generality, we can put the heat capacity to be equal to one, which yields the following equation for the Earth's temperature X:

$$\dot{X}(t) = -\frac{\partial}{\partial x} U(X(t), \frac{t}{2T}) \tag{1.3}$$

The time-dependence of $U(x,t)$ caused by the sin-term from (1.2) produces two effects. First, the extrema $X_i(t)$, $X_m(t)$ and $X_w(t)$ oscillate deterministically with small amplitude and a period of about 10^5 years. Second, the depths of the potential wells oscillate with the same period. Note that due to the smallness of A, the potential does not degenerate into one-well potential for all $t \geq 0$.

The extrema of the potential are the so-called metastable states of the dynamical system (1.3). The lower meta-stable state $X_i(t)$ describes the Earth's temperature in the Ice Age's regime, the upper state $X_w(t)$ in the warm age's regime. They are both stable, whereas the intermediate state $X_m(t)$ is not. Any solution of (1.3) is deterministic and is attracted to one of the meta-stable states X_i or X_w. Unfortunately, the system (1.3) allows no hopping between cold and warm regimes: the climate of the Earth is completely determined by the initial conditions.

To enable hops between the two regimes, we add noise to the system and consider

$$\dot{X}(t) = -U'(X(t), \frac{t}{2T}) + \sqrt{\varepsilon}\dot{W}_t, \tag{1.4}$$

where $\dot{W}$ is white noise and $U'(x,t) = \frac{\partial}{\partial x} U(x,t)$.

Now let us study the *stochastic* energy-balance equation with respect to the noise parameter ε. First, we should describe the potential U more precisely. R. Benzi et al. considered in their studies the periodically perturbed quartic potential $U(x,t) = x^4/4 - x^2/2 - Ax\sin 2\pi t$. For A small, this potential is double-well, 1-periodic in time and enjoys the space-time antisymmetry $U(x, t+\frac{1}{2}) = U(-x,t)$.

To be consistent with the further exposition, we now introduce the potential which will be used in our work. First, we consider the asymmetric time-independent double-well potential $U(x)$ with minima at ± 1 and a saddle point at 0. Then, the time-perturbed potential is given as $U(x,t) = U(x)$ on $t \in [0, \frac{1}{2})$, and $U(x,t) = U(-x, t+\frac{1}{2})$, $t \geq 0$, see Fig. 1.3. We study the qualitative behaviour of sample paths of (1.4) for fixed T and different values of ε. To emphasise the dependence of the solutions of (1.4) on ε and T we denote $X_t^{\varepsilon,T}(x)$ the solution of (1.4) starting at x at $t=0$. As was already mentioned, if we consider the

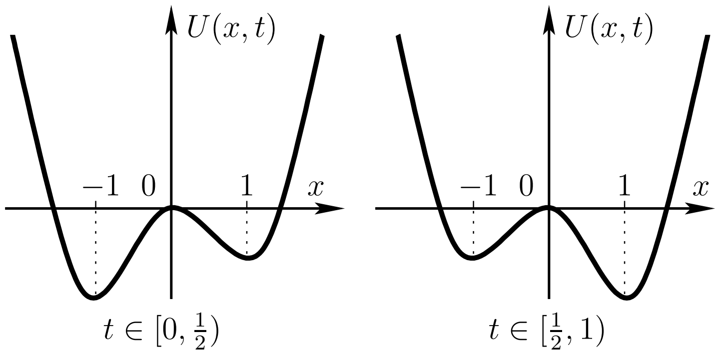

Fig. 1.3: The time-periodic double-well potential $U(x,t)$ on $t \in [0,1)$, $x \in \mathbb{R}$.

paths of $X_t^{0,T}(x)$ we can find that after some time they are exponentially close to the meta-stable states ± 1 depending on the initial point x. In our setting, these states are actually not only meta-stable, but *stable*. Moreover, observing the trajectories of $X^{0,T}$ we cannot say which well of the potential is deeper at the moment. In other words, this deterministic system is not sensible to the periodic influence caused by the time-dependent potential.

For $T > 0$ fixed we consider the paths of $X^{\varepsilon,T}$ for different values of ε, from very small to very large. If ε is very small, jumps between the wells are rare, the typical time between two jumps is much bigger than T. Thus, we observe the picture as on Fig. 1.4, i.e. we observe only the double-well structure of the potential, not time-periodicity. If $\varepsilon \gg 1$ the jumps occur very often, and the variance of $X^{\varepsilon,T}$ is so large (see Fig. 1.5) that we cannot say anything about the spatial geometry of the potential U and whether it depends on time or not. The most interesting case is the one of moderate ε. It turns out that there exists some 'optimal' value of ε such that the sample paths $X_t^{\varepsilon,T}$ become 'almost periodic' in time, with period $2T$. 'Almost periodic' means that although $X_t^{\varepsilon,T}$ is random, it is

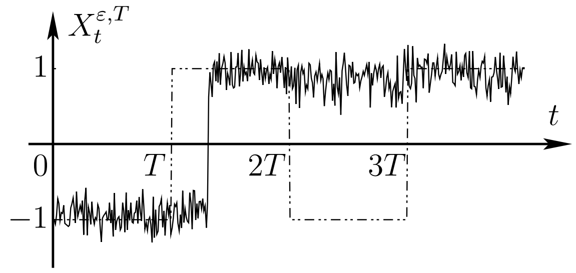

Fig. 1.4: The sample path of $X^{\varepsilon,T}$, $\varepsilon \sim 0$.

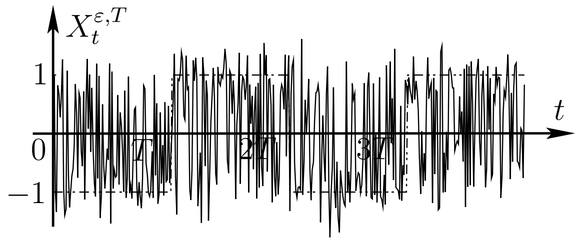

Fig. 1.5: The sample path of $X^{\varepsilon,T}$, $\varepsilon \gg 1$.

close to the minimum of the deepest well of U and as the wells switch their roles, $X_t^{\varepsilon,T}$ follows the global minimum of the potential, see Fig. 1.6. In the language of climate modelling this means that there is a theoretical possibility, that the Ice Ages are a result of an 'optimal tuning' of the solar system, that is, the small random fluctuations (of weather, solar activity etc.) and the small deterministic periodic influence of Jupiter might have produced the drastic cumulative effect of abrupt climate change on the Earth. On Fig. 1.6 we can see a completely

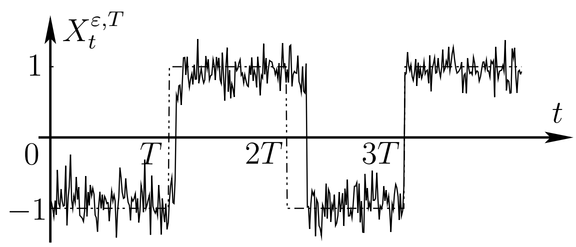

Fig. 1.6: The sample path of $X^{\varepsilon,T}$ for 'optimal' ε.

new regime. It cannot be observed if the potential does not have double-well structure. It also cannot be observed for the 'wrong' values of ε and without periodic perturbation of the potential.

Let us discuss whether we have lost any essential information when considering a temporally step-wise potential instead of a continuous one. According to large deviation theory (see the detailed explanations in Chapter 2) for large enough time scales T the solution trajectories $X^{\varepsilon,T}$ follow the *discontinuous deterministic* curve describing the actual location of the deep well's minimum. In a temporally step-wise potential, only the small periodic oscillations of the positions of the metastable states are neglected. We expect that the assumption of constant depth of wells during one half-period implicit in the step-wise potential will not affect the qualitative aspects of periodic tuning. This has to be looked at more closely in future work, especially with respect to the bifurcation phenomena discovered in [7, 8, 9].

To obtain Fig. 1.6 we have experimentally determined the 'optimal' noise level. The natural question arises: can one define a *measure of goodness* for the system (1.4) determining the noise intensity ε_0 of *optimal tuning* for which the trajectories $X^{\varepsilon_0,T}$ follow the periodic input signal in a 'best possible' way?

R. Benzi et al. studied the system (1.4) numerically with respect to the following measure. They considered sample paths of the system on long time intervals, Fourier analysed them and plotted the power of the spectral component with period $2T$ as a function of ε, of course, averaged over some hundreds of realizations. It turns out, that this curve has a strong peak near some $\varepsilon = \varepsilon_0$, which means that at this noise level the random trajectories on average have the largest possible periodic component of period $2T$. This spectral measure of goodness was called the *spectral power amplification* (SPA) coefficient. The phenomenon itself was named *stochastic resonance*. In comparison to the conventional resonance, where the amplitude of the system increases if the frequency of the external periodic driving force is close to the eigenfrequency of the system, stochastic resonance is an effect of amplification of the random output's periodic component as a reply to a weak periodic perturbation in the presence of noise. Although the term 'stochastic resonance' was found 'misleading' by some authors about twenty years ago [49, 17, 21] now it is well established, not to say popular.

The elementary climate model given by (1.4) was soon strongly disputed for lack of realistic assumptions. For instance, the intensity of insolation changes due to the eccentricity of the Earth's orbit is much too small to account for the observed geophysical records. The model also allows temperatures below absolute zero. Recently, stochastic resonance reappeared in more realistic reduced climate models (see Ganopolski, Rahmstorf [27]) in which an explanation of statistical properties of spontaneous intermediate warmings during glacial periods commonly known as Dansgaard-Oeschger events is given.

1.1.2 From the Schmitt trigger to recent times

In 1983, experimental studies by S. Fauve and F. Heslot [21] showed the phenomenon of stochastic resonance in a simple electronic device.

The Schmitt trigger investigated by them is a very simple and well-known electronic circuit, characterized by a two-state output and a hysteretic loop, extensively studied by physicists [21, 45, 47, 26, 42, 2, 43]. One supplies to the circuit the input voltage $w = w_t$, which is an arbitrary function of time. In the ideal Schmitt trigger the output voltage $Y = Y_t$ has only two possible values, say $-V$ and V. Let w increase from $-\infty$. Then $Y = V$ until w reaches the reference voltage level V_+. As this happens, the output jumps instantaneously to the level $-V$. Decreasing w does not affect the output Y until w reaches the reference voltage V_-. Then it jumps back. Therefore, the Schmitt trigger is a bistable system with hysteresis, see Fig. 1.7. The width of the hysteresis loop is $V_+ - V_-$. Applying a periodic voltage of small amplitude a and period $T > 0$, for example,

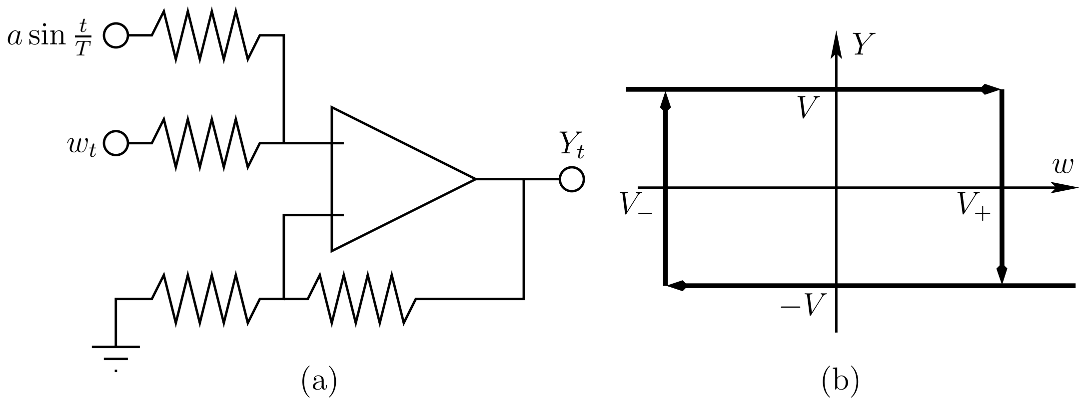

Fig. 1.7: A circuit diagram of the Schmitt trigger (a). The input-output characteristic (hysteresis loop) of the trigger (b).

to V_+, we periodically modulate the reference level. After adding a random noise at the input, the system is able to jump between the two states $\pm V$. As in the previous example of Ice Ages we can consider a discontinuous modulation, for instance given by $V_+(t) = a \operatorname{sgn} \sin\left(\frac{2\pi t}{2T}\right)$. The whole picture is now similar to the one in (1.4). Here the periodic modulation of the reference voltage corresponds to the tilting of the potential wells.

S. Fauve and F. Heslot [21] studied the power spectrum of the system and, as in the example with Ice Ages, established that the energy carried by the spectral component of Y at a given driving frequency has a local maximum for a certain intensity of the input noise.

The Schmitt trigger provides another interpretation to the phenomenon of stochastic resonance. A system displaying stochastic resonance can be considered as a sort of random amplifier. The weak periodic signal which cannot be detected in the absence of noise, can be successfully recovered if the system (the Schmitt trigger or (1.4)) is appropriately tuned. In other words, the weak underlying periodicity is exhibited at appropriately chosen non-zero levels of noise, and gets lost if noise is either too small or too large.

It was not before 1988 when stochastic resonance was used again in experiments performed on an optical system known as bidirectional ring laser [46].

Thereafter, the effect of stochastic resonance has been found in a variety of physical systems and studied by a variety of physical measures of quality of tuning: passive optical bistable systems [16], in experiments with magnetoelastic ribbons [57], in superconducting quantum interference devices [28]. It was also observed in chemical systems [40], as well as in biological ones [52, 34, 25].

Physical studies of stochastic resonance usually deal with real or numerical experiments. Theoretical approaches face a number of difficulties. To describe the diffusion system (1.4) one needs to solve the corresponding Fokker-Planck equation. In case of one-dimensional diffusion and white noise, this equation is two-dimensional due to the time-dependence of the potential. Despite the time-periodicity of the potential, in general it is difficult to study the solutions of the Fokker-Planck equation as functions of the noise intensity.

To overcome this difficulty, simpler two-state models were suggested in [49, 17, 45]. They reduce the dynamics of a diffusion in a double-well potential to the dynamics of a two-state process on the skeleton space consisting only of the metastable states. Indeed, as was already discussed, if the noise level is small, the diffusion sample paths are located near the local minima of the potential wells. Thus, if we identify the left and the right wells with, say -1 and 1, and consider the new discrete-space process on ± 1, it carries information about the *inter-well* dynamics of the diffusion. The *intra-well* small fluctuations of the diffusion in the potential minima are now filtered out.

B. McNamara and K. Wiesenfeld in [45] formally reduce the diffusion dynamics to a two-state process. It is known that for small noise intensity ε transitions between the potential wells occur at *Kramers' times.* To exponential order they are given by $e^{\frac{\Delta U}{\varepsilon}}$, where ΔU is the work to be done against the potential gradient to cross the barrier. Assuming that transitions of their two-state process are induced by these times and the height of the potential barrier is periodically modulated they approximately determine the spectral properties of the two-state process in the limit as the amplitude of the periodic signal vanishes.

For more information we address the reader to two big reviews on stochastic resonance from the physical point of view [26, 2] which contain hundreds of references.

Despite its popularity in the physical society, stochastic resonance attracted the attention of mathematicians relatively lately. The first mathematical paper on this subject by M. Freidlin [22] considers the phenomenon from the point of view of the Freidlin-Wentzell theory of large deviations.

In this paper diffusions in a general potential landscape with finitely many minima are considered. The attractor basins are subdivided into a hierarchy of cycles with main states corresponding to the deepest one among the cycle states. In the presence of periodic forcing with period time scale $e^{\lambda/\varepsilon}$, in the small noise limit $\varepsilon \to 0$ transitions between (the main states of) cycles with critical hopping work close to λ will be periodically observed. Transitions with smaller critical work may happen, but are negligible in the limit. So in the limit one observes quasi-deterministic periodic hopping between some cycles of potential minima. In the simplest case of two minima of potential depth $\frac{V}{2}$ and $\frac{v}{2}$, $v < V$, the role of which switches periodically with period T, for T larger than $e^{v/\varepsilon}$ it is shown that the diffusion will be close to the deterministic periodic function jumping between the locations of the deepest wells. As T exceeds this exponential order, many short excursions to the wrong well during one period will occur. They will not count on the exponential scale, but trajectories will look less and less periodic. It therefore becomes clear that Freidlin's condition of resonance will not be capable of interpreting the physicists' quality measures of goodness for tuning. Indeed, this condition gives only the lower bound for the period of exterior deterministic perturbation which leads to the periodicity of the trajectories. We shall sketch Freidlin's approach in Chapter 2.

A different step towards a mathematical understanding of stochastic resonance was made recently by N. Berglund and B. Gentz [7, 8, 9]. For parametrized deterministic dynamical systems passing through a pitchfork bifurcation point, the relaxation of solutions to stable equilibria are known to happen after well known delays. N. Berglund and B. Gentz exploit this observation to derive pathwise estimates for the trajectories of noisy perturbations of these systems. These results are applied to situations in which the parameter moves the system periodically or in hysteresis loops back and forth through bifurcation points, for example in periodically changing double-well potentials. The papers thus clarify some questions concerning the trajectorial behaviour in the context of stochastic resonance.

1.2 Aims and scope

This thesis deals with a mathematical foundation of the physical paradigm of stochastic resonance. As opposed to N. Berglund and B. Gentz [7, 8, 9] we study — in the physical jargon — measures of quality of tuning of the *output* to the periodic *input*. These concepts are mostly based on comparison of trajectories of the noisy system and the deterministic periodic curve describing the location of the relevant metastable states, averaged with respect to the equilibrium measure.

In the simple one-dimensional situation we consider the diffusion driven by the stochastic differential equation

$$\dot{X}_t^{\varepsilon,T} = -U'(X_t^{\varepsilon,T}, \frac{t}{2T}) + \sqrt{\varepsilon}\dot{W}_t, \quad t \geq 0,$$

where U is a time-periodic double-well potential with wells of unequal depths $\frac{V}{2}$ and $\frac{v}{2}$, considered in Section 1.1.1, see Fig. 1.3. On half-periods of length T the potential is time-independent. Our main tool in the study of stochastic resonance is the equilibrium (invariant) measure $\mu^{\varepsilon,T}(x,t)$ which lives on the infinite cylinder $\mathbf{R} \times [0, 2T)$. The most important measure of quality studied in the thesis is the *spectral power amplification* which plays an eminent role in the physical literature, see [26, 2]. It describes the spectral energy of the period $2T$ carried by the averaged trajectories of $X^{\varepsilon,T}$ and is defined as

$$\eta^X(\varepsilon, T) = \left| \int_0^1 \mathbf{E}_\mu X^{\varepsilon,T}_{2Ts} e^{2\pi i s} \right|^2,$$

where $\mathbf{E}_\mu$ denotes the expectation with respect to the invariant measure. This measure of goodness is considered as a function of the noise level ε and the half-period of periodic modulation T. Stochastic resonance with respect to the SPA coefficient occurs if $\eta^X(\varepsilon, T)$ has a local maximum in ε. We are going to determine the optimal tuning rate, i.e. the coordinate $\varepsilon(T)$ of this maximum if T is large. Freidlin's results suggest that $\varepsilon \geq \frac{v}{\log T}$, since only above this critical level the process is able to leave the shallow well at all and therefore show periodic behaviour.

The first step to find optimal tuning intensities ε consists in effectively reducing the dynamics of the diffusion to the potential minima. To mimic well this dynamics, we define two-state Markov chains with transition probabilities corresponding to the inverses of the Kramers' times for leaving the corresponding wells, and the states ± 1 corresponding to the local minima of the potential. This programme is completed for both discrete- and continuous-time Markov chains.

Chapter 3 of the thesis is devoted to the discrete-time case. To retain the essentials of the diffusion dynamics, we define time-periodic one-step transition probabilities such that on the first half-period the probability to jump from -1 to 1 is $\varphi = pe^{-V/\varepsilon}$ and the probability to jump back is $\psi = qe^{-v/\varepsilon}$, where $0 < p, q \leq 1$ parameters playing the role of pre-factors of smaller than exponential order which appear in the asymptotic expansion of laws of transition times or invariant densities of the diffusion. The exponential factor corresponds to Kramers' law. On the second half-period these probabilities change their roles which imitates the switching of the potential wells. Although the Markov chain is not time-homogeneous, enlarging the state space and considering the two-dimensional Markov process on the direct product of two discrete circles allows us to describe the invariant law of the Markov chain. The invariant law yields an explicit formula for the amplification coefficient (Theorem 3.2.1, p. 25). We study this coefficient as a function of ε, T and other parameters. It turns out, that for $T \to \infty$ there always exists an optimal tuning giving a local maximum for the SPA measure of goodness. On the exponential scale it is given by $\varepsilon \approx \frac{V+v}{2\log T}$. Theorem 3.3.1, p. 26, contains the complete analysis of the SPA coefficient and the optimal tuning rate.

The continuous-time case considered in Chapter 4 is very similar to its discrete counterpart, especially for small ε. However, continuous-time results are technically simpler and more transparent, and we study the continuous-time Markov chain in more detail. Analogously to Chapter 3 we define the infinitesimal probabilities to jump from one state to the other as φ and ψ. The time-periodic invariant measure is found as a solution of a corresponding forward Kolmogorov equation, and the SPA coefficient is finally found (up to a constant pre-factor) in Proposition 4.2.1, p. 35, as

$$\eta^Y(\varepsilon, T) = \frac{4}{\pi^2} \frac{T^2(\varphi - \psi)^2}{(\varphi + \psi)^2 T^2 + \pi^2}$$

It is no surprise, that for $T \to \infty$ the local maximum of $\eta^Y(\varepsilon, T)$ exists for all values of the parameters and is located at $\varepsilon \approx \frac{V+v}{2 \log T}$.

It needs to be emphasized that although we study the optimal tuning asymptotically, all formulae in Chapters 3 and 4 are valid for all values of the rates v and V, and all noise levels $\varepsilon > 0$ and modulation half-periods $T > 0$.

The invariant law provides a good basis for studying different measures of goodness of tuning. Thus, we study the *SPA-to-noise ratio*, which we define as SPA coefficient divided by the squared intensity of noise, the whole *energy* carried by the averaged trajectories, and the *energy-to-noise* ratio. These measures describe the spectral properties of the process and in their case optimality of tuning means maximality in ε. However, the optimal tuning rates are different from the spectral power amplification and range in leading order from $\frac{v}{\log T}$ to $\frac{V}{\log T}$.

We also investigate two more intuitive notions of quality of tuning: the entropy of the equilibrium measure and the entropy relative to the periodically changing point mass sitting in the deep well. As functions of ε they possess local minima and in this case optimality of tuning means minimality in ε. If we interpret entropy as a measure of chaos or randomness, the point of minimal entropy marks the noise level at which the system behaves the least random possible.

Finally, we study the measure of goodness suggested by M. Freidlin in [22], namely the cumulative time spent by the trajectory in the 'wrong' place. We consider it averaged with respect to the invariant law. It turns out to have a local minimum in ε (Proposition 4.6.2, p. 43) again giving the point of optimal tuning.

We next deal with the diffusion. Its equilibrium density is a solution of a forward Kolmogorov (Fokker-Planck) equation, which in this case is a parabolic partial differential equation. We study its solution using the Fourier method of separation of variables. To apply it, the spectral properties of the diffusion's infinitesimal generator have to be studied.

In Chapter 5 we establish that if the potential is not degenerate in its extrema, i.e. the curvatures do not vanish there, and increases at infinity, then the spectrum of the infinitesimal generator considered as an operator in $\mathcal{L}^2(\mathbb{R}, e^{-2U/\varepsilon}dx)$ is discrete and non-positive. The zeroth eigenvalue vanishes and the corresponding eigenfunction is constant. As $\varepsilon \to 0$, the first eigenvalue turns out to be exponentially small and the absolute value of its leading term yields Kramers' rate $\lambda_1 \approx \frac{\sqrt{U''(1)|U''(0)|}}{2\pi} e^{-v/\varepsilon}$. The first eigenfunction can be chosen to be exponentially close to 1 on the positive half-line, and $-\sqrt{\frac{U''(-1)}{U''(1)}}e^{-(V-v)/\varepsilon}$ on the negative half-line. The neighbourhood of zero constitutes the so-called internal layer. To obtain the first eigenvalue and eigenfunction we use variational principles and series expansions of the eigenfunctions.

Of central importance to the analysis of goodness of tuning is a *spectral gap* property of the second eigenvalue. We show that its absolute value is bounded below by some positive constant which is *independent* of noise intensity.

This has crucial implications for η^X and η^Y to be developed in Chapter 6. First of all, it means that only the first two terms in the series expansion of the invariant density matter in the description the SPA coefficient η^X. We obtain the following formula (up to a constant pre-factor)

$$\eta^X(\varepsilon, T) = \frac{4}{\pi^2}\left(\frac{\int_{\mathbb{R}} ye^{-2U(y)/\varepsilon}}{\int_{\mathbb{R}} e^{-2U(y)/\varepsilon}}\right)^2 \frac{T^2\lambda_1^2}{T^2\lambda_1^2+\pi^2} + r(\varepsilon, T) \tag{1.5}$$

The remainder term r is small under certain conditions, and the expression for η^X reminds us of an SPA coefficient for the Markov chain. Indeed, if we choose the pre-factors in the infinitesimal probabilities $p = \frac{\sqrt{U''(-1)|U''(0)|}}{2\pi}$ and $q = \frac{\sqrt{U''(1)|U''(0)|}}{2\pi}$, then η^X and η^Y look very similar, as the expression in the parenthesis in (1.5) is approximately 1 for small ε.

Surprisingly, however, dependencies on the fine structure of the potential function U in the minima beyond the expected curvature properties enter the game and lead to quite unexpected results. For example, a subtle drag away from the other well caused by the sign of the third derivative of U in the deep well suffices to prevent the spectral power amplification curve from having a local maximum in the parameter range suggested by the approximating Markov chain. Contrary to what intuition supposes one of the physicists' favourite quality functions does not show resonance effects at all in this case.

More precisely, we quote here the asymptotics from Theorem 6.3.1, p. 97. Let $\alpha \in [v+\delta, \Delta]$ for some $\delta > 0$ and $\Delta > v+\delta$. Then under some mild assumptions

$$\eta^X(\frac{\alpha}{\log T}, T) = \frac{4}{\pi^2}\left(1 + \frac{U^{(3)}(-1)}{2U''(-1)^2}\frac{\alpha}{\log T}\right) + \mathcal{O}\left(\frac{1}{\log^2 T}\right), \quad T\to\infty.$$

Taking for instance $\Delta = 2V$, this formula means that the SPA coefficient of a diffusion is either an increasing function, or a decreasing function of noise intensity. It has no local maximum in the region where its Markov chain counterpart has.

The reason for this unexpected dramatic deviation from the approximating Markov chain behaviour lies in the significance attributed to small fluctuations inside the potential wells by the spectral power amplification. If these fluctuations are cut off, the Markov chain is seen to be a good approximation in the small noise limit, and provides the optimal tuning rate.

Indeed, consider the modified SPA coefficient

$$\widetilde{\eta}^X(\varepsilon,T)=\left|\int_0^1 \mathbf{E}_\mu g(X^{\varepsilon,T}_{2Ts})e^{2\pi i s}\,ds\right|^2,$$

where the function g identifies small fluctuations of the diffusion near the well bottoms and is given by

$$g(x)=\begin{cases} x, & x\in(-\infty,x_1]\cup[x_2,y_1]\cup[y_2,\infty),\\ -1, & x\in[x_1,x_2],\\ 1, & x\in[y_1,y_2],\end{cases}$$

where $x_1<-1<x_2<0$ are such that $U(x_1)=U(x_2)=-\frac{V}{4}$, and $0<y_1<1<y_2$ are such that $U(y_1)=U(y_2)=-\frac{v}{4}$, see Fig. 6.3 on p. 99.

Then under some mild assumptions, for any positive and sufficiently small γ there is $T(\gamma)$ such that for $T>T(\gamma)$ the modified SPA coefficient $\widetilde{\eta}^X(\varepsilon,T)$ has a local maximum in ε on the interval $[\gamma^{-1}\frac{V+v}{2\log T},\gamma\frac{V+v}{2\log T}]$, i.e. the optimal tuning rate ε is exponentially equivalent to $\frac{V+v}{2\log T}$, $T\to\infty$ (Theorem 6.4.1, p. 101).

We finally mention that in Chapters 5 and 6 we extensively use Laplace's method of asymptotic evaluation of integrals depending on a parameter. In an Appendix we present all facts and formulae about this method used in this thesis.

Chapter 2

What Large Deviations Tell us about Tuning

Although more than four hundred physical papers concerning stochastic resonance were published in the last twenty years, a rigorous mathematical approach to the effect was given only recently in [22]. In this paper stochastic resonance was considered from the point of view of large deviations, and a lower bound for optimal tuning of the random output to the periodic input was given. In this chapter we briefly formulate the facts needed from Freidlin-Wentzell theory of perturbed dynamical systems [23] in the one-dimensional case and discuss the results of [22] concerning the 'optimal tuning' of the double-well oscillator.

2.1 Diffusion with small noise

Let $\varepsilon > 0$. Following [23] we consider the diffusion X^ε in $\mathbb{R}$ which is the solution of the SDE

$$dX_t^\varepsilon = -U'(X_t^\varepsilon)\, dt + \sqrt{\varepsilon}\, dW_t, \quad X_0^\varepsilon = x, \quad t \geq 0,$$

where W is a standard Brownian motion, and U is a smooth function. Note that in one-dimensional case the drift term can always be represented as a derivative of some potential; in higher dimensions this is not true.

For $T > 0$, we introduce the *action functional* on the space $\mathcal{C}[0,T]$ corresponding to our potential U

$$S_{0T}(h) = \begin{cases} \frac{1}{2}\int_0^T (\dot{h}_s + U'(h_s))^2\, ds, & h \text{ is absolutely continuous,} \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easy to see that $S_{0T} \geq 0$, and if $S_{0T}(h) = 0$ then h is a trajectory of the dynamical system $\dot{x} = -U'(x)$ on the interval $[0,T]$.

Let $x, y \in \mathbb{R}$. By means of the action functional we define the *quasipotential*

$$V(x,y) = \inf\{S_{0T}(h) \,:\, h \in \mathcal{C}[0,T],\, h_0 = x,\, h_T = y,\, T > 0\}.$$

The quasipotential describes the work done by a physical particle moving in the potential landscape given by U to get from x to y. More precisely, let $\tau_y^\varepsilon = \inf\{t > 0 : X_t^\varepsilon = y\}$ and denote by $\mathbf{P}_x$ the law of the diffusion starting at x. In these terms

$$V(x,y) = \lim_{T\to\infty}\lim_{\varepsilon\to 0} -\log \mathbf{P}_x(\tau_y^\varepsilon \le T).$$

By means of the quasipotential one can describe the asymptotic behaviour of the diffusion as $\varepsilon \to 0$. Describing the asymptotics of transition times, it contains a mathematical formulation of Kramers' law.

Theorem 2.1.1 ([23]) *Let $[a,b]$ be a finite interval, $0 \in [a,b]$ be the unique zero of $U'(x)$ on the interval, and 0 be the asymptotically stable point of the dynamical system $\dot{x} = -U'(x)$. Let $\tau^\varepsilon = \inf\{t > 0 : X_t^\varepsilon \notin [a,b]\}$. Then for any $x \in (a,b)$ the following holds:*

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbf{E}_x \tau^\varepsilon = \min\{V(0,a), V(0,b)\} = V_0,$$
$$\lim_{\varepsilon\to 0} \mathbf{P}_x(e^{(V_0-\delta)/\varepsilon} < \tau^\varepsilon < e^{(V_0+\delta)/\varepsilon}) = 1 \ \textit{for any}\ \delta > 0.$$

■

Let U be a double-well potential with minima at ± 1 and a saddle point at the origin, and assume $U(x) \to \infty$, as $|x| \to \infty$. Let $U(-1) = -\frac{V}{2}$, $U(1) = -\frac{v}{2}$, $U(0) = 0$, $0 < v < V$.

If x and y are in the same well it is easy to show that

$$V(x,y) = 2\max\{U(y) - U(x), 0\}. \tag{2.1}$$

The pre-factor 2 in (2.1) explains why we choose the depths as $\frac{V}{2}$ and $\frac{v}{2}$. In particular, if $U(y) < U(x)$, the quasipotential $V(x,y) = 0$ because we can go 'down' in the potential landscape from x to y along the deterministic trajectory, on which the action functional equals zero. On the other hand, the way 'up' from y to x costs twice the difference between $U(x)$ and $U(y)$.

Let x and y belong to different wells and suppose for example that $-1 \le x$, $y \le 1$. In this case, we have to overcome a potential barrier of height $U(0) - U(x)$ on the way between x and 0, and the way 'down' to y is free. Consequently, the quasipotential $V(x,y) = 2(U(0) - U(x))$. Analogously, $V(y,x) = 2(U(0)) - U(y)$. In particular, if $x = -1$ then $V(x,y) = V$, and if $y = 1$, we have $V(y,x) = v$. From Theorem 2.1.1 we can conclude that the mean time to jump from the left well to the right is exponentially large in ε, and has order $e^{V/\varepsilon}$. The mean time to jump back is smaller and has order $e^{v/\varepsilon}$. These asymptotics suggest that if we want to record the inter-well motion of the diffusion we should consider it on exponentially long time intervals. The following theorem describes this behaviour precisely.

Theorem 2.1.2 ([22]) *Let $T = T(\varepsilon)$ be such that*

$$\lim_{\varepsilon \to 0} \varepsilon \log T(\varepsilon) = \lambda > 0.$$

Let δ and A be arbitrary positive numbers, $x \in \mathbb{R}$, and Λ denote the Lebesgue measure on $\mathbb{R}$. Then

$$\Lambda(t \in [0, A] \, : \, |X^{\varepsilon}_{tT(\varepsilon)} + 1| > \delta) \to 0$$

in $\mathbf{P}_x$-probability as $\varepsilon \to 0$, if $\lambda > v$. If $\lambda < v$ then

$$\Lambda(t \in [0, A] \, : \, |X^{\varepsilon}_{tT(\varepsilon)} - \operatorname{sgn} x| > \delta) \to 0$$

in $\mathbf{P}_x$-probability as $\varepsilon \to 0$. ■

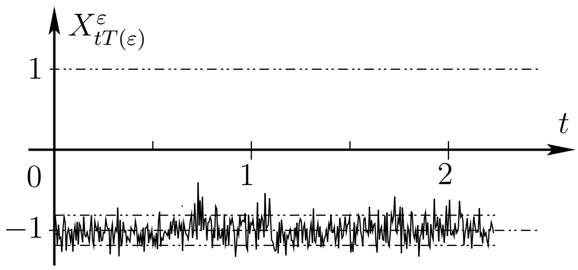

Fig. 2.1: The sample path of $X^{\varepsilon}_{tT(\varepsilon)}$, $\lambda > v$.

Indeed, on the time interval of length $e^{\lambda/\varepsilon}$ the diffusion always has enough time to reach the deep well if $\lambda > v$. Moreover, it can make jumps back to the shallow well, but the sum of the periods in which the trajectory is outside of a δ-neighbourhood of -1 has a probability which vanishes as $\varepsilon \to 0$ (see Fig. 2.1). If $\lambda < v$ then during the period of length $e^{\lambda/\varepsilon}$ the diffusion does not have enough time to leave the well where it has started, so it stays in the δ-neighbourhood of the corresponding local minimum with high probability.

2.2 Stochastic periodicity above critical noise level

Consider now the diffusion $X^{\varepsilon,T}$ with time-periodic drift which is a solution of the SDE

$$dX^{\varepsilon,T}_t = -U'(X^{\varepsilon,T}_t, \frac{t}{2T})\, dt + \sqrt{\varepsilon}\, dW_t, \quad X^{\varepsilon,T}_0 = x \in \mathbb{R}, \quad t \geq 0, \tag{2.2}$$

where the potential $U(x,t) = U(-x, t+\frac{1}{2})$ is a 1-periodic function of time ($U' = \frac{\partial U}{\partial x}$). We also assume that on the time interval $[0, \frac{1}{2})$ the function U is a double-well potential described in the previous section (see Fig. 1.3, p. 4). The parameter T denotes the half-period with which the drift term in (2.2) changes its form. On the intervals $[kT, (k+1)T)$, $k = 0, 1, \dots$ the diffusion $X^{\varepsilon,T}$ is time-homogeneous, one can calculate the quasipotentials, and if the half period T is long enough, one can apply Theorem 2.1.2 on each of the intervals. Hence, the following holds:

Theorem 2.2.1 ([22]) *For $\varepsilon > 0$, let the half-period $T = T(\varepsilon)$ be such that*

$$\lim_{\varepsilon \to 0} \varepsilon \log T(\varepsilon) = \lambda > 0$$

and $\delta > 0$ and $A > 0$ be arbitrary, $x \in \mathbb{R}$. Let the function

$$\phi(t) = \begin{cases} -1, & t \in [k, k+\frac{1}{2}), \\ 1, & t \in [k+\frac{1}{2}, k+1), \quad k = 0, 1, 2, \dots, \end{cases}$$

be a periodic and deterministic function of time, and Λ denote the Lebesgue measure on $\mathbb{R}$. Then, if $\lambda > v$,

$$\Lambda(t \in [0, A] \, : \, |X^{\varepsilon,T}_{2T(\varepsilon)t} - \phi(t)| > \delta) \to 0 \tag{2.3}$$

in $\mathbf{P}_x$-probability as $\varepsilon \to 0$.

If $\lambda < v$ then

$$\Lambda(t \in [0, A] \, : \, |X^{\varepsilon,T}_{2T(\varepsilon)t} - \operatorname{sgn} x| > \delta) \to 0$$

in $\mathbf{P}_x$-probability as $\varepsilon \to 0$. ■

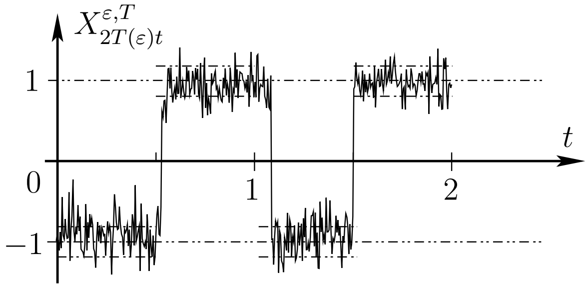

Fig. 2.2: The sample path of $X^{\varepsilon,T}_{2T(\varepsilon)t}$, $\lambda > v$.

The statement of Theorem 2.2.1 is clear: the function $\phi(t)$ exhibits the location of the global minimum of U at time $2T(\varepsilon)t$. If the exponential order of $T(\varepsilon)$ is bigger than v, then independently of the initial point the trajectory has enough time to reach the deepest well during the half-period. In this case we observe a sort of stochastic periodicity. If $\lambda < v$, then $T(\varepsilon)$ is too short for the diffusion to 'feel' the change of the potential, and it stays in the same well with very high probability.

Theorem 2.2.1 gives us one possible measure for periodicity of the trajectories. This is the Lebesgue measure of the time the diffusion spends outside of the δ-tube of the deterministic function ϕ. A sort of optimal tuning is also found: if the half-period $T(\varepsilon)$ is such that $\lim_{\varepsilon\to 0} \varepsilon \log T(\varepsilon) < v$ we observe no reply to the periodic perturbation of the drift in (2.2). Periodicity in the diffusion paths appears only if $\lim_{\varepsilon\to 0} \varepsilon \log T(\varepsilon) > v$, and thus we have obtained the *lower bound* for tuning rate. We note also that the lower bound for tuning does not depend on the absolute values of v and V. It is only important that $v < V$.

Theorem 2.2.1 gives no upper bound for $T(\varepsilon)$. However, the upper bound should exist, maybe, for some other measure of quality of the paths. Indeed, let $\lambda \gg v$. The corresponding value $T(\varepsilon)$ is then exponentially larger than both the mean times to jump from the left to the right and back. This means that $X^{\varepsilon,T}$ can make many excursions to the shallow well, the cumulative duration of which is exponentially small in comparison with the time spent in the deep well. The Lebesgue measure in (2.3) still goes to zero in probability, but if we look at the sample paths, periodicity is destroyed, at least in the common sense of the word (see Fig. 2.3).

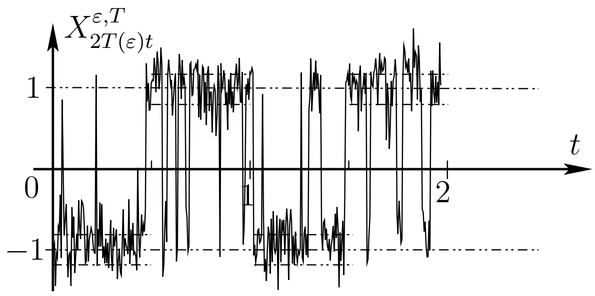

Fig. 2.3: The sample path of $X^{\varepsilon,T}_{2T(\varepsilon)t}$, $\lambda \gg v$.

A natural question arises: what other measures of quality for periodicity of the paths can be considered, and what is the optimal tuning rate for them? We shall try to answer these questions in the next chapters, and compare the answers with Theorem 2.2.1.

Chapter 3

Stochastic Resonance in Two-State Markov Chains with Discrete Time

Having in mind our final goal — the description of diffusion with time-periodic drift — in this chapter we introduce a two-state Markov chain with discrete time. As we have seen in the previous chapter (Theorems 2.1.2 and 2.2.1), the small noise diffusion in a double-well potential predominantly 'lives' in the local minima of the potential. If we identify the neighbourhoods of the minima (or even the wells) with the states of the Markov chain and define its transition probabilities in an appropriate way, then, at least intuitively, such a Markov chain can give a rough description of the *inter-well* behaviour of the underlying diffusion.

The idea of such an approximation was discussed in the physical literature from the point of view of the statistics of jump times of the random process between the states [17], however, without analysing the dependence of the results on the noise intensity. In [45] the power spectrum of the random output of the periodically modulated two-state process was studied on the physics level of rigour in case the amplitude of the deterministic component tends to zero and the noise parameter is small, so that one can apply Kramers' law [38].

In our approach we do not impose restrictions on the 'noise' and 'amplitude' parameters. Our main tool is the invariant law of the Markov chain determined in Section 3.1. In Section 3.2 we interpret the Markov chain as a random amplifier and introduce a measure of goodness — the *spectral power amplification* coefficient (SPA) which has a clear physical meaning [26] and measures the strength of the low-frequency spectral component in the averaged random output. Section 3.3 is devoted to the analysis of the SPA coefficient as a function of noise. It turns out that it has a local maximum. The coordinate of this maximum representing the *optimal tuning* in the sense of the SPA is determined.

This chapter is a revised, corrected and enlarged version of [33].

3.1 Markov chains with time-periodic transition probabilities

For $m \in \mathbb{N}$, consider a time-inhomogeneous Markov chain $Z_m = (Z_m(k))_{k\geq 0}$ on the state space $\mathcal{S}^Z = \{-1, 1\}$. Let $P_m(k)$ be the matrix of one-step transition probabilities at time k. If we denote

$$\pi_m^-(k) = \mathbf{P}(Z_m(k) = -1),$$
$$\pi_m^+(k) = \mathbf{P}(Z_m(k) = 1),$$

and write P^* for the transposed matrix, we obtain

$$\begin{pmatrix} \pi_m^-(k+1) \\ \pi_m^+(k+1) \end{pmatrix} = P_m^*(k) \begin{pmatrix} \pi_m^-(k) \\ \pi_m^+(k) \end{pmatrix}.$$

In order to mimic the periodic switching of the double-well potential in our Markov chain, we define the transition matrix $P_m(\cdot)$ to be periodic in time with half-period m. More precisely,

$$P_m(k) = \begin{cases} P_1, & 0 \leq k(\text{mod } 2m) \leq m-1, \\ P_2, & m \leq k(\text{mod } 2m) \leq 2m-1, \end{cases} \tag{3.1}$$

with P_1 and P_2 to be defined below.

As the diffusion given by (2.2), the Markov chain defined in (3.1) is time-homogeneous on intervals corresponding to half-periods. On these intervals the behaviour of the diffusion is governed by the potential $U(\cdot)$ or $U(-\cdot)$ (see Fig. 1.3) and the evolution of the Markov chain is given by the stochastic matrices P_1 and P_2. The integer $m > 0$ is clearly the half-period of the deterministic modulation. The spatial antisymmetry of the potential function can be transferred to the Markov chain by setting

$$P_1 = \begin{pmatrix} 1-\varphi & \varphi \\ \psi & 1-\psi \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 1-\psi & \psi \\ \varphi & 1-\varphi \end{pmatrix}. \tag{3.2}$$

We now have to define the transition probabilities φ and ψ figuring in P_1, P_2 in such a way that the Markov chain retains the dynamical behaviour of the diffusion reduced to its metastable states. In our setting, the states ± 1 of the process Z_m correspond to the right respectively left well of the potential. The Freidlin-Wentzell theory [23] (recall Theorem 2.1.1) states that for a time-homogeneous diffusion with small noise in a double-well potential U the mean times to jump from one well to the other are exponentially large in the noise intensity ε as $\varepsilon \to 0$. Moreover, the mean time to leave the deep well of depth $\frac{V}{2}$ has the order $e^{\frac{V}{\varepsilon}}$ and the mean time to leave the shallow well of depth $\frac{v}{2} < \frac{V}{2}$ is of order $e^{\frac{v}{\varepsilon}}$.

On the other hand, for the time-homogeneous Markov chain governed, for example, by the stochastic matrix P_1 the mean time to jump from -1 to 1 equals φ^{-1} and the mean time to jump from 1 to -1 equals ψ^{-1} which suggests the following expressions for φ and ψ:

$$\begin{gathered} \varphi = pe^{-\frac{V}{\varepsilon}}, \qquad \psi = qe^{-\frac{v}{\varepsilon}}, \\ \varepsilon > 0, \quad 0 < v < V, \quad 0 < p, q \leq 1. \end{gathered} \tag{3.3}$$

The parameter ε in (3.3) is associated with the noise intensity in (2.2), which tends to zero in the Freidlin-Wentzell theory. In our model we consider all possible intensities and assume that $\varepsilon > 0$. The values $0 < v < V$ correspond to the double depths of the potential wells or the values of the quasipotential in the metastable states. The pre-factors p and q, which do not appear in the large deviations' statements, are introduced here to justify the equalities in (3.3) instead of the exponential equivalence given by large deviations (Theorem 2.1.1). In this chapter we assume p and q to be constants adding some asymmetry to the picture and taking values between 0 and 1 to guarantee the positivity of the transition probabilities in (3.2). The latter restriction will be weakened in the next chapter where we consider the Markov chain with continuous time. In Chapter 6 we shall discuss the link between the diffusion and the two-state Markov process and show that these pre-factors have a clear geometrical meaning and can be expressed in terms of the curvatures of the potential in the saddle point and the minima.

It is well known that for a time-homogeneous Markov chain on $\mathcal{S}^Z$ with transition matrix P one can talk about *equilibrium*, given by the stationary distribution, to which the law of the chain converges exponentially fast. The stationary distribution can be found by solving the matrix equation $\pi = P^*\pi$ with normalizing condition $\pi^- + \pi^+ = 1$.

For time-inhomogeneous Markov chains with time periodic transition matrix, the situation is quite similar. After enlarging the state space $\mathcal{S}^Z$ to $\mathcal{S}^{\mathbf{Z}} = \mathcal{S}^Z \times \mathbb{Z}_{2m} = \{-1, 1\} \times \{0, 1, \ldots, 2m-1\}$, we recover a time-homogeneous chain by setting

$$\mathbf{Z}_m(k) = (Z_m(k), k \ (\mathrm{mod}\ 2m)), \quad k \geq 0,$$

to which the previous remarks apply. Topologically, the enlarged state space $\mathcal{S}^{\mathbf{Z}}$ is a direct product of two discrete circles. For convenience of notation, we assume the enlarged state space to be ordered in the following way:

$$\mathcal{S}^{\mathbf{Z}} = [(-1,0), (1,0), (-1,1), (1,1), \ldots, (-1, 2m-1), (1, 2m-1)].$$

Writing A_m for the matrix of one-step transition probabilities of $\mathbf{Z}_m$, the stationary distribution $Q = (q(i,j))^*$, $i = \pm 1$, $j = 0, \ldots, 2m-1$, is obtained as a normalized solution of the matrix equation

$$(\mathrm{A}_m^* - E)Q = 0, \tag{3.4}$$

E being the $(2m \times 2m)$-unit matrix. We shall be dealing with the following variant of stationary measure, which is not normalized in time.

Definition 3.1.1 *Let* $\pi_m(k) = (\pi_m^-(k), \pi_m^+(k))^* = 2m(q(-1,k), q(1,k))^*$, $0 \le k \le 2m-1$. *We call* $\pi_m = (\pi_m(k))_{0 \le k \le 2m-1}$ *the stationary measure of the Markov chain* Z_m.

The reason why we consider the renormalized measure is quite simple. The main object of our interest is the process Z_m, and $\mathbf{P}(Z_m = 1) + \mathbf{P}(Z_m = -1) = 1$ for all $k \ge 0$. The process $\mathbf{Z}_m$ is an auxiliary object used to justify the existence of the invariant law for Z_m.

The stationary measure will be the main tool in our subsequent analysis. In the following theorem we find the explicit formula for π_m.

Theorem 3.1.1 *For every* $m \ge 1$, *the stationary measure* π_m *of* Z_m *with matrices of one-step probabilities defined in (3.2) is given by*

$$\begin{cases} \pi_m^-(l) &= \dfrac{\psi}{\varphi+\psi} + \dfrac{\varphi-\psi}{\varphi+\psi}\,\dfrac{(1-\varphi-\psi)^l}{1+(1-\varphi-\psi)^m}, \\ \pi_m^+(l) &= \dfrac{\varphi}{\varphi+\psi} - \dfrac{\varphi-\psi}{\varphi+\psi}\,\dfrac{(1-\varphi-\psi)^l}{1+(1-\varphi-\psi)^m}; \end{cases}$$
$$\begin{cases} \pi_m^-(l+m) = \pi_m^+(l), \\ \pi_m^+(l+m) = \pi_m^-(l), \qquad 0 \le l \le m-1. \end{cases} \tag{3.5}$$

Proof: The probabilities $\pi_m^{\pm}(k)$ differ from their counterparts $q(\pm 1, k)$ by the factor $2m$, and $\pi_m^+(k) + \pi_m^-(k) = 1$ for all $k = 0, 1, \dots, 2m-1$. To find the vector Q we transform the matrix $\mathrm{A}_m^* - E$ into a more convenient form. The matrix A_m of one-step transition probabilities of $\mathbf{Z}_m$ is explicitly given by

$$\mathrm{A}_m = \left(\begin{array}{ccccccc|ccccccc}
0 & P_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & P_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & P_1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & P_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & P_1 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & P_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & P_2 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & P_2 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & P_2 \\
P_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right).$$

A_m has block structure. In our notation 0 means a (2×2)-matrix with all entries equal to zero, P_1, and P_2 are the 2-dimensional matrices defined in (3.2).

It is easy to see how the periodicity of transition probabilities is interpreted by the matrix A_m. From the states $(\pm 1, k)$, $k = 0, \ldots, 2m-1$ the process $\mathbf{Z}_m$ passes to the state $(\pm 1, k+1 \pmod{2m})$ with probability one. The probability to pass to other states equals zero.

The vector Q of the invariant law of the process $\mathbf{Z}_m$ satisfies the matrix equation (3.4) with

$$
\mathrm{A}_m^* - E = \left(\begin{array}{ccccccc|cccccc}
-E & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & P_2^* \\
P_1^* & -E & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & P_1^* & -E & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\multicolumn{7}{c|}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
0 & 0 & 0 & 0 & \cdots & -E & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & P_1^* & -E & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \cdots & 0 & P_1^* & -E & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & P_2^* & -E & 0 & \cdots & 0 & 0 \\
\multicolumn{7}{c|}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & -E & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & P_2^* & -E
\end{array}\right).
$$

By elementary operations on the rows of $\mathrm{A}_m^* - E$ we eliminate P_2^* in the last column of the upper row to obtain the equivalent matrix A'_m

$$
\mathrm{A}'_m = \left(\begin{array}{ccccccc|cccccc}
\widehat{P} - E & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
P_1^* & -E & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & P_1^* & -E & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\multicolumn{7}{c|}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
0 & 0 & 0 & 0 & \cdots & -E & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & P_1^* & -E & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \cdots & 0 & P_1^* & -E & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & P_2^* & -E & 0 & \cdots & 0 & 0 \\
\multicolumn{7}{c|}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & -E & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & P_2^* & -E
\end{array}\right),
$$

where $\widehat{P} = P_2^* P_2^* \cdots P_1^* = (P_2^*)^m (P_1^*)^m$. A'_m is a block-wise upper-triangular matrix, and so $A'_m Q = 0$ can be solved as usual.

The probabilities $\pi_m(0)$ satisfy the matrix equation $((P_2^*)^m (P_1^*)^m - E)\pi_m(0) = 0$ with additional condition $\pi_m^-(0) + \pi_m^+(0) = 1$. To calculate $(P_2^*)^m (P_1^*)^m$, we use a formula for the m-th power of (2×2) stochastic matrices [56, Chapter I, §12], which results in

$$(P_1^*)^m = \frac{1}{\varphi+\psi}\begin{pmatrix}\psi & \psi\\ \varphi & \varphi\end{pmatrix} + \frac{(1-\varphi-\psi)^m}{\varphi+\psi}\begin{pmatrix}\varphi & -\psi\\ -\varphi & \psi\end{pmatrix},$$
$$(P_2^*)^m = \frac{1}{\varphi+\psi}\begin{pmatrix}\varphi & \varphi\\ \psi & \psi\end{pmatrix} + \frac{(1-\varphi-\psi)^m}{\varphi+\psi}\begin{pmatrix}\psi & -\varphi\\ -\psi & \varphi\end{pmatrix}.$$

Using matrix multiplication we find

$$\begin{aligned}(P_2^*)^m(P_1^*)^m &= \frac{1}{\varphi+\psi}\begin{pmatrix}\varphi & \varphi\\ \psi & \psi\end{pmatrix}\\ &+ (1-\varphi-\psi)^m\frac{\varphi-\psi}{\varphi+\psi}\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix} + \frac{(1-\varphi-\psi)^{2m}}{\varphi+\psi}\begin{pmatrix}\varphi & -\psi\\ -\varphi & \psi\end{pmatrix},\end{aligned}$$

from which a straightforward calculation yields

$$\begin{cases}\pi_m^-(0) = \dfrac{\varphi+\psi(1-\varphi-\psi)^m}{(\varphi+\psi)(1+(1-\varphi-\psi)^m)},\\ \pi_m^+(0) = \dfrac{\psi+\varphi(1-\varphi-\psi)^m}{(\varphi+\psi)(1+(1-\varphi-\psi)^m)}.\end{cases}$$

To compute the remaining entries, we use $\pi_m(l) = (P_1^*)^l\pi_m(0)$ for $0 \le l \le m-1$, and $\pi_m(l) = (P_2^*)^l(P_1^*)^m\pi_m(0)$ for $m \le l \le 2m-1$ to obtain (3.5). Note also the symmetry $\pi_m^-(l+m) = \pi_m^+(l)$ and $\pi_m^+(l+m) = \pi_m^-(l)$, $0 \le l \le m-1$. ■

3.2 Spectral power amplification

The chain Z_m can be interpreted as amplifier of a signal. The stochastic system may be seen to receive a deterministic periodic input signal which switches the powers in the transition probabilities (3.3), i.e.

$$I_m(l) = \begin{cases}V, & 0 \le l \pmod{2m} \le m-1,\\ v, & m \le l \pmod{2m} \le 2m-1.\end{cases}$$

The output is the random process $Z_m(k)$.

The input signal I_m admits a spectral representation

$$I_m(k) = \frac{1}{2m}\sum_{a=0}^{2m-1} c_m(a)e^{-\frac{2\pi ik}{2m}a}, \quad k\in\mathbb{Z},$$

where $c_m(a) = \frac{1}{2m}\sum_{l=0}^{2m-1} I_m(l)e^{\frac{2\pi ia}{2m}l}$ is the Fourier coefficient of frequency $a/2m$. The quantity $|c_{2m}(a)|^2$ measures the power carried by this Fourier component.

We are only interested in the component of the input frequency $1/2m$. Its power is given by

$$\begin{aligned} |c_m(1)|^2 &= \frac{1}{4m^2}\left|\sum_{l=0}^{m-1} V e^{\frac{2\pi i}{2m}l} + \sum_{l=m}^{2m-1} v e^{\frac{2\pi i}{2m}l}\right|^2 = \frac{(V-v)^2}{4m^2}\left|\sum_{l=0}^{m-1} e^{\frac{2\pi i}{2m}l}\right|^2 \\ &= \frac{(V-v)^2}{4m^2}\frac{|1-e^{\pi i}|^2}{|1-e^{\frac{\pi i}{m}}|^2} = \frac{(V-v)^2}{m^2}\csc^2\left(\frac{\pi}{2m}\right). \end{aligned} \tag{3.6}$$

In the stationary regime, i.e. if the law of Z_m is given by the measure π_m, the power carried by the output at frequency $a/2m$ is a random variable

$$\xi_m(a) = \frac{1}{2m}\sum_{l=0}^{2m-1} Z_m(l) e^{\frac{2\pi i a}{2m}l}.$$

We define the *spectral power amplification* as the relative power carried by the averaged component of the output with half-period m. In the next section we shall study the SPA coefficient as a function of ε and m. We emphasize this in the following definition although ε does not appear there explicitly.

Definition 3.2.1 *The* spectral power amplification *(SPA) coefficient of the Markov chain Z_m with half-period $m \geq 1$ is given by*

$$\eta^Z(\varepsilon, m) = \frac{|\mathbf{E}_\pi \xi_m(1)|^2}{|c_m(1)|^2}.$$

Here $\mathbf{E}_\pi$ denotes expectation w.r.t. the stationary measure π_m.

The explicit description of the invariant measure now readily yields the following formula for the SPA coefficient.

Theorem 3.2.1 *Let $m \geq 1$. The SPA coefficient of the Markov chain Z_m with one-step transition probabilities* (3.2) *equals*

$$\eta^Z(\varepsilon, m) = \frac{4}{(V-v)^2}\frac{(\varphi-\psi)^2 m^2}{(\varphi+\psi)^2 m^2 + \pi^2 + \kappa(\varepsilon, m)}, \tag{3.7}$$

where $\kappa(\varepsilon, m) = 4(1-\varphi-\psi)m^2 \sin^2\left(\frac{\pi}{2m}\right) - \pi^2$.

Proof: Using (3.5) one immediately gets

$$
\begin{aligned}
\mathbf{E}_\pi \xi_m(1) &= \frac{1}{2m}\sum_{k=0}^{2m-1} \mathbf{E}_\pi Z_m(k) e^{\frac{2\pi i}{2m}k} = \frac{1}{2m}\sum_{k=0}^{2m-1}(\pi_m^+(k)-\pi_m^-(k))e^{\frac{2\pi i}{2m}k} \\
&= \frac{1}{2m}\sum_{k=0}^{m-1}(\pi_m^+(k)-\pi_m^-(k))e^{\frac{\pi i}{m}k} + \frac{1}{2m}\sum_{k=0}^{m-1}(\pi_m^+(k+m)-\pi_m^-(k+m))e^{\frac{\pi i}{m}(k+m)} \\
&= \frac{1-e^{\pi i}}{2m}\sum_{k=0}^{m-1}(\pi_m^+(k)-\pi_m^-(k))e^{\frac{\pi i}{m}k} = \frac{1}{m}\frac{\varphi-\psi}{\varphi+\psi}\sum_{k=0}^{m-1}\left[1-\frac{2(1-\varphi-\psi)^k}{1+(1-\varphi-\psi)^m}\right]e^{\frac{\pi i}{m}k} \\
&= \frac{1}{m}\frac{\varphi-\psi}{\varphi+\psi}\left[\frac{1-e^{\pi i}}{1-e^{\frac{\pi i}{m}}} - \frac{2}{1+(1-\varphi-\psi)^m}\frac{1-(1-\varphi-\psi)^m e^{\pi i}}{1-(1-\varphi-\psi)e^{\frac{\pi i}{m}}}\right] \\
&= \frac{2}{m}\frac{\varphi-\psi}{\varphi+\psi}\left[\frac{1}{1-e^{\frac{\pi i}{m}}} - \frac{1}{1-(1-\varphi-\psi)e^{\frac{\pi i}{m}}}\right].
\end{aligned}
$$

Combining the previous formula with (3.6) yields

$$
\begin{aligned}
\eta^Z(\varepsilon,m) &= \frac{4}{m^2}\left(\frac{\varphi-\psi}{\varphi+\psi}\right)^2\left|\frac{1}{1-e^{\frac{\pi i}{m}}} - \frac{1}{1-(1-\varphi-\psi)e^{\frac{\pi i}{m}}}\right|^2 \frac{m^2}{(V-v)^2}|1-e^{\frac{\pi i}{m}}|^2 \\
&= \frac{4}{(V-v)^2}\frac{(\varphi-\psi)^2}{|1-(1-\varphi-\psi)e^{\frac{\pi i}{m}}|^2} = \frac{4}{(V-v)^2}\frac{(\varphi-\psi)^2}{(\varphi+\psi)^2+4(1-\varphi-\psi)\sin^2(\frac{\pi}{2m})},
\end{aligned}
$$

from which (3.7) follows immediately. ■

Recall that the one-step probabilities P_1 and P_2 depend on the parameters $0 < p, q \le 1$ and, what is especially important, on $\varepsilon > 0$ which is interpreted as noise level. Our next goal is to *tune* the parameter ε to a value which maximizes the SPA coefficient as a function of ε.

3.3 Extrema and zeros of η^Z

In this section we study some features of the function $\eta^Z(\varepsilon, m)$ and its dependence on $m \in \mathbb{N}, 0 < v < V < \infty$ and the pre-factors $0 < p, q \le 1$. Especially, we are interested in the behaviour of the SPA coefficient for small values of ε, i.e. when the Markov chain approaches the jump rate between -1 and 1 of the diffusion.

The following theorem contains the main result on optimal tuning.

Theorem 3.3.1 *Let $m \ge 1$, $\varepsilon > 0$, $0 < v < V$, and $0 < p, q \le 1$. Denote $\beta = \frac{v}{V} < 1$, $a_m = \sin^2(\frac{\pi}{2m})$, and consider the function*

$$
p_-(q) = \frac{s(q,\beta,m) + \sqrt{s(q,\beta,m)^2 + 2a_m q\beta(2-q)u(q,\beta,m)}}{u(q,\beta,m)} \tag{3.8}
$$

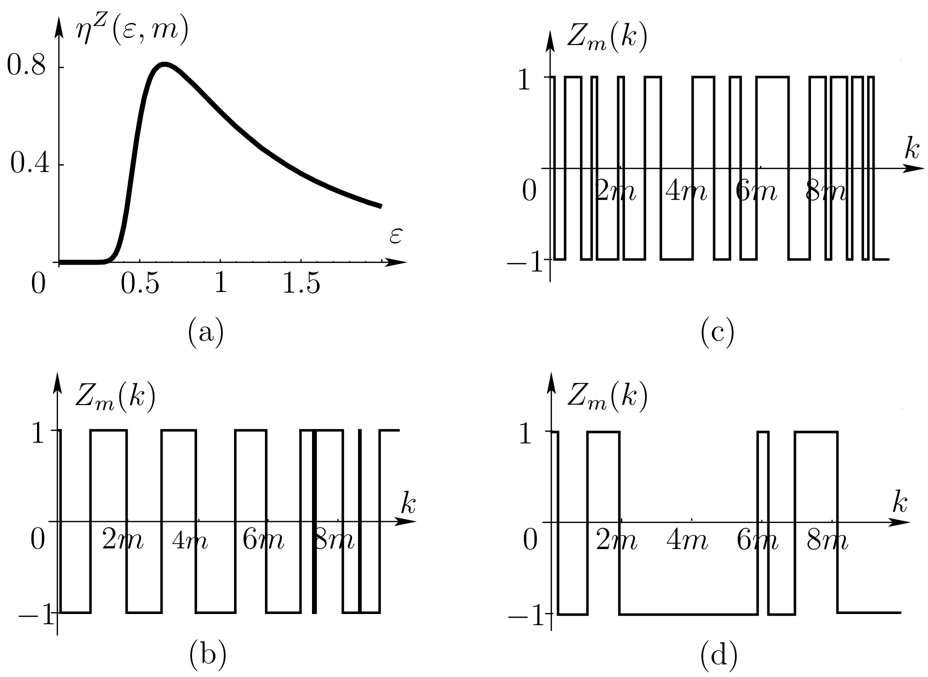

Fig. 3.1: The spectral power amplification coefficient $\eta^Z(\varepsilon, m)$ for $p = q = 0.5$, $m = 500$, $v = 2$, $V = 4$ (a). Numerical simulations of $Z_m(k)$ for different values of the 'noise' parameter $\varepsilon = 0.63$ (b), $\varepsilon = 0.9$ (c) and $\varepsilon = 0.4$ (d).

on the interval $q \in (0, 1]$, *where* $s(q, \beta, m) = a_m(3q(1 - \beta) - 2) - q^2(1 - \beta)$ *and* $u(q, \beta, m) = 2(q(1 - \beta) - a_m)$. *The function* $p_-(q)$ *is smooth and satisfies* $0 \le p_-(q) \le \beta q$, $q \in (0, 1]$, *and* $p'_-(0) = \beta$.

For any $\delta > 0$ *we consider the domain* $U^\delta = \{(p, q) : 0 < p \le 1, \delta \le q \le 1\}$, *which is a union of three disjoint domains (see Fig. 3.2)*

$$U_0^\delta = \{(p, q) : 0 < p < p_-(q), \delta \le q \le 1\},$$
$$U_1^\delta = \{(p, q) : p_-(q) \le p < q, \delta \le q \le 1\},$$
$$U_2^\delta = \{(p, q) : q \le p \le 1, \delta \le q \le 1\}.$$

For any δ *and* β *fixed there exists* $m(\beta, \delta) > 0$ *such that for* $m \ge m(\beta, \delta)$ *the following dependence of the SPA coefficient* $\eta^Z(\varepsilon, m)$ *on the pre-factors* p *and* q *holds:*

1. *If* $(p, q) \in U_0^\delta$ *then* $\varepsilon \mapsto \eta^Z(\varepsilon, m)$ *is strictly increasing function on* $(0, +\infty)$.
2. *If* $(p, q) \in U_1^\delta$ *then* $\varepsilon \mapsto \eta^Z(\varepsilon, m)$ *has a unique local maximum on* $(0, +\infty)$.
3. *If* $(p, q) \in U_2^\delta$ *then* $\varepsilon \mapsto \eta^Z(\varepsilon, m)$ *has a unique local maximum on* $(0, +\infty)$ *and a root in* $\widehat{\varepsilon} = (V - v)/\log\left(\frac{p}{q}\right)$.

Moreover, the optimal tuning rate which gives the maximum of $\eta^Z(\cdot, m)$ *satisfies*

$$m(\varepsilon) = \frac{\pi}{\sqrt{2pq}}\sqrt{\frac{v}{V-v}}e^{\frac{V+v}{2\varepsilon}}\left(1 + \mathcal{O}(e^{-\frac{\min\{v, V-v\}}{\varepsilon}})\right) \quad \textit{for} \quad \varepsilon \to 0$$

and

$$\eta^Z(\varepsilon, m(\varepsilon)) \to \frac{4}{\pi^2(V-v)^2} \quad \textit{as} \quad \varepsilon \to 0.$$

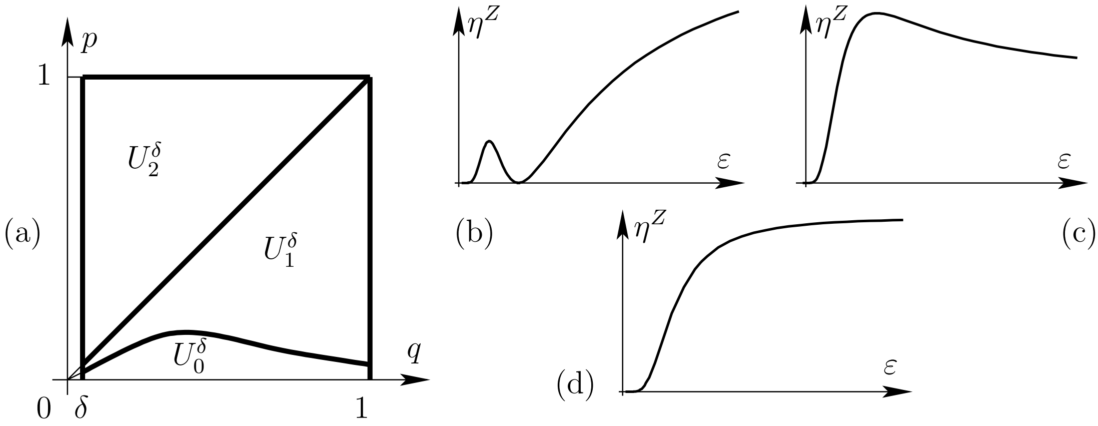

Fig. 3.2: Typical form of the domains U_0^δ, U_1^δ and U_2^δ (a). Typical form of $\eta^Z(\varepsilon, m)$ when (p, q) belongs to U_2^δ (b), U_1^δ (c) and U_0^δ (d).

Proof: A direct analysis of (3.7) leads to

$$\lim_{\varepsilon \to 0} \eta^Z(\varepsilon, m) = 0 \text{ for any } m > 0,$$

and $\eta^Z(\varepsilon, m) = 0$ if $\varphi = \psi$, i.e. for $\widehat{\varepsilon} = (V-v)/\log\left(\frac{p}{q}\right)$ which is positive for $p > q$.

Let us introduce the new variable $t \in (0, 1]$ given by

$$e^{-\frac{V}{\varepsilon}} = t \quad \text{and} \quad e^{-\frac{v}{\varepsilon}} = t^\beta.$$

Formula (3.7) takes now the form

$$\eta^Z(t, m) = \frac{4}{(V-v)^2}\frac{(pt - qt^\beta)^2}{(pt + qt^\beta)^2 + 4(1 - pt - qt^\beta)a_m}, \quad t \in (0, 1). \tag{3.9}$$

To study the extrema of the SPA coefficient we find zeros of the derivative of (3.9) in t. A straightforward calculation yields that the zeros of $\frac{d}{dt}\eta^Z(t, m)$ are zeros of the equation

$$\begin{aligned}(pt - qt^\beta)\big[pq(1-\beta)t(pt + qt^\beta) + a_m q^2\beta t^\beta + 2a_m pt^{1-\beta} \\ - a_m(2q\beta + p^2t^{2-\beta} + 3pq(1-\beta)t)\big] = 0.\end{aligned} \tag{3.10}$$

The first parenthesis gives the root of the derivative $t = (q/p)^{\frac{1}{1-\beta}}$ which is at the same time the root of (3.9).

Denote the function in square brackets in (3.10) as $F(t)$, $t \in (0, 1]$. It follows directly from (3.10) that $F(0) = -2a_m q\beta < 0$. Let us show that if $0 < \beta < 1$ and $\delta > 0$ are fixed, and $(p, q) \in U^\delta$ then $F(t)$ is strictly increasing on $(0, 1)$ for m bigger than some $m(\delta, \beta)$.

Consider the derivative

$$\begin{aligned}
&F'(t) = f^+(t) - f^-(t), \text{ where}\\
&f^+(t) = pq^2(1-\beta^2)t^\beta + 2p^2q(1-\beta)t + a_m q^2\beta^2 t^{\beta-1} + 2a_m p(1-\beta)t^{-\beta},\\
&f^-(t) = a_m p^2(2-\beta)t^{1-\beta} + 3a_m pq(1-\beta).
\end{aligned}$$

The functions f^+ and f^- are positive on $(0, 1)$. For $0 < t \le \hat{t} = \left(\frac{2}{5}(1-\beta)\right)^{1/\beta}$ the chain of inequalities

$$f^+(t) > 2a_m p(1-\beta)t^{-\beta} \ge 5a_m p > f^-(t)$$

holds, and therefore $F'(t) > 0$ on the interval $(0, \hat{t}]$.

Consider $F'(t)$ on $t \in [\hat{t}, 1]$. Let $(p, q) \in U^\delta$, i.e. $q \ge \delta$. Then

$$f^+(t) > pq^2(1-\beta^2)t^\beta + 2p^2q(1-\beta)t \ge p\delta^2(1-\beta^2)\hat{t}^\beta + 2p^2\delta(1-\beta)\hat{t},$$

and on the other hand

$$f^-(t) < 3a_m\delta p + 2a_m p^2.$$

Comparing terms with p and p^2 in the right-hand sides of the two previous formulae and using the inequality $\sin x \ge \frac{x}{2}$ for small positive x yields that $F'(t) > 0$ if

$$m \ge m(\beta, \delta) = \max\{\frac{\pi}{4\sqrt{\delta(1-\beta)\hat{t}}}, \frac{\pi}{2\sqrt{2\delta(1-\beta^2)\hat{t}^\beta}}\}.$$

Finally, $F(t)$ is an increasing function on the interval $(0, 1]$ for $(p, q) \in U^\delta$ and $m \ge m(\beta, \delta)$. It has a unique root on the interval if and only if $F(1) \ge 0$; this occurs if

$$p^2\big(q(1-\beta) - a_m\big) - p\big(a_m(3q(1-\beta) - 2) - q^2(1-\beta)\big) + a_m q\beta(q-2) \ge 0.$$

Solving this inequality with respect to p results in (3.8). The value of the derivative of $p_-(q)$ at zero and the inequality $0 \le p_-(q) \le \beta q$ follow from straightforward calculation.

Let us study the behaviour of the local maximum of $\varepsilon \mapsto \eta^Z(\varepsilon, m)$ for $m \to \infty$. The equation $F(t) = 0$ can be rewritten as

$$\sin\frac{\pi}{2m} = \left(\frac{pq^2(1-\beta)t^{1+\beta} + p^2q(1-\beta)t^2}{2q\beta + p^2t^{2-\beta} + 3pq(1-\beta)t - q^2\beta t^\beta - 2pt^{1+\beta}}\right)^{\frac{1}{2}}.$$

Extract the leading term $t^{1+\beta}$ from the right-hand side of the preceding formula an recall $t = e^{-\frac{V}{\varepsilon}}$. Then one obtains

$$m(\varepsilon) = \frac{\pi}{\sqrt{2pq}} \sqrt{\frac{v}{V-v}} e^{\frac{V+v}{2\varepsilon}} \left(1 + \mathcal{O}(e^{-\frac{\min\{v, V-v\}}{\varepsilon}}\right) \quad \text{for} \quad \varepsilon \to 0. \tag{3.11}$$

This is a formula for the 'optimal tuning' for the SPA coefficient as a measure of goodness for stochastic resonance. It expresses that in order to obtain the maximum of $\eta^Z(\varepsilon, m)$ at ε, the half period m must be given by (3.11).

The limiting value of $\eta^Z(\varepsilon, m)$ is found by inserting (3.11) into (3.7). ■

Chapter 4

Stochastic Resonance in Two-State Markov Chains with Continuous Time. Noise-Induced Amplification and Stabilization.

In this chapter we study the continuous-time version of the reduction of diffusions as in (2.2) to two-state motions on the metastable states. The continuous-time Markov chains obtained this way are in a sense close to the corresponding diffusions. Instead of one-step transition probabilities we define time-periodic infinitesimal generator and determine the invariant measure of the corresponding random process. Continuous time allows us to consider the process for all positive noise levels ε, half-periods T, and parameters p and q.

In this setting we study in detail several measures of quality of tuning. They can be subdivided into two groups: measures of *amplification* and measures of *stabilization*. Measures such as the SPA coefficient, the SPA-to-noise ratio, the energy, and the energy-to-noise ratio describe noise-induced amplification of a deterministic periodic signal and have a characteristic local maximum at a critical noise level which determines the respective resonance point. The out-of-phase measure, the relative entropy and the entropy of the invariant measure are referred to as measures of the noise-induced stabilization and have a characteristic local minimum in ε.

SPA coefficient is well-known in the physics literature [26]. It will also be investigated as a measure of goodness for diffusions in a periodically changing potential landscape in Chapter 6. The out-of-phase measure is an averaged version of the quality measure from [22]. Versions of the entropy measures were studied in the physical papers [48, 24, 2]. For all measures of goodness we derive explicit formulae which allow us to determine the 'optimal tuning' rate, i.e. the noise level ε which gives the corresponding local maximum/minimum.

In comparison with Chapter 3, where we consider Markov chains with discrete time, the continuous-time results of this chapter take a simpler form.

4.1 The Markov chain and its invariant measure

Consider a family of Markov chains $Y^{\varepsilon,T} = (Y_t^{\varepsilon,T})_{t\geq 0}$ on the state space $\mathcal{S}^Y = \{-1, 1\}$. The variable $t \in \mathbb{R}_+$ denotes time, and $0 < \varepsilon, T < \infty$ parametrize the family in the following way. Let us introduce the transition probabilities

$$p_{ij}(s,t;\varepsilon,T) = \mathbf{P}(Y_t^{\varepsilon,T} = j | Y_s^{\varepsilon,T} = i), \quad i,j = \pm 1, \quad 0 \leq s \leq t. \tag{4.1}$$

To do this, similarly to Chapter 3 we define time-periodic infinitesimal probabilities of $Y^{\varepsilon,T}$, which are the continuous-time analogue of the one-step transition probabilities. In other words, the process $Y^{\varepsilon,T}$ is temporarily inhomogeneous and its infinitesimal generator $Q_{\varepsilon,T}(t)$ is temporarily periodic with period $2T$ and equals

$$Q_{\varepsilon,T}(2Tt) = \begin{cases} Q_1 = \begin{pmatrix} -\varphi & \varphi \\ \psi & -\psi \end{pmatrix}, & 0 \leq t \pmod 1 < \frac{1}{2}, \\ Q_2 = \begin{pmatrix} -\psi & \psi \\ \varphi & -\varphi \end{pmatrix}, & \frac{1}{2} \leq t \pmod 1 < 1, \end{cases} \tag{4.2}$$

where we assume as in Chapter 3 that

$$\begin{gathered} \varphi = pe^{-V/\varepsilon}, \qquad \psi = qe^{-v/\varepsilon}, \\ \varepsilon > 0, \qquad p,q > 0, \qquad 0 < v < V. \end{gathered} \tag{4.3}$$

Note that we have weakened the conditions on p and q used in Chapter 3.

On the intervals $[kT, (k+1)T)$, $k \geq 0$, the process $Y^{\varepsilon,T}$ is time-homogeneous and its transition probabilities (4.1) can be expressed in terms of φ and ψ:

$$\begin{aligned} p_{-1,1}(t, t+h; \varepsilon, T) &= \varphi h + o(h), \\ p_{1,-1}(t, t+h; \varepsilon, T) &= \psi h + o(h), \quad 0 \leq t \pmod{2T} < T, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} p_{-1,1}(t, t+h; \varepsilon, T) &= \psi h + o(h), \\ p_{1,-1}(t, t+h; \varepsilon, T) &= \varphi h + o(h), \quad T \leq t \pmod{2T} < 2T, \end{aligned} \tag{4.5}$$

The goal of this chapter is to study the 'periodic' properties of $Y^{\varepsilon,T}$ induced by the periodicity of the infinitesimal probabilities.

To determine the invariant law of the process $Y^{\varepsilon,T}$ we proceed as in Chapter 3 and consider a new two-dimensional Markov process

$$\mathbf{Y}_t^{\varepsilon,T} = \left(Y_{2Tt}^{\varepsilon,T}, \frac{t}{2T} \pmod 1 \right), \quad t \geq 0,$$

on the state space $\mathcal{S}^{\mathbf{Y}} = \{-1,1\} \times S^1$, which topologically is the product of two circles.

The process $\mathbf{Y}^{\varepsilon,T}$ is time-homogeneous. Note that we have compressed time: it is convenient to have the time scale independent of the parameter T. The infinitesimal generator of $\mathbf{Y}^{\varepsilon,T}$ is

$$B_{\varepsilon,T} f(x,\theta) = \lim_{h\to 0} \frac{\mathbf{E}_{x,\theta} f\left(Y^{\varepsilon,T}_{2T(\theta+h)}, \frac{\theta+h}{2T} \pmod 1\right) - f(x,\theta)}{h}, \quad (x,\theta) \in \mathcal{S}^{\mathbf{Y}}. \tag{4.6}$$

As $x = \pm 1$ we may think that $B_{\varepsilon,T}$ is defined on the space of vectors $f(\theta) = (f^-(\theta), f^+(\theta))^*$ with smooth components. Using (4.4) and (4.5) gives

$$B_{\varepsilon,T} f = \frac{1}{2T}\frac{d}{d\theta} f + Q_{\varepsilon,T}(\theta) f.$$

Let the vector $\nu_{\varepsilon,T} = (\nu^-_{\varepsilon,T}(\theta), \nu^+_{\varepsilon,T}(\theta))^*$, $\theta \in [0,1]$, denote the invariant density of $\mathbf{Y}^{\varepsilon,T}$ w.r.t. the product of counting measure on $\{-1,1\}$ and Lebesgue measure on a circle S^1 normalized so that $\nu^-_{\varepsilon,T}(\theta) + \nu^+_{\varepsilon,T}(\theta) = 1$. We shall call $\nu_{\varepsilon,T}$ the *invariant law* of the process $Y^{\varepsilon,T}$. Indeed, for $\theta \in [0,1]$

$$\mathbf{P}_\nu(Y^{\varepsilon,T}_{2T\theta} = \pm 1) = \nu^{\pm}_{\varepsilon,T}(\theta).$$

The invariant measure satisfies the forward Kolmogorov equation

$$B^*_{\varepsilon,T}\nu_{\varepsilon,T} = 0,$$

and the continuity condition $\nu_{\varepsilon,T}(0) = \nu_{\varepsilon,T}(1)$, where the adjoint operator is given by

$$B^*_{\varepsilon,T} f = -\frac{1}{2T}\frac{d}{d\theta} f + Q^*_{\varepsilon,T}(\theta) f.$$

From the symmetry between Q_1 and Q_2 in (4.2) we deduce the following

Proposition 4.1.1 *The invariant measure of the process $\mathbf{Y}^{\varepsilon,T}$ has the following symmetry property: $\nu^{\pm}_{\varepsilon,T}(\theta) = \nu^{\mp}_{\varepsilon,T}(\theta + \frac{1}{2})$, $0 \le \theta \le \frac{1}{2}$.*

Proof: The statement follows easily from the fact that if, for example, $\nu = (\nu^-, \nu^+)^*$ is a solution of $-\frac{1}{2T}\dot{\nu} + Q_1^*\nu = 0$, then $\overline{\nu} = (\nu^+, \nu^-)^*$ satisfies $-\frac{1}{2T}\dot{\overline{\nu}} + Q_2^*\overline{\nu} = 0$, together with the continuity condition and the uniqueness of the invariant measure. Here, $\dot{\nu} = \frac{d}{d\theta}\nu$. ■

It follows from Proposition 4.1.1 that in order to find the invariant measure, it is enough to solve the boundary value problem

$$\begin{cases} -\dfrac{1}{2T}\dfrac{d}{d\theta}\nu_{\varepsilon,T} + Q_1^*\nu_{\varepsilon,T} = 0, \\ \nu^-_{\varepsilon,T}(0) = \nu^+_{\varepsilon,T}(\frac{1}{2}), \\ \nu^-_{\varepsilon,T}(\theta) + \nu^+_{\varepsilon,T}(\theta) = 1, \\ \nu^{\pm}_{\varepsilon,T}(\theta) > 0, \quad \theta \in [0, \frac{1}{2}], \end{cases} \tag{4.7}$$

which is done in the following

Proposition 4.1.2 *For $T > 0$ and φ and ψ defined in* (4.3)*, the invariant measure of the process $Y^{\varepsilon,T}$ equals*

$$\begin{cases} \nu^-_{\varepsilon,T}(\theta) = \dfrac{\psi}{\varphi+\psi} + \dfrac{\varphi-\psi}{\varphi+\psi}\dfrac{e^{-2(\varphi+\psi)T\theta}}{1+e^{-(\varphi+\psi)T}}, \\ \nu^+_{\varepsilon,T}(\theta) = \dfrac{\varphi}{\varphi+\psi} - \dfrac{\varphi-\psi}{\varphi+\psi}\dfrac{e^{-2(\varphi+\psi)T\theta}}{1+e^{-(\varphi+\psi)T}}; \end{cases}$$
$$\begin{cases} \nu^-_{\varepsilon,T}(\theta+\frac{1}{2}) = \nu^+_{\varepsilon,T}(\theta), \\ \nu^+_{\varepsilon,T}(\theta+\frac{1}{2}) = \nu^-_{\varepsilon,T}(\theta), \quad 0 \le \theta \le \frac{1}{2}. \end{cases} \tag{4.8}$$

Proof: Solving the differential equation in (4.7) and using the normalizing condition gives the following general solution

$$\nu^-_{\varepsilon,T}(\theta) = \frac{\psi}{\varphi+\psi} + Ae^{-2(\varphi+\psi)T\theta}, \quad \nu^+_{\varepsilon,T}(\theta) = \frac{\varphi}{\varphi+\psi} - Ae^{-2(\varphi+\psi)T\theta}$$

where A is an arbitrary constant. Applying the boundary condition leads to

$$A = \frac{\varphi-\psi}{\varphi+\psi}\frac{1}{1+e^{-(\varphi+\psi)T}},$$

which concludes the proof. ■

Note that $\nu^\pm_{\varepsilon,T}(\theta)$ is a sum of two parts. For $\theta \in [0, \frac{1}{2}]$, the time-independent pair $(\frac{\psi}{\varphi+\psi}, \frac{\varphi}{\varphi+\psi})$ is the invariant measure of the time-homogeneous Markov chain with infinitesimal generator Q_1, and $(\nu^-_{\varepsilon,T}(\frac{1}{2}), \nu^+_{\varepsilon,T}(\frac{1}{2})) \to (\frac{\psi}{\varphi+\psi}, \frac{\varphi}{\varphi+\psi})$ exponentially fast with rate $\varphi+\psi$ as $T \to \infty$. This is an illustration of a classical result about convergence of the law of a stochastic process to its invariant law.

On the second half-period, the Markov chain is governed by the infinitesimal generator Q_2 and therefore $(\nu^-_{\varepsilon,T}(1), \nu^+_{\varepsilon,T}(1)) \to (\frac{\varphi}{\varphi+\psi}, \frac{\psi}{\varphi+\psi})$, $T \to \infty$, which is the invariant law of the Markov process with generator Q_2.

The invariant measure found in Proposition 4.1.2 is our main tool in studying the periodic properties of the process $Y^{\varepsilon,T}$. In the following sections we shall introduce, study and compare several measures of goodness of stochastic resonance.

4.2 Spectral power amplification coefficient

As in Section 3.2 we define the *spectral power amplification* coefficient as the ratio between the component of period $2T$ of the power carried by the averaged trajectory of $Y^{\varepsilon,T}$ and the corresponding component of the power carried by the deterministic signal

$$I_T(t) = \begin{cases} V, & 0 \le t \pmod{2T} < T, \\ v, & T \le t \pmod{2T} < 2T. \end{cases}$$

The signal I_T 'switches' the powers in the infinitesimal probabilities φ and ψ.

Precisely, we define the SPA coefficient as

$$\eta^Y(\varepsilon,T) = \left| \frac{\int_0^1 \mathbf{E}_\nu Y^{\varepsilon,T}_{2Ts} e^{2\pi i s}\, ds}{\int_0^1 I_T(2Ts) e^{2\pi i s}\, ds} \right|^2 = \frac{\pi^2}{(V-v)^2} \left| \int_0^1 \mathbf{E}_\nu Y^{\varepsilon,T}_{2Ts} e^{2\pi i s}\, ds \right|^2, \tag{4.9}$$

where $\mathbf{E}_\nu$ denotes the expectation w.r.t. the invariant measure $\nu_{\varepsilon,T}$.

The goal of this section is to study the behaviour of the SPA coefficient as a function of ε and T.

Proposition 4.2.1 *The SPA coefficient is given by*

$$\eta^Y(\varepsilon,T) = \frac{4}{(V-v)^2} \frac{T^2(\varphi-\psi)^2}{(\varphi+\psi)^2 T^2 + \pi^2}, \tag{4.10}$$

with φ and ψ defined in (4.3).

Proof: We use (4.8) to obtain

$$\int_0^1 \mathbf{E}_\nu Y^{\varepsilon,T}_{2Ts} e^{2\pi i s}\, ds = \int_0^1 (\nu^+_{\varepsilon,T}(s) - \nu^-_{\varepsilon,T}(s)) e^{2\pi i s}\, ds$$
$$= 2\int_0^{\frac{1}{2}} (\nu^+_{\varepsilon,T}(s) - \nu^-_{\varepsilon,T}(s)) e^{2\pi i s}\, ds = 2\frac{\varphi-\psi}{\varphi+\psi}\left(\frac{i}{\pi} + \frac{1}{\pi i - (\varphi+\psi)T}\right)$$

which directly leads to (4.10). ∎

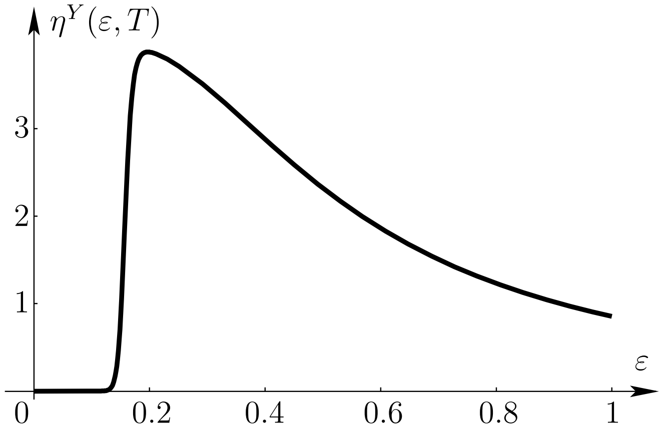

Fig. 4.1: If $p = q = 1$, $V = 3$, $v = 2$ and $T = 10^6$ the SPA coefficient $\eta^Y(\varepsilon,T)$ has a local maximum at $\varepsilon \approx 0.197$.

Let us compare formula (4.10) with its discrete counterpart (3.7). If we set $m = T$ the formulae differ only in the term $\kappa(\varepsilon,m) = 4(1-\varphi-\psi)m^2 \sin^2(\frac{\pi}{2m}) - \pi^2$. This term reflects the difference between discrete and continuous time and vanishes in the limit $T \to \infty$ and $\varepsilon \to 0$. This suggests that the optimal tuning for $Y^{\varepsilon,T}$ should be close to the tuning of Z_m, at least in the small noise limit.

Proposition 4.2.2 *a)* $\eta^Y(\varepsilon, T) \geq 0$, $\eta^Y(0,T) = 0$;

b) Let $0 < \beta = \frac{v}{V} < 1$ *and* $T > 0$ *be fixed. Consider the function*

$$\widehat{p}(q) = \frac{2\pi^2 q\beta}{2q^2T^2(1-\beta) + \pi^2 + \sqrt{(2q^2T^2(1-\beta)+\pi^2)^2 + 8\pi^2q^2T^2\beta(1-\beta)}}, \quad (4.11)$$

such that $0 \leq \widehat{p}(q) \leq \beta q$, *for* $q > 0$, *and* $\widehat{p}'(0) = \beta$.

Let us consider three disjoint domains (see Fig. 4.2)

$$\begin{aligned} U_0 =& \{(p,q) : 0 < p \leq \widehat{p}(q), q > 0\}, \\ U_1 =& \{(p,q) : \widehat{p}(q) < p \leq q, q > 0\}, \\ U_2 =& \{(p,q) : 0 < q < p\}. \end{aligned}$$

Then the following dependence of the SPA coefficient $\eta^Y(\varepsilon, T)$ *on the pre-factors*

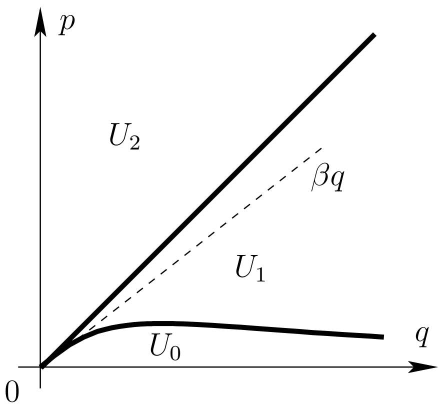

Fig. 4.2: Typical form of the domains U_0, U_1 and U_2.

p and q holds:

1. *if* $(p,q) \in U_0$ *then* $\varepsilon \mapsto \eta^Y(\varepsilon, T)$ *is a strictly increasing function on* $(0, +\infty)$;
2. *if* $(p,q) \in U_1$ *then* $\varepsilon \mapsto \eta^Y(\varepsilon, T)$ *has a unique local maximum on* $(0, +\infty)$;
3. *if* $(p,q) \in U_2$ *then* $\varepsilon \mapsto \eta^Y(\varepsilon, T)$ *has a unique local maximum and vanishes at* $\widehat{\varepsilon} = (V - v)/\log\left(\frac{p}{q}\right)$.

Moreover, for $T \to \infty$ *the domain* $U_0 \to \emptyset$, *and the 'optimal tuning'* $T_\eta = T_\eta(\varepsilon)$ *which gives the maximum of* $\eta^Y(\varepsilon, T)$ *satisfies*

$$T_\eta(\varepsilon) = \frac{\pi}{\sqrt{2pq}}\sqrt{\frac{v}{V-v}}e^{\frac{V+v}{2\varepsilon}}\left\{1 + \mathcal{O}(e^{-\frac{V-v}{\varepsilon}})\right\}$$

and

$$\eta^Y(\varepsilon, T_\eta(\varepsilon)) \to \frac{4}{\pi^2(V-v)^2} \quad as \quad \varepsilon \to 0. \quad (4.12)$$

Proof: Statement a) follows directly from (4.10).

Let us introduce the variable $t = \exp\left(-\frac{V}{\varepsilon}\right) \in [0,1]$. Then $\varphi = pt$ and $\psi = qt^\beta$, and

$$\eta^Y(\varepsilon, T) = \eta_T^Y(t) = \frac{4}{(V-v)^2} \frac{T^2(pt - qt^\beta)^2}{(pt + qt^\beta)^2 T^2 + \pi^2}.$$

Taking the derivative of $\eta_T^Y(t)$ in t we find that the extrema of $\eta_T^Y(t)$ are the roots of the equation

$$t^\beta(pt - qt^\beta)\left\{\pi^2(q\beta - pt^{1-\beta}) - 2pqT^2(1-\beta)t(qt^\beta + pt)\right\} = 0. \tag{4.13}$$

The first parenthesis of (4.13) gives the root $t = (q/p)^{\frac{1}{1-\beta}}$ which does not depend on T, is less than 1 if $p > q$, and corresponds to $\widehat{\varepsilon} = (V - v)/\log\left(\frac{p}{q}\right)$.

Let us show that the function in the second parenthesis of (4.13) has exactly one root if $(p,q) \in U_1 \cup U_2$, and has no root if $(p,q) \in U_0$. Denote the second parenthesis of (4.13) by $F(t)$. Note that $F(0) = \pi^2 q\beta > 0$,

$$F'(t) = -\pi^2 p(1-\beta)t^{-\beta} - 2pq^2T^2(1-\beta^2)t^\beta - 4p^2qT^2(1-\beta)t \le 0,$$

and consequently, $F(t)$ is monotonically decreasing on $(0,1]$. Hence, F has a unique root on $t \in (0,1]$ if and only if $F(1) \le 0$. The latter inequality can be rewritten as the following quadratic inequality w.r.t. p

$$2p^2qT^2(1-\beta) + (2q^2T^2(1-\beta) + \pi^2)p - q\beta\pi^2 \ge 0. \tag{4.14}$$

Solving this inequality results in (4.11). The properties of $\widehat{p}(q)$ follow from a straightforward calculation.

To find an 'optimal tuning' $T = T(\varepsilon)$ we note that $F(t) = 0$ is a linear equation in T^2 from which we get that

$$T_\eta = \frac{\pi}{\sqrt{2pq}} \sqrt{\frac{\beta}{1-\beta}}\, t^{-\frac{1+\beta}{2}} \left(\frac{1 - \frac{p}{q}\frac{1}{\beta}t^{1-\beta}}{1 + \frac{p}{q}t^{1-\beta}}\right)^{\frac{1}{2}}. \tag{4.15}$$

It is clear that as $T \to \infty$ the noise parameter $\varepsilon \to 0$ and (4.15) can be rewritten as

$$T_\eta = \frac{\pi}{\sqrt{2pq}} \sqrt{\frac{v}{V-v}}\, e^{\frac{V+v}{2\varepsilon}} \left\{1 + \mathcal{O}(e^{-\frac{V-v}{\varepsilon}})\right\} \tag{4.16}$$

Note that for large T the solution exists *always*, for any v, V, p, q, hence $U_0 \to \emptyset$ as $\varepsilon \to 0$. The limit (4.12) is obtained by inserting (4.16) into (4.10). ■

4.3 SPA-to-noise ratio

In this section we study another measure of quality of tuning, the SPA-to-noise ratio (SPN), which can be defined as the ratio of the 'amplitude' of the output signal and noise 'amplitude'. This notion is related to another popular measure of stochastic resonance, the so-called signal-to-noise ratio [21, 26, 45].

Formally, this measure of goodness is given by

$$\mathrm{SPN}(\varepsilon, T) = \frac{1}{\varepsilon^2} \left| \int_0^1 \mathbf{E}_\nu Y_{2Ts}^{\varepsilon,T} e^{2\pi i s}\, ds \right|^2 = \frac{(V-v)^2}{\pi^2} \frac{\eta^Y(\varepsilon, T)}{\varepsilon^2}. \tag{4.17}$$

The SPA-to-noise ratio compares the spectral component of the random output with the noise intensity and measures the proportion of the noise energy transferred to the component of the output corresponding to period $2T$.

Proposition 4.3.1 *a) The SPA-to-noise ratio is given by*

$$\mathrm{SPN}(\varepsilon, T) = \frac{4}{\pi^2 \varepsilon^2} \frac{T^2(\varphi - \psi)^2}{(\varphi + \psi)^2 T^2 + \pi^2}, \tag{4.18}$$

with φ and ψ defined in (4.3).

b) If $p > q$, $\mathrm{SPN}(\cdot, T)$ *vanishes in* $\widehat{\varepsilon} = (V - v)/\log\left(\frac{p}{q}\right)$. *For T large enough $\varepsilon \mapsto \mathrm{SPN}(\varepsilon, T)$ has a local maximum, and the optimal tuning occurs for*

$$T_{\mathrm{SPN}}(\varepsilon) = \frac{\pi\sqrt{v}}{q\sqrt{\varepsilon}} e^{\frac{v}{\varepsilon}} (1 + \mathcal{O}(\varepsilon)). \tag{4.19}$$

Moreover,

$$\mathrm{SPN}(\varepsilon, T_{\mathrm{SPN}}(\varepsilon)) = \frac{4}{\pi^2 \varepsilon^2}(1 + \mathcal{O}(\varepsilon)), \quad \text{as} \quad \varepsilon \to 0. \tag{4.20}$$

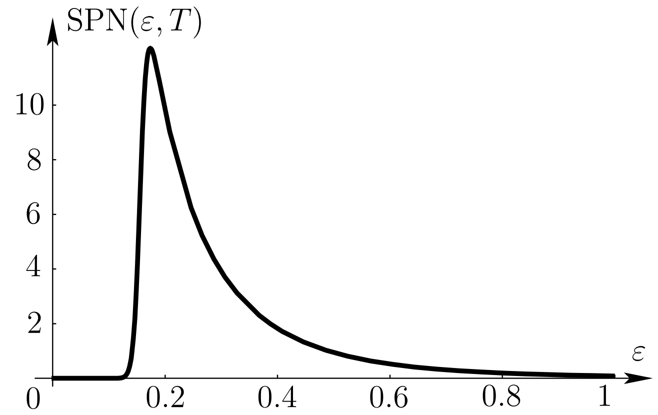

Fig. 4.3: If $p = q = 1$, $V = 3$, $v = 2$ and $T = 10^6$ the SPA-to-noise ratio $\mathrm{SPN}(\varepsilon, T)$ has a local maximum at $\varepsilon \approx 0.173$.

Proof: The formula (4.18) is an obvious consequence of (4.10) and (4.17).

To study the extrema of the $\mathrm{SPN}(\varepsilon, T)$ with respect to ε let us intoduce the variable $t = e^{-V/\varepsilon}$ taking values in $(0, 1]$. Then, $\varphi = pt$ and $\psi = qt^\beta$, where $\beta = \frac{v}{V} < 1$. In terms of t the formula (4.18) takes the form

$$\mathrm{SPN}(t, T) = \frac{4}{\pi^2 V^2} \frac{T^2 (pt - qt^\beta)^2 \log^2 t}{(pt + qt^\beta)^2 T^2 + \pi^2}, \tag{4.21}$$

and taking the derivative in t we find that the extrema of (4.21) are zeros of the following equation:

$$\begin{aligned}(pt - qt^\beta) \Big[&\pi^2 (q\beta t^\beta - pt) |\log t| - \pi^2 (qt^\beta - pt) \\ &- 2pq(1-\beta) T^2 t^{1+\beta} (qt^\beta + pt) |\log t| - T^2 (qt^\beta - pt)(qt^\beta + pt)^2 \Big] = 0.\end{aligned}$$

The expression in the first parenthesis gives the root $\widehat{\varepsilon} = (V - v)/\log\left(\frac{p}{q}\right)$ which is non-negative for $p > q$ and is also a root of $\mathrm{SPN}(\varepsilon, T)$.

The equation in brackets can be solved for T. We get

$$\begin{aligned} T &= \left(\frac{\pi^2 (q\beta t^\beta - pt)|\log t| - \pi^2 (qt^\beta - pt)}{(qt^\beta - pt)(qt^\beta + pt)^2 + 2pq(1-\beta) t^{1+\beta} (qt^\beta + pt) |\log t|} \right)^{\frac{1}{2}} \\ &= \frac{\pi}{q} t^{-\beta} \sqrt{\beta |\log t|} \left(\frac{1 - \frac{p}{q\beta} t^{1-\beta} - \frac{1}{\beta |\log t|} + \frac{p}{q\beta |\log t|} t^{1-\beta}}{(1 - \frac{p}{q} t^{1-\beta})(1 + \frac{p}{q} t^{1-\beta})^2 + 2\frac{p}{q}(1-\beta) t^{1-\beta} (1 + \frac{p}{q} t^{1-\beta}) |\log t|} \right)^{\frac{1}{2}}. \end{aligned}$$

As $t \to 0$, the latter expression can be expanded as

$$T = \frac{\pi}{q} t^{-\beta} \sqrt{\beta |\log t|} \left(1 + \mathcal{O}(|\log t|^{-1}) \right),$$

which yields (4.19). The limiting value (4.20) is obtained by inserting (4.19) into (4.18). ■

As we see, the SPA-to-noise ratio has a local maximum in ε. This means that on the optimal noise level the noise energy feeds the periodic component of the averaged random output.

4.4 Energy

The SPA coefficient η^Y discussed in Section 4.2 describes the averaged power carried by the longest harmonic in the Fourier decomposition of $Y^{\varepsilon,T}$. Let us consider the *energy* carried by all harmonics, which, due to Parseval's equality, is given by

$$\mathrm{En}(\varepsilon, T) = \int_0^1 (\mathbf{E}_\nu Y^{\varepsilon,T}_{2Ts})^2 \, ds.$$

Proposition 4.4.1 *a) The energy carried by the averaged Markov chain $Y^{\varepsilon,T}$ equals*

$$\mathrm{En}(\varepsilon,T) = \left(\frac{\varphi-\psi}{\varphi+\psi}\right)^2 \left[1 - \frac{2}{T(\varphi+\psi)} \tanh \frac{T(\varphi+\psi)}{2}\right]. \tag{4.22}$$

b) If $p > q$, then $\mathrm{En}(\varepsilon,T)$ has a root in $\widehat{\varepsilon} = (V-v)/\log\left(\frac{p}{q}\right)$. For T large enough, $\varepsilon \mapsto \mathrm{En}(\varepsilon,T)$ has a local maximum. An optimal tuning rate at which the maximum is attained is given by

$$T_{\mathrm{En}}(\varepsilon) = \frac{1}{2p}\frac{v}{V-v} e^{\frac{V}{\varepsilon}}\{1 + \mathcal{O}(e^{-\frac{V-v}{\varepsilon}})\} \tag{4.23}$$

and

$$\mathrm{En}(\varepsilon, T_{\mathrm{En}}(\varepsilon)) \to 1 \quad \textit{as} \quad \varepsilon \to 0. \tag{4.24}$$

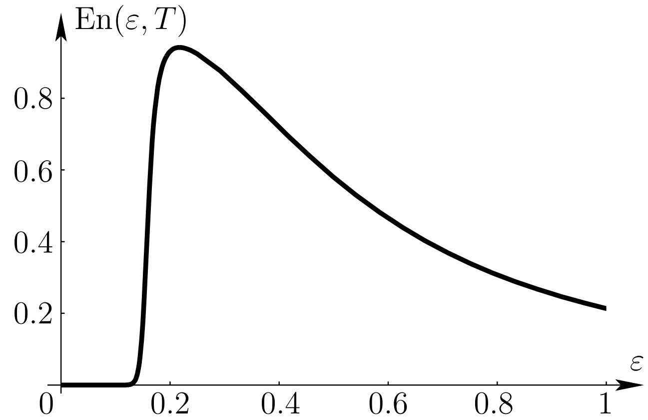

Fig. 4.4: If $p = q = 1$, $V = 3$, $v = 2$ and $T = 10^6$ the energy $\mathrm{En}(\varepsilon,T)$ has a local maximum at $\varepsilon \approx 0.217$.

Proof: The proof is analogous to the proof of Propositions 4.2.1 and 4.3.1. To obtain (4.22) we note that

$$\begin{aligned}
\mathrm{En}(\varepsilon,T) &= \int_0^1 (\mathbf{E}_\nu Y^{\varepsilon,T}_{2Ts})^2\, ds = \int_0^1 (\nu^+_{\varepsilon,T}(s) - \nu^-_{\varepsilon,T}(s))^2\, ds \\
&= 2\int_0^{\frac{1}{2}} (\nu^+_{\varepsilon,T}(s) - \nu^-_{\varepsilon,T}(s))^2\, ds \\
&= 2\left(\frac{\varphi-\psi}{\varphi+\psi}\right)^2 \int_0^{\frac{1}{2}} \left(1 - \frac{2e^{-2(\varphi+\psi)Ts}}{1+e^{-(\varphi+\psi)T}}\right)^2 ds \\
&= \left(\frac{\varphi-\psi}{\varphi+\psi}\right)^2 \left[1 - \frac{2}{T(\varphi+\psi)} \tanh\frac{T(\varphi+\psi)}{2}\right].
\end{aligned}$$

To find extrema of $\mathrm{En}(\varepsilon, T)$ we denote $\beta = \frac{v}{V}$ and $t = e^{-\frac{V}{\varepsilon}} \in (0, 1]$ and take the derivative of (4.22). A straightforward calculation yields the equation $(qt^\beta - pt)F_T(t) = 0$ for the extrema, where

$$\begin{aligned} F_T(t) &= T(p^2t^2 - q^2t^{2\beta})(pt + q\beta t^\beta)\operatorname{sech}^2\left[\tfrac{T}{2}(pt + qt^\beta)\right] \\ &\quad - 4Tpq(1-\beta)t^{1+\beta}(qt^\beta + pt) \\ &\quad - 2(p^2t^2 - q^2\beta t^{2\beta} + 5pq(1-\beta)t^{1+\beta})\tanh\left[\tfrac{T}{2}(pt + qt^\beta)\right]. \end{aligned} \tag{4.25}$$

The first parenthesis of the equation of the extrema gives the root $\widehat{\varepsilon} = (V - v)/\log\left(\frac{p}{q}\right)$. Next, we parametrize $T_t = \frac{a}{t}$, $a > 0$, and consider $F_{T_t}(t)$ as $t \to 0$. It is clear that $\operatorname{sech}\left[\frac{T_t}{2}(qt^\beta + pt)\right] = \operatorname{sech}\left[\frac{a}{2}(qt^{\beta-1} + p)\right] \to 0$ exponentially fast, whereas $\tanh\left[\frac{T_t}{2}(qt^\beta + pt)\right] \to 1$ exponentially fast as $t \to 0$. This yields

$$F_{T_t}(t) = 2q^2t^{2\beta}\left(\beta - 2ap(1-\beta) + \mathcal{O}(t^{1-\beta})\right).$$

For $a < \frac{\beta}{2p(1-\beta)}$ we have $F_{T_t}(t) > 0$, and for $a > \frac{\beta}{2p(1-\beta)}$ we have $F_{T_t}(t) < 0$ as $t \to 0$, and we can localize the optimal value of T in the interval $\left[\frac{\beta}{3p(1-\beta)t}, \frac{\beta}{p(1-\beta)t}\right]$. Therefore, in this interval we can asymptotically solve the equation for T with the result:

$$T = \frac{1}{2p}\frac{\beta}{1-\beta}t^{-1}(1 + \mathcal{O}(t^{1-\beta})).$$

This gives (4.23). The limit (4.24) is obtained with the help of (4.22) and (4.23). ■

4.5 Energy-to-noise ratio

As a similar notion to the SPA-to-noise ratio let us next consider the *energy-to-noise ratio* which is defined by

$$\mathrm{ENR}(\varepsilon, T) = \frac{1}{\varepsilon^2}\int_0^1 (\mathbf{E}_\nu Y_{2Ts}^{\varepsilon,T})^2\, ds = \frac{\mathrm{En}(\varepsilon, T)}{\varepsilon^2}. \tag{4.26}$$

Proposition 4.5.1 *a) The energy-to-noise ratio is given by*

$$\mathrm{ENR}(\varepsilon, T) = \frac{1}{\varepsilon^2}\left(\frac{\varphi - \psi}{\varphi + \psi}\right)^2\left[1 - \frac{2}{T(\varphi + \psi)}\tanh\frac{T(\varphi + \psi)}{2}\right], \tag{4.27}$$

with φ and ψ defined by (4.3).

b) If $p > q$, $\mathrm{ENR}(\varepsilon, T)$ vanishes at $\widehat{\varepsilon} = (V - v)/\log\left(\frac{p}{q}\right)$. For T large enough, $\varepsilon \mapsto \mathrm{ENR}(\varepsilon, T)$ has a local maximum. An optimal tuning rate, at which this maximum is attained is given by

$$T_{\mathrm{ENR}}(\varepsilon) = \frac{v}{q\varepsilon}e^{\frac{v}{\varepsilon}}(1 + \mathcal{O}(\varepsilon)) \tag{4.28}$$

and

$$\mathrm{ENR}(\varepsilon, T_{\mathrm{ENR}}(\varepsilon)) = \frac{1}{\varepsilon^2} + \mathcal{O}(\varepsilon^{-1}) \quad as \quad \varepsilon \to 0. \tag{4.29}$$

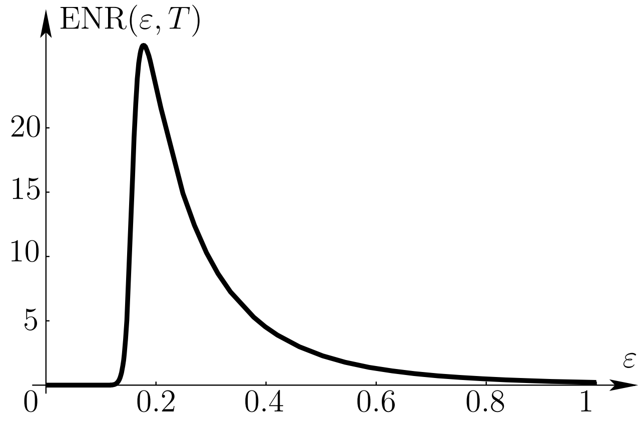

Fig. 4.5: If $p = q = 1$, $V = 3$, $v = 2$ and $T = 10^6$ the energy-to-noise ratio $\mathrm{ENR}(\varepsilon, T)$ has a local maximum at $\varepsilon \approx 0.178$.

Proof: The formula (4.27) is a direct consequence of (4.26) and (4.22). After denoting $t = e^{-\frac{V}{\varepsilon}}$ and $\beta = \frac{v}{V}$ and taking the derivative of (4.27) we find that it vanishes at $\widehat{\varepsilon} = (V - v)/\log(\frac{p}{q})$ and the other extrema (if any) of the energy-to-noise ratio satisfy the equation $F_T(t) = 0$, where

$$\begin{aligned} F_T(t) &= T(p^2t^2 - q^2t^{2\beta})(pt + q\beta t^\beta)\,\mathrm{sech}^2\,[\tfrac{T}{2}(pt + qt^\beta)]\log t \\ &\quad - 2T(pt + qt^\beta)(p^2t^2 - q^2t^{2\beta} + 2pq(1-\beta)t^{1+\beta}\log t) \\ &\quad + 4q^2t^{2\beta} - 2p^2t^2 + 2(p^2t^2 - 5pq(1-\beta)t^{1+\beta} - q^2\beta t^{2\beta})\tanh\,[\tfrac{T}{2}(pt + qt^\beta)]\log t. \end{aligned}$$

We parametrize $T_t = at^{-\beta}|\log t|$ and consider $F_{T_t}(t)$ as $t \to 0$. With this choice, it is clear again that $\mathrm{sech}\,[\frac{T_t}{2}(qt^\beta + pt)] = \mathrm{sech}\,[\frac{a}{2}(qt^{\beta-1} + p)] \to 0$ exponentially fast, whereas $\tanh\,[\frac{T_t}{2}(qt^\beta + pt)] \to 1$ exponentially fast as $t \to 0$. This yields

$$F_{T_t}(t) = 2q^2t^{2\beta}|\log t|(\beta - aq + \mathcal{O}(|\log t|^{-1})).$$

Therefore, the optimal T belongs to the interval $[\frac{\beta|\log t|}{2qt^\beta}, \frac{2\beta|\log t|}{qt^\beta}]$ as $t \to 0$. This allows us to find the asymptotic value for

$$T = \frac{\beta}{q}t^{-\beta}|\log t|(1 + \mathcal{O}(|\log t|^{-1})),$$

which is equivalent to (4.28). The asymptotics (4.29) follows from (4.27) and (4.28) immediately. ∎

ENR characterizes the rate with which the noise feeds the random output with the energy.

4.6 Out-of-phase measure

The approach by M.Freidlin [22] is valid for a quite general class of stochastic systems. Theorem 2.2.1 describes the periodic responce of the diffusion. The measure of quality in this theorem is the Lebesgue measure of the time which the trajectory spends outside some neighbourhood of the minimum of the deep well, i.e. the total amount of time spent in the 'wrong' place. The same results hold for the Markov chain $Y^{\varepsilon,T}$ for small values of ε.

Proposition 4.6.1 ([22]) *For $\varepsilon > 0$, let the half-period $T = T(\varepsilon)$ be such that*

$$\lim_{\varepsilon\to 0} \varepsilon \log T(\varepsilon) = \lambda > 0.$$

Let the function

$$\phi(t) = \begin{cases} -1, & t \pmod 1 \in [0, \frac{1}{2}), \\ 1, & t \pmod 1 \in [\frac{1}{2}, 1), \end{cases}$$

be a periodic and deterministic function of time, and Λ denote the Lebesgue measure on $[0, 1]$. Then, if $\lambda > v$,

$$\Lambda(t \in [0, 1] \,:\, Y^{\varepsilon,T}_{2T(\varepsilon)t} \neq \phi(t)) \to 0 \tag{4.30}$$

in probability as $\varepsilon \to 0$.

If $\lambda < v$ then

$$\Lambda(t \in [0, 1] \,:\, Y^{\varepsilon,T}_{2T(\varepsilon)t} \neq Y^{\varepsilon,T}_0) \to 0$$

in probability as $\varepsilon \to 0$. ■

Consider the Lebesgue measure from (4.30) as a function of ε and T

$$\Lambda(\varepsilon, T) = \Lambda(t \in [0, 1] : Y^{\varepsilon,T}_{2Tt} \neq \phi(t)).$$

Note that $\Lambda(\varepsilon, T)$ is a random variable. We introduce the *out-of-phase measure* by

$$d(\varepsilon, T) = \mathbf{E}_\nu \Lambda(\varepsilon, T) = \mathbf{E}_\nu \int_0^1 \mathbf{I}(Y^{\varepsilon,T}_{2Ts} \neq \phi(s))\, ds.$$

The out-of-phase measure describes how much time on average is spent by the Markov chain in the 'wrong' state.

Proposition 4.6.2 *a) The out-of-phase measure is given by*

$$d(\varepsilon, T) = \frac{1}{\varphi + \psi}\left[\varphi - \frac{1}{T}\frac{\varphi - \psi}{\varphi + \psi} \tanh \frac{T(\varphi + \psi)}{2}\right]. \tag{4.31}$$

b) For T large enough, $\varepsilon \mapsto d(\varepsilon, T)$ has a local minimum. The optimal tuning rate, at which this minimum is attained is given by

$$T_d(\varepsilon) = \frac{1}{p}\frac{v}{V - v} e^{\frac{V}{\varepsilon}} \{1 + \mathcal{O}(e^{-\frac{V-v}{\varepsilon}})\} \tag{4.32}$$

and we have

$$d(\varepsilon, T_d(\varepsilon)) \to 0 \quad as \quad \varepsilon \to 0. \tag{4.33}$$

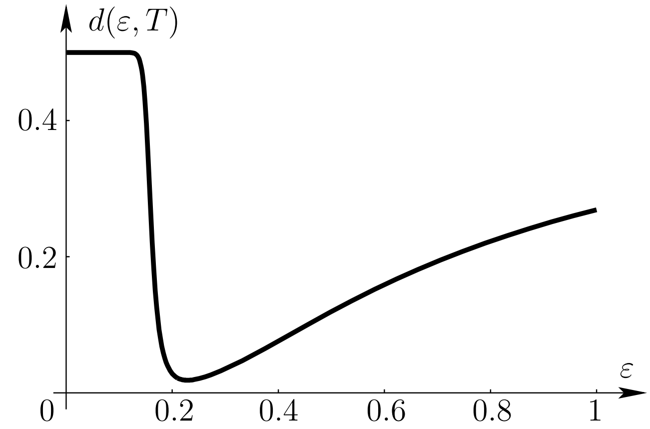

Fig. 4.6: If $p = q = 1$, $V = 3$, $v = 2$ and $T = 10^6$ the out-of-phase measure $d(\varepsilon, T)$ has a local minimum at $\varepsilon \approx 0.229$.

Proof: To obtain (4.31) we notice that

$$\begin{aligned} d(\varepsilon, T) &= \mathbf{E}_\nu \int_0^1 \mathbf{I}(Y_{2Ts}^{\varepsilon,T} \neq \phi(s))\, ds \\ &= \int_0^{\frac{1}{2}} \mathbf{P}_\nu(Y_{2Ts}^{\varepsilon,T} = 1)\, ds + \int_{\frac{1}{2}}^1 \mathbf{P}_\nu(Y_{2Ts}^{\varepsilon,T} = -1)\, ds \\ &= 2\int_0^{\frac{1}{2}} \nu_{\varepsilon,T}^{+}(s)\, ds = \frac{1}{\varphi + \psi}\left[\varphi - \frac{1}{T}\frac{\varphi - \psi}{\varphi + \psi}\tanh\left(\frac{T(\varphi + \psi)}{2}\right)\right]. \end{aligned}$$

After substituning $t = e^{-\frac{V}{\varepsilon}}$ into (4.31) and taking the derivative in t we find that the coordinates of extrema satisfy the equation $F_T(t) = 0$, where

$$\begin{aligned} F_T(t) =& T(q\beta t^\beta + pt)(q^2 t^{2\beta} - p^2 t^2)\operatorname{sech}^2\left[\tfrac{T}{2}(qt^\beta + pt)\right] \\ &+ 2Tpq(1-\beta)t^{1+\beta}(qt^\beta + pt) \\ &+ 2(p^2t^2 - q^2\beta t^{2\beta} - 3pq(1-\beta)t^{1+\beta})\tanh\left[\tfrac{T}{2}(qt^\beta + pt)\right], \end{aligned}$$

where $0 < \beta = \frac{v}{V} < 1$. Let us again localize the values of T such that $F_T(t) = 0$ has a root as $t \to 0$. We parametrize $T_t = \frac{a}{t}$, $a > 0$, and consider $F_{T_t}(t)$ as $t \to 0$. It is clear that $\operatorname{sech}\left[\frac{T_t}{2}(qt^\beta + pt)\right] = \operatorname{sech}\left[\frac{a}{2}(qt^{\beta-1} + p)\right] \to 0$ exponentially fast, whereas $\tanh\left[\frac{T_t}{2}(qt^\beta + pt)\right] \to 1$ exponentially fast as $t \to 0$. Hence, $F_{T_t}(t)$ can be asymptotically expanded as

$$F_{T_t}(t) = 2q^2t^{2\beta}(ap(1-\beta) - \beta) + \mathcal{O}(t^{1+\beta}).$$

Observe that $F_{T_t}(t) < 0$ if $a < \frac{\beta}{p(1-\beta)}$, and $F_{T_t}(t) > 0$ if $a > \frac{\beta}{p(1-\beta)}$ as $t \to 0$. In other words, as $t \to 0$, the half-period T we look for belongs to the interval $[\frac{\beta}{2p(1-\beta)t}, \frac{2\beta}{p(1-\beta)t}]$. On this interval we can asymptotically solve the equation $F_T(t) = 0$ for T which results in

$$T = \frac{\beta}{p(1-\beta)} t^{-1}(1 + \mathcal{O}(t^{1-\beta})) \quad \text{as} \quad t \to 0.$$

This is equivalent to (4.32).

The limit (4.33) is obtained directly from (4.31) and (4.32). ■

Remark 4.6.1 The out-of-phase measure can be rewritten as

$$d(\varepsilon, T) = \frac{1}{4}\int_0^1 \mathbf{E}_\nu (Y_{2Ts}^{\varepsilon,T} - \phi(s))^2 \, ds,$$

which represents the mean square deviation of $Y^{\varepsilon,T}$ from the deterministic function ϕ appearing in Proposition 4.6.1.

4.7 Relative entropy

Let us consider the point mass $\delta_{\phi(t)}$, $t \geq 0$, in $\phi(t)$ with ϕ according to the preceding section. We have $\delta_{\phi(t)} = \phi^-(t)\delta_{-1} + \phi^+(t)\delta_1$, where $\phi^-(t) = 1$, $\phi^+(t) = 0$ if $t \pmod 1 \in [0, \frac{1}{2})$ and $\phi^-(t) = 0$, $\phi^+(t) = 1$ if $t \pmod 1 \in [\frac{1}{2}, 1)$. In the language of diffusion, minimum of the deep well always has mass 1.

We consider the *relative entropy* of the invariant measure $\nu_{\varepsilon,T}$ with respect to δ_ϕ defined by

$$H_{\phi|\nu}(\varepsilon, T) = \int_0^1 \sum_{\alpha=+,-} \phi^\alpha(s) \log \frac{\phi^\alpha(s)}{\nu^\alpha_{\varepsilon,T}(s)} \, ds.$$

Proposition 4.7.1 *a) The relative entropy is given by*

$$\begin{aligned} H_{\phi|\nu}(\varepsilon, T) = &-\log \frac{\psi}{\varphi + \psi} + \\ &\frac{1}{T(\varphi+\psi)}\left[\mathrm{Li}_2\left(\frac{\psi-\varphi}{\psi}\frac{e^{-T(\varphi+\psi)}}{1+e^{-T(\varphi+\psi)}}\right) - \mathrm{Li}_2\left(\frac{\psi-\varphi}{\psi}\frac{1}{1+e^{-T(\varphi+\psi)}}\right)\right], \end{aligned} \tag{4.34}$$

where $\mathrm{Li}_2(x)$ *is the dilogarithm function defined by* $\mathrm{Li}_2(x) = \int_x^0 \frac{\log(1-y)}{y}\, dy$, $x \leq 1$.

b) For T large enough, $\varepsilon \mapsto H_{\phi|\nu}(\varepsilon, T)$ has a local minimum, and an optimal tuning rate is given by

$$T_{\phi|\nu}(\varepsilon) = \frac{\pi^2}{6p}\frac{v}{V-v} e^{\frac{V}{\varepsilon}}(1 + \mathcal{O}(\varepsilon^{-1} e^{-\frac{V-v}{\varepsilon}})). \tag{4.35}$$

Moreover,

$$H_{\phi|\nu}(\varepsilon, T_{\phi|\nu}(\varepsilon)) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.36}$$

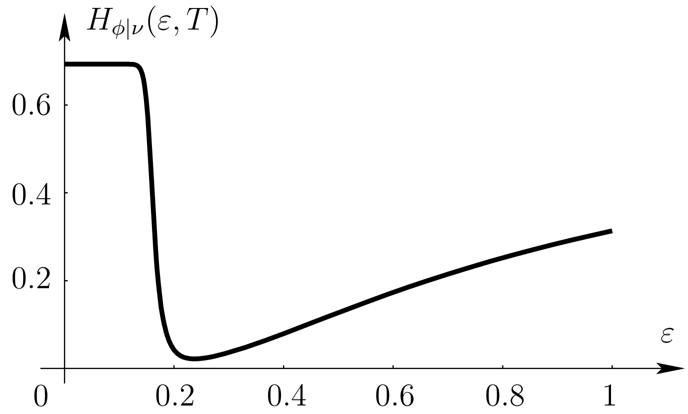

Fig. 4.7: If $p = q = 1$, $V = 3$, $v = 2$ and $T = 10^6$ the relative entropy $H_{\phi|\nu}(\varepsilon, T)$ has a local minimum at $\varepsilon \approx 0.238$.

Proof: To obtain (4.34) we use (4.8) and the definition of the relative entropy. This gives

$$\begin{aligned} H_{\phi|\nu}(\varepsilon, T) &= \int_0^1 \sum_{\alpha=+,-} \phi^\alpha(s) \log \frac{\phi^\alpha(s)}{\nu^\alpha_{\varepsilon,T}(s)}\, ds \\ &= \int_0^{\frac{1}{2}} \log \frac{1}{\nu^-_{\varepsilon,T}(s)}\, ds + \int_{\frac{1}{2}}^1 \log \frac{1}{\nu^+_{\varepsilon,T}(s)}\, ds \\ &= -2 \int_0^{\frac{1}{2}} \log \nu^-_{\varepsilon,T}(s)\, ds \\ &= -2 \int_0^{\frac{1}{2}} \log \left(\frac{\psi}{\varphi + \psi} + \frac{\varphi - \psi}{\varphi + \psi} \frac{e^{-2(\varphi+\psi)Ts}}{1 + e^{-(\varphi+\psi)T}} \right) ds. \end{aligned}$$

The latter integral is evaluated by means of the formula:

$$\int \log\left(a + be^{-2cs}\right) ds = s \log a + \frac{1}{2c} \mathrm{Li}_2\left(-\frac{be^{-2cs}}{a}\right), \quad a > 0, b > -a, c > 0.$$

As usual, we denote $t = e^{-\frac{V}{\varepsilon}}$ and $\beta = \frac{v}{V}$ to find the derivative of $H_{\phi|\nu}$ with respect to t. This results in an algebraic equation $F_T(t) = 0$ for the extrema of

$H_{\phi|\nu}$, where

$$\begin{aligned} F_T(t) = (pt + qt^\beta)\Bigg[&-(1-\beta)pTt(pt - qt^\beta)\left(1 + e^{(pt+qt^\beta)T}\right) \\ &+ \left(p^2Tt^2 - q^2\beta Tt^{2\beta} + (1-\beta)pt\left(1 - qTt^\beta + e^{(pt+qt^\beta)T}\right)\right) \\ &\qquad\qquad \times \log\left(\frac{q + pt^{1-\beta}e^{(pt+qt^\beta)T}}{q + qe^{(pt+qt^\beta)T}}\right) \\ &+ \left(T(p^2t^2 - q^2\beta t^{2\beta})e^{(pt+qt^\beta)T} - (1-\beta)pt\left(1 + (1 + qTt^\beta)e^{(pt+qt^\beta)T}\right)\right) \\ &\qquad\qquad \times \log\left(1 - \frac{q - pt^{1-\beta}}{q + qe^{(pt+qt^\beta)T}}\right)\Bigg] \\ &- (pt - qt^\beta)(pt + q\beta t^\beta)\left(1 + e^{(pt+qt^\beta)T}\right)\mathrm{Li}_2\left(\frac{q - pt^{1-\beta}}{q + qe^{(pt+qt^\beta)T}}\right) \\ &+ (pt - qt^\beta)(pt + q\beta t^\beta)\left(1 + e^{(pt+qt^\beta)T}\right)\mathrm{Li}_2\left(\frac{(q - pt^{1-\beta})e^{(pt+qt^\beta)T}}{q + qe^{(pt+qt^\beta)T}}\right). \end{aligned} \tag{4.37}$$

We localize the zero of $F_T(t) = 0$ for $t \to 0$. Let us parametrize $T_t = \frac{a}{t}$ for some $a > 0$ and consider F_{T_t} as $t \to 0$. It is immediate that as $t \to 0$, $\exp\left((pt + qt^\beta)T_t\right) \to \infty$ exponentially fast, hence the log-terms in $F_{T_t}(t)$ converge to 0, as does the first Li_2-term, whereas the second one converges to $\frac{\pi^2}{6} = -\int_0^1 \frac{\log(1-y)}{y}\,dy = \sum_{k=1}^\infty \frac{x^k}{k^2}$.

The terms of (4.37), which have a factor $\exp\{(pt + qt^\beta)T_t\}$ will be dominating as $t \to 0$, and therefore the following expansion holds:

$$F_{T_t}(t) = e^{a(p+qt^{\beta-1})}q^2t^{2\beta}\left(ap(1-\beta) - \frac{\pi^2}{6}\beta + \mathcal{O}(t^{1-\beta}\log t)\right).$$

Therefore, $F_{T_t}(t) < 0$ if $a < \frac{\pi^2\beta}{6p(1-\beta)}$, and $F_{T_t}(t) > 0$ if $a > \frac{\pi^2\beta}{6p(1-\beta)}$, as $t \to 0$. In other words, we have localized the optimal half-period T on the interval $[\frac{\pi^2\beta}{7p(1-\beta)t}, \frac{\pi^2\beta}{5p(1-\beta)t}]$ as $t \to 0$. In this interval we asymptotically solve the equation $F_T(t) = 0$ for T, which gives

$$T = \frac{\pi^2\beta}{6p(1-\beta)}t^{-1}(1 + \mathcal{O}(t^{1-\beta}\log t)), \quad t \to 0.$$

After rewriting it in terms of ε, the latter formula gives (4.35). The limit (4.36) follows from (4.34) and (4.35). ∎

4.8 Entropy

Define the *entropy* of the invariant measure $\nu^\pm_{\varepsilon,T}$ at time $t \in [0,1]$ as

$$H_t(\varepsilon, T) = -\nu^-_{\varepsilon,T}(t)\log\nu^-_{\varepsilon,T}(t) - \nu^+_{\varepsilon,T}(t)\log\nu^+_{\varepsilon,T}(t), \quad t \in [0,1],$$

and consider its *entire entropy* by

$$H(\varepsilon,T) = \int_0^1 H_s(\varepsilon,T)\,ds$$

Proposition 4.8.1 *a) The entire entropy of $\nu_{\varepsilon,T}$ is given by*

$$\begin{aligned} H(\varepsilon,T) = {} & \log(\varphi+\psi) - \frac{\varphi\log\varphi + \psi\log\psi}{\varphi+\psi} \\ & + \frac{\varphi}{T(\varphi+\psi)^2}\left[\mathrm{Li}_2\left(\frac{\varphi-\psi}{\varphi}\frac{1}{1+e^{-T(\varphi+\psi)}}\right) - \mathrm{Li}_2\left(\frac{\varphi-\psi}{\varphi}\frac{e^{-T(\varphi+\psi)}}{1+e^{-T(\varphi+\psi)}}\right)\right] \\ & + \frac{\psi}{T(\varphi+\psi)^2}\left[\mathrm{Li}_2\left(\frac{\psi-\varphi}{\psi}\frac{1}{1+e^{-T(\varphi+\psi)}}\right) - \mathrm{Li}_2\left(\frac{\psi-\varphi}{\psi}\frac{e^{-T(\varphi+\psi)}}{1+e^{-T(\varphi+\psi)}}\right)\right], \end{aligned} \tag{4.38}$$

where $\mathrm{Li}_2(x)$ *is the dilogarithm function,* $\mathrm{Li}_2(x) = \int_x^0 \frac{\log(1-y)}{y}\,dy$, $x \le 1$.

b) If $p > q$ the entropy has a local maximum at $\widehat{\varepsilon} = (V-v)/\log\left(\frac{p}{q}\right)$ and $H(\widehat{\varepsilon},T) = \log 2$ for all $T > 0$. For T large enough $H(\varepsilon,T)$ always has a local minimum, and an optimal tuning rate is given by

$$T_H(\varepsilon) = \frac{\pi^2}{6p}\frac{v\varepsilon}{(V-v)^2}e^{\frac{V}{\varepsilon}}(1+\mathcal{O}(\varepsilon)). \tag{4.39}$$

Moreover,

$$H(\varepsilon, T_H(\varepsilon)) \to 0 \quad as \quad \varepsilon \to 0. \tag{4.40}$$

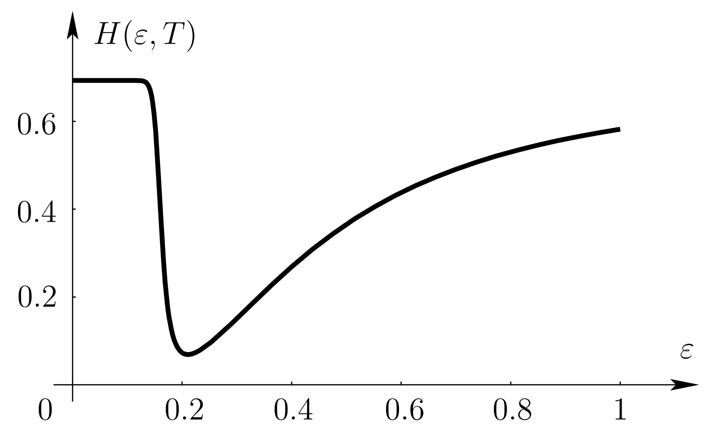

Fig. 4.8: If $p = q = 1$, $V = 3$, $v = 2$ and $T = 10^6$ the entropy $H(\varepsilon,T)$ has a local minimum at $\varepsilon \approx 0.211$.

Proof: To obtain (4.38) we use (4.8) and the integration formula

$$\int (a+be^{-2cs})\log(a+be^{-2cs})\,ds = \frac{b}{2c}e^{-2cs}\left(1-\log(a+be^{-2cs})\right)$$
$$+as\log a - \frac{a}{2c}\log(a+be^{-2cs}) + \frac{a}{2c}\,\mathrm{Li}_2\left(-\frac{be^{-2cs}}{a}\right),$$
$$a>0, b>-a, c>0.$$

We denote as usual $t=e^{-\frac{V}{\varepsilon}}$ and $\beta=\frac{v}{V}$ so that $\varphi=pt$ and $\psi=qt^{\beta}$ and take the derivative of H in t to determine the extrema. The straightforward and tedious calculation yields that they are given by the roots of the equation $F_T(t)=0$, with

$$F_T(t) = e^{2(pt+qt^{\beta})T}\xi_2(t,T) + e^{(pt+qt^{\beta})T}\xi_1(t,T) + \xi_0(t,T), \tag{4.41}$$

where

$$\begin{aligned}
\xi_2(t,T) &= pq(1-\beta)^2Tt^{1+\beta}(q^2t^{2\beta}-p^2t^2)\log\left(\tfrac{p}{q}t\right)\\
&\quad + pq(1-\beta)t^{1+\beta}(pt+qt^{\beta})\left[\log(pt) - \log\left(\frac{(qt^{\beta}+pt)e^{(pt+qt^{\beta})T}}{1+e^{(pt+qt^{\beta})T}}\right)\right]\\
&\quad + qTt^{\beta}(q^2t^{2\beta}-p^2t^2)\log\left(1-\frac{q-pt^{1-\beta}}{q(1+e^{(pt+qt^{\beta})T})}\right)\\
&\quad - pt(pt-qt^{\beta})(pt-q(1-2\beta)t^{\beta})\\
&\qquad \times\left[\mathrm{Li}_2\left(\frac{(pt-qt^{\beta})e^{(pt+qt^{\beta})T}}{pt(1+e^{(pt+qt^{\beta})T})}\right) - \mathrm{Li}_2\left(\frac{pt-qt^{\beta}}{pt(1+e^{(pt+qt^{\beta})T})}\right)\right]\\
&\quad - qt^{\beta}(qt^{\beta}-pt)(q\beta t^{\beta}+p(2-\beta)t)\\
&\qquad \times\left[\mathrm{Li}_2\left(\frac{qt^{\beta}-pt}{qt^{\beta}(1+e^{(pt+qt^{\beta})T})}\right) - \mathrm{Li}_2\left(\frac{(qt^{\beta}-pt)e^{(pt+qt^{\beta})T}}{qt^{\beta}(1+e^{(pt+qt^{\beta})T})}\right)\right],
\end{aligned}$$

and $\xi_1(t,T)$ and $\xi_0(t,T)$ are similar combinations of powers of t and logarithmic and dilogarithmic functions of the same arguments as in ξ_2. The functions ξ_0, ξ_1 and ξ_2 vanish if $pt=qt^{\beta}$. This yields the local maximum of the entropy at $\widehat{\varepsilon}=(V-v)/\log(\frac{p}{q})$. Note that at $\widehat{\varepsilon}$ the infinitesimal probabilities φ and ψ are equal and it follows from (4.38) that $H(\widehat{\varepsilon},T)=\log 2$ for all $T>0$.

We shall determine the root of $F_T(t)=0$ for small t. Let us parametrize $T_t=\frac{a}{t|\log t|}$ for $a>0$ and expand (4.41) in t. It is evident that as $t\to 0$, $\exp((pt+qt^{\beta})T_t)\to\infty$ exponentially fast, hence the log-terms containing the exponentials will vanish in the limit, as is the case for the first three Li_2-terms. The fourth Li_2-term, however, converges to $\frac{\pi^2}{6}$. Collecting the leading $t^{3\beta}$-terms in ξ_2 then results in

$$F_{T_t}(t) = e^{2a(p+qt^{\beta-1})|\log t|^{-1}}q^3t^{3\beta}\left(\frac{\pi^2}{6}\beta - ap(1-\beta)^2 + \mathcal{O}(|\log t|^{-1})\right)$$

Therefore, $F_{T_t} < 0$ for $a > \frac{\pi^2\beta}{6p(1-\beta)^2}$ and $F_{T_t} > 0$ for $a < \frac{\pi^2\beta}{6p(1-\beta)^2}$ as $t \to 0$. Hence we have localized the local minimum of the entropy, for example on the interval $[\frac{\pi^2\beta}{7p(1-\beta)^2 t|\log t|}, \frac{\pi^2\beta}{5p(1-\beta)^2 t|\log t|}]$, as $t \to 0$. Solving $F_T(t) = 0$ for T on this interval gives the asymptotic formula

$$T = \frac{\pi^2\beta}{6p(1-\beta)^2}(t|\log t|)^{-1}\left(1 + \mathcal{O}(|\log t|^{-1})\right).$$

In terms of ε this is equivalent to (4.39). The limit (4.40) is an application of (4.39) to (4.38). ∎

4.9 Overview and concluding remarks

For the convenience of the reader we represent the optimal tuning rates for the quality measures studied in this chapter in the following table.

	Measure of quality of stochastic resonance	Optimal tuning $T(\varepsilon)$, $\varepsilon \to 0$	Value at $\widehat{\varepsilon}$
1.	SPA coefficient, η^Y	$T_\eta(\varepsilon) \approx \frac{\pi}{\sqrt{2pq}}\sqrt{\frac{v}{V-v}}e^{\frac{V+v}{2\varepsilon}}$	0
2.	SPA-to-n. ratio, SPN	$T_{\mathrm{SPN}}(\varepsilon) \approx \frac{\pi\sqrt{v}}{q\sqrt{\varepsilon}}e^{\frac{v}{\varepsilon}}$	0
3.	Energy, En	$T_{\mathrm{En}}(\varepsilon) \approx \frac{1}{2p}\frac{v}{V-v}e^{\frac{V}{\varepsilon}}$	0
4.	E.-n. ratio, ENR	$T_{\mathrm{ENR}}(\varepsilon) \approx \frac{v}{q\varepsilon}e^{\frac{v}{\varepsilon}}$	0
5.	Out-of-phase, d	$T_d(\varepsilon) \approx \frac{1}{p}\frac{v}{V-v}e^{\frac{V}{\varepsilon}}$	$\frac{1}{2}$
6.	Relative entropy, $H_{\phi\|\nu}$	$T_{H_{\phi\|\nu}}(\varepsilon) \approx \frac{\pi^2}{6p}\frac{v}{V-v}e^{\frac{V}{\varepsilon}}$	$\log 2$
7.	Entropy, H	$T_H(\varepsilon) \approx \frac{\pi^2}{6p}\frac{\varepsilon v}{(V-v)^2}e^{\frac{V}{\varepsilon}}$	$\log 2$

Theorem 2.2.1 and Proposition 4.6.1 provide a lower bound for the exponential rate $T(\varepsilon)$ above which in the small noise limit $\varepsilon \to 0$ the randomly perturbed system produces trajectories with periodicity properties. Our results for the Markov chain obtained in this chapter all agree with this bound and determine the exact rates of growth for $T(\varepsilon)$ as $\varepsilon \to 0$ together with pre-factors.

It is interesting to remark that if the exponential rate is v, i.e. corresponds to the depth of the shallow well, then the pre-factors contain only the parameter q, which stands for the geometry of the shallow well. Analogously, if $T(\varepsilon) \sim e^{V/\varepsilon}$, then the pre-factor of the optimal tuning rate only contains p, the geometrical factor of the deep well, and not q. The only exception is the SPA coefficient η^Y whose optimal tuning rate contains all parameters of the system.

The measures of quality considered can be subdivided into two groups. The SPA coefficient, the SPA-to-noise ratio, the energy and the ENR belong to the first group. They are based on the interpretation of the Markov chain as a random amplifier and describe the spectral properties of the averaged output. The key result which may seem counterintuitive is as follows: an optimal transfer of a deterministic periodic signal through a random system is not guaranteed by the elimination of noise, but rather by tuning it in on some essentially non-zero intensity level.

The same result holds for the second group of measures which can be considered as measures of stabilization. Indeed, the out-of-phase measure and the relative entropy determine the deviation of the random output from a deterministic function ϕ. Again, *increasing* the noise makes the random output *less random.* The final measure — the entropy of the invariant measure of the process — is, roughly speaking, the measure of randomness. The bigger the entropy, the more chaotic is the system. The fact that non-zero noise minimizes the entropy is a very good example of noise-induced order.

Let us next briefly discuss the particular noise level $\widehat{\varepsilon} = (V - v)/\log\left(\frac{p}{q}\right)$. At this level, the infinitesimal probabilities φ and ψ are equal, the generators Q_1 and Q_2 are indistinguishable from each other for all $t, T > 0$, and the process $Y^{\widehat{\varepsilon},T}$ is time-homogeneous with invariant measure $(\frac{1}{2}, \frac{1}{2})$. In other words, at $\widehat{\varepsilon}$ the Markov chain is a symmetric telegraph process which lives in -1 and 1 with equal probability. No periodicity is observed.

The out-of-phase measure and the relative entropy, of course, do not vanish at $\widehat{\varepsilon}$, but are far from their maxima. Indeed, the process which spends only half of the time in the 'right' does not constitute the worst case for these measures. The worst case is rather the deterministic process $1 - \phi(t)$ which is in the 'wrong' place for all t. The entropy of the invariant measure takes its maximum at $\widehat{\varepsilon}$ because for any t the random variable $Y_t^{\widehat{\varepsilon},T}$ has a symmetric Bernoulli law which is the most 'chaotic' among binary laws (see, [56, Chapter1, §5]).

Finally, we briefly outline the programme for the remaining two chapters. In Chapter 6 we investigate the SPA coefficient for the diffusion driven by the SDE (2.2). Its asymptotic properties as $\varepsilon \to 0$ and $T \to \infty$ will be derived with the help of the invariant density of the diffusion. To determine the invariant density, we have to solve the forward Kolmogorov (Fokker-Planck) equation which requires a spectral analysis of the diffusion's infinitesimal generator. The following Chapter 5 is devoted to this study. It includes a discussion of the zeroth and first eigenvalues and eigenfunctions. The discovery of a spectral gap between the first and second eingenvalues will be crucial for the analysis of the SPA coefficient in Chapter 6.

Chapter 5

Spectral Analysis of the Operator of Small Noise Diffusion

In this chapter we study spectral properties of the infinitesimal generator of a one-dimensional diffusion with small noise level in a double-well potential. The first section is designed to motivate our approach. In particular, we give heuristic derivations of the first eigenvalues and eigenfunctions. Moreover, we employ the theory of Schrödinger operators for potentials increasing at infinity to prove the discreteness of the spectrum and the existence of a complete system of eigenfunctions. Variational principles are evoked for the study of the eigenvalues and eigenfunctions. Eigenvalue and eigenfunction of order zero are easy to derive exactly, while the first eigenvalue λ_1 is determined approximately. It is known to be exponentially small in ε. To find a good estimate of the first eigenfunction Φ_1, we use a series representation of Φ_1 in powers of λ_1. In fact, we show that there is a spectral gap between λ_1 and a second eigenvalue λ_2: the second eigenvalue λ_2 is bounded from below by some positive constant not depending on ε. In the last section we briefly discuss the multi-dimensional case.

5.1 The heuristics

In this section we present the heuristic derivation of the first eigenvalue and eigenfunction of the differential operator

$$L_\varepsilon = \frac{\varepsilon}{2}\frac{d^2}{dx^2} - U'\frac{d}{dx}$$

which is the infinitesimal generator of a time-homogeneous diffusion driven by the SDE

$$dX_t^\varepsilon = -U'(X_t^\varepsilon)\,dt + \sqrt{\varepsilon}\,dW_t, \quad t \geq 0, \varepsilon > 0, \tag{5.1}$$

with a purely space dependent potential function U.

For convenience of notation, in this and the next chapter we fix and omit the subscript ε in the notation of the operators L_ε, L_ε^* etc. and their spectral data.

The results about the spectrum of the operator L we shall derive are needed in Chapter 6, where we have to solve the forward Kolmogorov (Fokker-Planck) equation for the invariant density of the diffusion (2.2) with time-periodic potential. The equation to solve is given by

$$\frac{1}{2T}\frac{\partial}{\partial t}f(x,t) = L^* f(x,t), \tag{5.2}$$

where

$$L^* f = \frac{\varepsilon}{2}\frac{d^2}{dx^2}f + U'\frac{d}{dx}f + U'' f$$

is the formal adjoint of L, i.e. a differential operator such that for all $f, g \in \mathcal{C}_0^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} Lf \cdot g\, dx = \int_{\mathbb{R}} f \cdot L^* g\, dx$$

and U a temporally periodic potential function switching between two spatially symmetric states, recall Fig. 1.3. Thus, equation (5.2) is time-homogeneous on half-intervals. To obtain its solution for all t we will continuously knit the solutions corresponding to time-independent states of the potential. In the time-independent setting of the present section, U is supposed to take one of these two states at all times.

Assume that we know that the spectrum of the operator L^* is discrete, and let us denote $\{-\lambda_k, \Psi_k\}_{k\geq 0}$ its eigenvalues and eigenfunctions. We assume that the spectrum is non-positive, i.e. $0 \leq \lambda_0 < \lambda_1 < \cdots$, and has no finite condensation point. If such a spectrum exists the method of separation of variables (or the Fourier method) can be applied and a solution of (5.2) can be given in the form of

$$f(x,t) = \sum_{k=0}^{\infty} a_k \Psi_k(x) e^{-2T\lambda_k t}, \quad x \in \mathbb{R}, t \geq 0,$$

where the real numbers a_k are determined from the initial and boundary conditions (see, e.g. [19, 51]).

Thus, we face the problem of describing the spectrum of the time-independent operator L^*, at least for small ε. It turns out that it can be solved by studying the spectral properties of L. Indeed, one easily notes that

$$L^* f = e^{-\frac{2U}{\varepsilon}} L(f e^{\frac{2U}{\varepsilon}}).$$

This equality means that L^* and L have the same eigenvalues, and the eigenfunctions of the operator L^* can be obtained as $\Psi_k = e^{-\frac{2U}{\varepsilon}}\Phi_k$, where Φ_k are the eigenfunctions of the diffusion generator L, $k \geq 0$.

Spectral analysis for generators of small noise diffusions was addressed in a big number of papers. Diffusions in bounded domains are particularly well understood [12, 11, 13, 14, 15, 23, 30, 31, 58]. In a nutshell, the essential result states that in the small noise limit the lowest eigenvalue corresponds to the inverse of the expected diffusion exit time from the bounded domain. The corresponding eigenfunction is almost constant on any compact subset of the domain.

Our state space in contrast is the non-compact real line. Here the situation is somewhat more complicated.

We start with the following basic assumptions about the potential U:

(**S**) $U \in \mathcal{C}^\infty(\mathbb{R})$;

(**G**) there exists $R > 0$ such that $U(x) = x^4/4$ for $|x| \geq R$;

(**M**) U has exactly two local minima at $x = \pm 1$ and one local maximum at $x = 0$; moreover,
$$U(-1) = -\frac{V}{2}, \quad U(0) = 0, \quad U(1) = -\frac{v}{2}, \quad \frac{2}{3} < \frac{v}{V} < 1;$$
the extrema are non-degenerate, i.e.
$$U''(\pm 1) = \omega_\pm > 0, \quad U''(0) = -\omega_0 < 0.$$

The condition $\frac{2}{3} < \frac{v}{V} < 1$ is a rather weak condition. It allows the depths of the two potential wells to be arbitrarily close and just requires the depths not to be too different. This agrees with the interpretation of stochastic resonance as optimal amplification of weak periodic signals by noise. Here 'weakness' means small amplitude of periodic signal which in our case corresponds to $\frac{V-v}{2}$ (see Chapters 3 and 4).

The operator L can be extended to a self-adjoint operator on the domain $\mathcal{D}_L$ consisting of functions from $\mathcal{L}^2(\mathbb{R}, \rho^{-1}dx)$, $\rho = e^{\frac{2U}{\varepsilon}}$, with locally absolutely continuous first derivative and square-integrable second derivative. Self-adjointness and the fast increase of the potential U at infinity imply the existence of the discrete non-positive spectrum and an orthogonal system of eigenfunctions lying in $\mathcal{L}^2(\mathbb{R}, \rho^{-1}dx)$.

The zeroth eigenvalue can be found directly from the variational principle [29]:

$$-\lambda_0 = \inf_{0 \neq \varphi \in \mathcal{D}_L} \frac{-\int_{\mathbb{R}} \varphi L \varphi e^{-\frac{2U}{\varepsilon}}\, dx}{\int_{\mathbb{R}} \varphi^2 e^{-\frac{2U}{\varepsilon}}\, dx} = \inf_{0 \neq \varphi \in \mathcal{D}_L} \frac{\varepsilon}{2} \frac{\int_{\mathbb{R}} (\varphi')^2 e^{-\frac{2U}{\varepsilon}}\, dx}{\int_{\mathbb{R}} \varphi^2 e^{-\frac{2U}{\varepsilon}}\, dx} = 0.$$

To verify the latter formula it is enough to take $\varphi = \text{Const}$. Thus, the zeroth eigenfunction Φ_0 can be chosen to be identically equal to 1.

To determine the first eigenfunction Φ_1 and eigenvalue λ_1 for small values of ε we use the intuitive ideas displayed in [53, 44].

The only a priori knowledge about λ_1 we use follows from [35] or the Freidlin-Wentzell theory [23]. It states that the first eigenvalue is exponentially small in ε and $\lim_{\varepsilon \to 0} \varepsilon \log \lambda_1 = -v$.

Let us next consider the eigenvalue problem for the first eigenvalue λ_1

$$\frac{\varepsilon}{2}\varphi''(x) - U'(x)\varphi'(x) = -\lambda_1\varphi(x), \quad x \in \mathbb{R} \tag{5.3}$$

for $\varepsilon \to 0$.

The smallness of λ_1 implies that the terms containing φ'' and φ in (5.3) vanish as $\varepsilon \to 0$, and the equation (5.3) reduces to

$$U'\varphi' = 0. \tag{5.4}$$

Equation (5.4) expresses the fact that φ is approximately constant on the intervals of constant sign of U'.

Let $\delta > 0$ be a small number tending to 0 together with ε, and let us study the consequences of the previous equation by determining the constants $c_1, \dots, c_4$ in the following ansatz for an approximation $\Phi_1^{(\delta)}$ of Φ_1:

$$\Phi_1^{(\delta)}(x) = \begin{cases} c_1, & x < -1-\delta, \\ c_2, & -1+\delta < x < \delta, \\ c_3, & \delta < x < 1-\delta, \\ c_4, & x > 1+\delta. \end{cases}$$

If, for example, $c_1 \neq c_2$ then the so-called *internal layer* may exist in the δ-neighbourhood of the point $x = -1$. In this neighbourhood we perform the stretching transformation by introducing the new coordinate

$$\xi = \frac{x+1}{\sqrt{\varepsilon}}, \quad |x+1| \leq \delta, \quad \varepsilon > 0.$$

This change of variables gives $\frac{d}{dx}\varphi(x) = \frac{1}{\sqrt{\varepsilon}}\frac{d}{d\xi}\varphi(\xi)$ and $\frac{d^2}{dx^2}\varphi(x) = \frac{1}{\varepsilon}\frac{d^2}{d\xi^2}\varphi(\xi)$. The drift term can be expanded in a Taylor series with respect to x. Since -1 is a critical point, this leads to $U'(x) = \omega_-(x+1) + \mathcal{O}((x+1)^2)$, where $\omega_- > 0$ is the curvature of U in -1. In terms of ξ this gives $U'(\xi) = \omega_-\sqrt{\varepsilon}\xi + \mathcal{O}(\varepsilon)$.

Using this asymptotics in (5.3) together with the exponential smallness of λ_1, we therefore obtain

$$\frac{1}{2}\varphi''(\xi) - \omega_-\xi\varphi'(\xi) = 0 \tag{5.5}$$

and the continuity conditions

$$\begin{aligned} \varphi(-\delta/\sqrt{\varepsilon}) &\to c_1, \\ \varphi(\delta/\sqrt{\varepsilon}) &\to c_2, \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

We solve equation (5.5) to obtain the general solution

$$\varphi(\xi) = d_1 \int_0^{\xi} e^{\omega_- y^2}\, dy + d_2$$

with constants d_1, d_2. The continuity conditions immediately imply that $d_1 = 0$, and $c_1 = d_2 = c_2$. Hence there is no internal layer near -1.

Analogously, one rules out the existence of an internal layer near 1. Denote now the common value of c_1, c_2 by C_1, the common value of c_3, c_4 by C_2.

We may further determine asymptotically C_1, C_2 in the ansatz

$$\Phi_1^{(\delta)}(x) = \begin{cases} C_1, & x < -\delta, \\ C_2, & x > \delta. \end{cases}$$

In a δ-neighbourhood of 0 it follows from (5.3) that the equation for the possible internal layer takes the form

$$\frac{1}{2}\varphi''(\xi) + \omega_0 \xi \varphi'(\xi) = 0,$$

where $U''(0) = -\omega_0 < 0$. The continuity conditions become

$$\begin{aligned} \varphi(-\delta/\sqrt{\varepsilon}) &\to C_1, \\ \varphi(\delta/\sqrt{\varepsilon}) &\to C_2, \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

Solving this equation gives the general solution

$$\varphi(\xi) = d_1 \int_0^{\xi} e^{-\omega_0 y^2}\, dy + d_2,$$

with some constants d_1, d_2 to be determined from the continuity conditions. We use the limiting value of the integral

$$\lim_{\xi \to +\infty} \int_0^{\xi} e^{-\omega_0 y^2}\, dy = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\omega_0 y^2}\, dy = \frac{1}{2}\sqrt{\frac{\pi}{\omega_0}}.$$

Hence the continuity conditions give

$$\begin{aligned} C_1 &= -\frac{1}{2}\sqrt{\frac{\pi}{\omega_0}} d_1 + d_2, \\ C_2 &= \frac{1}{2}\sqrt{\frac{\pi}{\omega_0}} d_1 + d_2. \end{aligned}$$

This results in the following formula for the internal layer near 0:

$$\begin{aligned} \varphi_\varepsilon(x) &= \sqrt{\frac{\omega_0}{\pi}}(C_2 - C_1) \int_0^{x/\sqrt{\varepsilon}} e^{-\omega_0 y^2}\, dy + \frac{C_1 + C_2}{2} \\ &= \sqrt{\frac{\omega_0}{\pi}}(C_2 - C_1) \int_0^{x} e^{-\frac{\omega_0 y^2}{\varepsilon}}\, dy + \frac{C_1 + C_2}{2}. \end{aligned}$$

Thus, we have determined an approximation of a solution of (5.3) by

$$\Phi_1^{(\delta)}(x) = \begin{cases} C_1, & x < -\delta, \\ \varphi_\varepsilon(x), & x \in [-\delta, \delta], \\ C_2, & x > \delta. \end{cases} \tag{5.6}$$

Let us now specify asymptotic values for the constants C_1 and C_2. First, we normalise $\Phi_1^{(\delta)}$ by putting $C_2 = 1$. Second, we use the orthogonality condition

$$(\Phi_0, \Phi_1^{(\delta)})_{\mathcal{L}^2(\mathbb{R}, \rho^{-1}dx)} = 0,$$

which is equivalent to

$$\int_{\mathbb{R}} \Phi_1^{(\delta)}(y) e^{-\frac{2U(y)}{\varepsilon}}\, dy = 0.$$

Let us rewrite the latter equation as

$$C_1 \int_{-\infty}^{-\delta} e^{-\frac{2U(y)}{\varepsilon}}\, dy + \int_{-\delta}^{\delta} \varphi_\varepsilon(y) e^{-\frac{2U(y)}{\varepsilon}} + \int_{\delta}^{\infty} e^{-\frac{2U(y)}{\varepsilon}}\, dy = 0. \tag{5.7}$$

Using Laplace's method (see Appendix) we asymptotically evaluate the first and the third summands in the previous formula. The second summand can be estimated by

$$\left| \int_{-\delta}^{\delta} \varphi_\varepsilon(y) e^{-\frac{2U(y)}{\varepsilon}} \right| \leq 2\delta(|C_1| + 1) e^{-\frac{2}{\varepsilon} \min_{[-\delta,\delta]} U} = \mathcal{O}(e^{\frac{\delta'}{\varepsilon}})$$

for some $\delta' > 0$, which can be made arbitrary small by reducing δ. Equation (5.7) takes the form

$$C_1 \sqrt{\frac{\pi\varepsilon}{\omega_-}} e^{\frac{V}{\varepsilon}} (1 + \mathcal{O}(\varepsilon)) + \mathcal{O}(e^{\frac{\delta'}{\varepsilon}}) + \sqrt{\frac{\pi\varepsilon}{\omega_+}} e^{\frac{v}{\varepsilon}} (1 + \mathcal{O}(\varepsilon)) = 0.$$

This leads to the following value of C_1

$$C_1 \approx -\sqrt{\frac{\omega_-}{\omega_+}} e^{-\frac{V-v}{\varepsilon}}.$$

So finally we obtained some heuristic knowledge of asymptotic properties of the first eigenfunction Φ_1 in the small noise limit (see Fig. 5.1). It is constant and exponentially small in ε for negative x, equals 1 for positive x, and there exists an internal layer which makes Φ_1 smooth on the whole axis. Note that our arguments to this point just used the information that λ_1 is exponentially small. We can continue along these lines to heuristically deduce its pre-exponential order, using our knowledge of Φ_1. We apply the variational principle to obtain

$$\lambda_1 = \frac{\varepsilon}{2} \frac{\int_{\mathbb{R}} (\Phi_1'(y))^2 e^{-\frac{2U(y)}{\varepsilon}}\, dy}{\int_{\mathbb{R}} (\Phi_1(y))^2 e^{-\frac{2U(y)}{\varepsilon}}\, dy}.$$

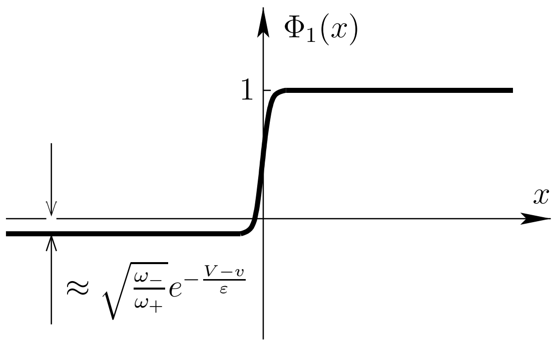

Fig. 5.1: The first eigenfunction Φ_1 of the operator L_ε for small ε.

A straightforward application of Laplace's method gives

$$\int_{\mathbb{R}} (\Phi_1'(y))^2 e^{-\frac{2U(y)}{\varepsilon}}\, dy = \frac{\omega_0}{\pi}(1-C_1)^2 \int_{-\delta}^{\delta} e^{-\frac{\omega_0 y^2}{\varepsilon}}\, dy \approx \frac{\omega_0}{\pi}\sqrt{\frac{\pi\varepsilon}{\omega_0}} = \sqrt{\frac{\omega_0\varepsilon}{\pi}},$$

$$\int_{\mathbb{R}} (\Phi_1(y))^2 e^{-\frac{2U(y)}{\varepsilon}}\, dy \approx \sqrt{\frac{\pi\varepsilon}{\omega_+}} e^{\frac{v}{\varepsilon}},$$

and therefore finally

$$\lambda_1 \approx \frac{\sqrt{\omega_0\omega_+}}{2\pi} e^{-\frac{v}{\varepsilon}}.$$

This result is in agreement with the Freidlin-Wentzell theory and Theorem 5.4.1 by S. Jacquot as far as the exponential order of the related quantities is concerned. Furthermore, we have also found a constant pre-factor. This result reminds us of Kramers' rate [38] describing the probability with which a Brownian particle overcomes a potential barrier of height $\frac{v}{2}$.

The argument presented in this section is not rigorous. The existence of the discrete spectrum of L is assumed and not proved. The passage from (5.3) to (5.4) and (5.5) is mathematically not sound. The use of internal layers near critical points also has to be made precise. The mathematical justification of these qualitatively correct results will be the main task of the following sections.

5.2 A Schrödinger operator and the discreteness of its spectrum

On the domain $\mathcal{D}_0 = \mathcal{C}_0^\infty(\mathbb{R})$ we consider the operator $(L, \mathcal{D}_0)$, where L is the differential operator of second order

$$L = \frac{\varepsilon}{2}\frac{d^2}{dx^2} - U'(x)\frac{d}{dx}.$$

The double-well potential U is supposed to fulfil the properties (**S**), (**G**) and (**M**) described in the previous section.

As was already mentioned above, the formal adjoint of the operator $(L, \mathcal{D}_0)$ with respect to $\mathcal{L}^2(\mathbb{R})$ is given by the differential operator

$$L^* = \frac{\varepsilon}{2}\frac{d^2}{dx^2} + U'(x)\frac{d}{dx} + U''(x)$$

on the domain $\mathcal{D}_0$. Indeed, for any $f, g \in \mathcal{C}_0^\infty(\mathbb{R})$ integration by parts yields

$$\begin{aligned}(Lf, g) &= \int_{\mathbb{R}} Lf \cdot g\, dx = \int_{\mathbb{R}} (\frac{\varepsilon}{2}f'' - U'f')g\, dx \\ &= -\frac{\varepsilon}{2}\int_{\mathbb{R}} f'g'\, dx + \int_{\mathbb{R}} f(U'g' + U''g)\, dx \\ &= \int_{\mathbb{R}} f(\frac{\varepsilon}{2}g'' + U'g' + U''g)\, dx = \int_{\mathbb{R}} f \cdot L^*g\, dx = (f, L^*g).\end{aligned}$$

A simple straightforward calculation shows that these operators are related in the following way

$$Lf = e^{\frac{2U}{\varepsilon}} L^*(fe^{-\frac{2U}{\varepsilon}}), \quad f, g \in \mathcal{D}_0.$$

The operator $(L, \mathcal{D}_0)$ is symmetric in $\mathcal{L}^2(\mathbb{R}, e^{-\frac{2U}{\varepsilon}}dx)$, the operator $(L^*, \mathcal{D}_0)$ is symmetric in $\mathcal{L}^2(\mathbb{R}, e^{\frac{2U}{\varepsilon}}dx)$, so they can be closed in these spaces. We denote the closures as $(L, \mathcal{D}_L)$ and $(L^*, \mathcal{D}_{L^*})$. In order to determine the spectral properties of the operators we consider still another operator $(l, \mathcal{D}_0)$ with

$$l = \frac{d^2}{dx^2} - w(x),$$

where the function

$$w = \frac{(U')^2}{\varepsilon^2} - \frac{U''}{\varepsilon} \in \mathcal{C}^\infty(\mathbb{R}). \tag{5.8}$$

The operator $(l, \mathcal{D}_0)$ is a one-dimensional Schrödinger operator. The function w is therefore called Schrödinger potential.

The Schrödinger operator $(l, \mathcal{D}_0)$ is symmetric in $\mathcal{L}^2(\mathbb{R})$, i.e.

$$(lf, g) = \int_{\mathbb{R}} lf \cdot g\, dx = \int_{\mathbb{R}} f \cdot lg\, dx = (f, lg), \quad f, g \in \mathcal{D}_0.$$

The set $\mathcal{C}_0^\infty$ is dense in $\mathcal{L}^2(\mathbb{R})$, therefore $(l, \mathcal{D}_0)$ can be closed in $\mathcal{L}^2(\mathbb{R})$. We denote the closure as $(l, \mathcal{D}_l)$.

In what follows, for the sake of brevity, we refer to the closures $(l, \mathcal{D}_l)$, $(L, \mathcal{D}_L)$ and $(L^*, \mathcal{D}_{L^*})$ as l, L and L^* respectively.

The Schrödinger operator l is related to L and L^* by the formulae

$$lf = \frac{2}{\varepsilon} e^{-\frac{U}{\varepsilon}} L(f e^{\frac{U}{\varepsilon}}), \tag{5.9}$$

$$lf = \frac{2}{\varepsilon} e^{\frac{U}{\varepsilon}} L^*(f e^{-\frac{U}{\varepsilon}}), \quad f \in \mathcal{D}_l. \tag{5.10}$$

These equations imply that the eigenvalues of l have to be multiplied by $\frac{\varepsilon}{2}$ to obtain those of L or L^*, and the eigenfunctions of L and L^* can be obtained from those of l by multiplying by $e^{\frac{U}{\varepsilon}}$ or $e^{-\frac{U}{\varepsilon}}$ respectively.

The Schrödinger potential w is smooth on the whole line, and

$$w(x) \to \infty, \quad |x| \to \infty. \tag{5.11}$$

More precisely, $w(x) = \frac{x^6}{\varepsilon^2} - \frac{3x^2}{\varepsilon} \approx \frac{x^6}{\varepsilon^2}$, as $|x| \to \infty$.

The problem of determining the spectrum of l is called a *singular Sturm-Liouville problem for the operator with increasing potential.* The theory for these problems is well developed. We formulate the main facts concerning the spectral properties of l following [41, 6, 29, 19, 18].

The following theorem by Sears gives a sufficient condition for l to be self-adjoint.

Theorem 5.2.1 ([54, 6]) *Let w satisfy*

$$w(x) \geq -Q(x), \quad x \in \mathbb{R},$$

where Q is a positive, even and continuous function on $\mathbb{R}$ that is non-decreasing on $[0, \infty)$ and satisfies

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{Q(2x)}} = \infty.$$

Then $(l, \mathcal{D}_0)$ is essentially self-adjoint (that is, its closure l is self-adjoint).

The conditions for the Sears theorem obviously hold for the potential w given by (5.8) due to assumption (**G**). Indeed, w is continuous and increases at infinity. So Q can be chosen, for example, to be identical to the constant $Q = |\min_{x\in\mathbb{R}} w(x)| + 1$.

Moreover, we can state that the domain D_l consists of all continuous square-integrable functions f with absolutely continuous first derivative on any finite interval, such that $f'' - w(x)f$ is also square-integrable on the whole line. From this also follows that f' is also square-integrable function on $\mathbb{R}$, see [41].

The general theory of self-adjoint operators imply that the eigenvalues of l are real and the eigenfunctions corresponding to different eigenvalues are orthogonal in $\mathcal{L}^2(\mathbb{R})$.

The *regular case* of the Sturm-Liouville problem is well understood. On the closed finite interval $[a, b]$, the operator l under the boundary conditions

$y(a)\cos\alpha + y'(a)\sin\alpha = 0$, $y(b)\cos\beta + y'(b)\sin\beta = 0$ has a complete orthonormal system of eigenfunctions f_k, $k \geq 0$. The corresponding eigenvalues a_k are simple, $a_0 > a_1 > \cdots$, $a_k \to -\infty$ as $k \to \infty$, and f_k has exactly k simple zeros on the open interval (a, b).

More generally, in the *singular case*, the operator l may have a continuous spectrum, but the increase of the potential w at infinity guaranteed by condition (5.11) makes the eigenvalue problem very similar to the regular one.

Theorem 5.2.2 ([6]) *If* (5.11) *is satisfied, then l has a discrete spectrum. More precisely, there is a complete orthonormal system of eigenfunctions y_k of l belonging to $\mathcal{L}^2(\mathbb{R})$ whose eigenvalues μ_k, $k \geq 0$, tend to $-\infty$ as $k \to \infty$.*

Theorem 5.2.3 ([6]) *Let* (5.11) *be satisfied. If μ is an eigenvalue of l, then its eigenspace is one-dimensional.*

Theorem 5.2.4 ([6]) *Let* (5.11) *be satisfied. Then every eigenfunction of l has a finite number of zeros. Moreover, if y_1, y_2 are eigenfunctions with eigenvalues μ_1, μ_2 and n_1, n_2 are the numbers of zeros, then $\mu_2 < \mu_1$ implies $n_2 > n_1$.*

Let the eigenfunctions

$$y_0,\, y_1,\, y_2, \ldots, y_k, \ldots \tag{5.12}$$

be arranged in order of increasing number of their roots, that is, if n_k is the number of zeros of y_k, then

$$n_0 < n_1 < n_2 < \cdots < n_k < \cdots \tag{5.13}$$

The corresponding eigenvalues satisfy

$$\mu_0 > \mu_1 > \mu_2 > \cdots > \mu_k > \cdots \tag{5.14}$$

Theorem 5.2.5 ([6]) *Let* (5.11) *be satisfied. Arrange the eigenfunctions* (5.12) *so that* (5.13) *and* (5.14) *hold. Then y_k has precisely k zeros, that is, in* (5.13) *we have $n_k = k$.*

The above formulated statements are easily transferred to the operators L and L^* with the help of (5.9) and (5.10). More precisely, the $\mathcal{L}^2(\mathbb{R}, e^{-\frac{2U}{\varepsilon}}dx)$-closure of L is self-adjoint on the domain $\mathcal{D}_L$ which consists of all continuous functions f from $\mathcal{L}^2(\mathbb{R}, e^{-\frac{2U}{\varepsilon}}dx)$, such that f' is locally absolutely continuous and $f'' - w(x)f$ is also in $\mathcal{L}^2(\mathbb{R}, e^{-\frac{2U}{\varepsilon}}dx)$.

L has a discrete spectrum $\{-\lambda_k\}_{k\geq 0}$ and its eigenfunctions $\{\Phi_k\}_{k\geq 0}$ are orthogonal in $\mathcal{L}^2(\mathbb{R}, e^{-\frac{2U}{\varepsilon}}dx)$, that is

$$\left(\frac{\Phi_i}{\|\Phi_i\|_{\rho^{-1}}}, \frac{\Phi_j}{\|\Phi_j\|_{\rho^{-1}}}\right)_{\rho^{-1}} = \int_{\mathbb{R}} \frac{\Phi_i(y)}{\|\Phi_i\|_{\rho^{-1}}} \frac{\Phi_j(y)}{\|\Phi_j\|_{\rho^{-1}}} e^{-\frac{2U(y)}{\varepsilon}}\, dy = \delta_{ij}, \quad i, j = 0, 1, 2, \ldots$$

It will turn out to be convenient for us to consider non-normalised eigenfunctions.

The operator L^* has the same eigenvalues $\{-\lambda_k\}_{k\geq 0}$ and its eigenfunctions $\Psi_k = e^{-\frac{2U}{\varepsilon}}\Phi_k$ are orthogonal in the space $\mathcal{L}^2(\mathbb{R}, e^{\frac{2U}{\varepsilon}}dx)$.

It is necessary to mention that the diffusion operator L is negatively definite, i.e. $(Lf, f)_{\rho^{-1}} \leq 0$. Indeed, by using integration by parts on $\mathcal{D}_L$ one can obtain

$$(Lf, f)_{\rho^{-1}} = \int_{\mathbb{R}} (\frac{\varepsilon}{2}f'' - U'f')fe^{-\frac{2U}{\varepsilon}}\,dy = -\frac{\varepsilon}{2}\int_{\mathbb{R}} (f')^2 e^{-\frac{2U}{\varepsilon}}\,dy \leq 0. \tag{5.15}$$

The results of this section give us only general information about the self-adjointness and spectrum of L. In the next section we formulate two variational principles which allow us to determine the first two eigenvalues and eigenfunctions of L (and consequently of L^*) for small values of the parameter ε.

5.3 Variational principles for the eigenvalues

Denote $\{\Phi_k, -\lambda_k\}_{k\geq 0}$ the eigensystem of the operator L, where the real numbers λ_k are arranged in increasing order. The eigenvalues and eigenfunctions of L can be obtained as solutions of the following variational problems.

Theorem 5.3.1 (Recursive definition, [29]) *The zeroth eigenvalue and eigenfunction of L satisfy*

$$\lambda_0 = \frac{(-L\Phi_0, \Phi_0)_{\rho^{-1}}}{(\Phi_0, \Phi_0)_{\rho^{-1}}} = \inf_{\substack{f\in\mathcal{D}_L\\ f\neq 0}} \frac{(-Lf, f)_{\rho^{-1}}}{(f, f)_{\rho^{-1}}}. \tag{5.16}$$

For $k \geq 1$ we have

$$\begin{aligned}
\lambda_1 &= \frac{(-L\Phi_1, \Phi_1)_{\rho^{-1}}}{(\Phi_1, \Phi_1)_{\rho^{-1}}} = \inf_{\substack{f\in\mathcal{D}_L\\ (\Phi_0,f)_{\rho^{-1}}=0\\ f\neq 0}} \frac{(-Lf, f)_{\rho^{-1}}}{(f, f)_{\rho^{-1}}},\\
&\dots\\
\lambda_k &= \frac{(-L\Phi_k, \Phi_k)_{\rho^{-1}}}{(\Phi_k, \Phi_k)_{\rho^{-1}}} = \inf_{\substack{f\in\mathcal{D}_L\\ (\Phi_j,f)_{\rho^{-1}}=0\\ 0\leq j\leq k-1\\ f\neq 0}} \frac{(-Lf, f)_{\rho^{-1}}}{(f, f)_{\rho^{-1}}}.
\end{aligned} \tag{5.17}$$

Theorem 5.3.2 (Courant's minimax principle, [6, 18]) *The eigenvalues of the operator L satisfy*

$$\begin{aligned}
\lambda_0 &= \inf_{\substack{f\in\mathcal{D}_L\\ f\neq 0}} \frac{(-Lf, f)_{\rho^{-1}}}{(f, f)_{\rho^{-1}}},\\
\lambda_k &= \sup_{\substack{M_k\\ \dim M_k = k}} \inf_{\substack{f\in\mathcal{D}_L\cap M_k^{\perp}\\ 0\leq j\leq k\\ f\neq 0}} \frac{(-Lf, f)_{\rho^{-1}}}{(f, f)_{\rho^{-1}}}, \quad k\geq 1,
\end{aligned} \tag{5.18}$$

where M_k is a subspace in $\mathcal{L}^2(\mathbb{R}, \rho^{-1}dx)$, $M_k^{\perp}$ denotes the orthogonal complement of M_k in $\mathcal{L}^2(\mathbb{R}, \rho^{-1}dx)$.

Remark 5.3.1 The variational principles can be also formulated in terms of the operator $(L, \mathcal{D}_0)$. In this case we have to consider the inf over $\mathcal{D}_0 = \mathcal{C}_0^\infty(\mathbb{R})$ instead of $\mathcal{D}_L$, that is, over a domain on which $(L, \mathcal{D}_0)$ is essentially self-adjoint.

Using Theorem 5.3.1 we easily determine the zeroth eigenfunction and eigenvalue of L.

Theorem 5.3.3 *The zeroth eigenvalue of the operator L equals* 0*, and the corresponding eigenfunction is given by* $\Phi_0 = 1$.

Proof: It follows from (5.15) for any function in $\mathcal{D}_L$

$$\frac{(-Lf, f)_{\rho^{-1}}}{(f, f)_{\rho^{-1}}} = \frac{\varepsilon}{2}\frac{(f', f')_{\rho^{-1}}}{(f, f)_{\rho^{-1}}} \geq 0$$

and consequently $\lambda_0 \geq 0$. Taking $f = \text{Const}$ yields $\lambda_0 = 0$. Hence, any constant function can be taken as the zeroth eigenfunction of L. For convenience we choose $\Phi_0 = 1$. ■

Recalling that the eigenfunctions Ψ_k of the operator L^* are expressed in terms of Φ_k as $\Psi_k = e^{-\frac{2U}{\varepsilon}}\Phi_k$ we conclude that $\Psi_0 = e^{-\frac{2U}{\varepsilon}}$. It is helpful to remark that

$$\mu_0(x) = \frac{\Psi_0(x)}{\|\Psi_0\|_\rho^2} = \frac{e^{-\frac{2U(x)}{\varepsilon}}}{\int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\, dy}, \quad x \in \mathbb{R},$$

is the invariant density of the diffusion (5.1).

5.4 The first eigenfunction and eigenvalue of L

In this section we shall investigate the first eigenvalue λ_1 and the first eigenfunction Φ_1 for small values of ε. Our study is again based on Theorem 5.3.1. The first eigenfunction of L is uniquely determined up to a normalizing factor by the equation

$$\frac{\varepsilon}{2}\varphi''(x) - U'(x)\varphi'(x) = -\lambda_1\varphi(x), \quad x \in \mathbb{R}, \tag{5.19}$$

and its orthogonality to Φ_0 in $\mathcal{L}^2(\mathbb{R}, \rho^{-1})$, i.e.

$$\int_{\mathbb{R}} \varphi(y)e^{-\frac{2U(y)}{\varepsilon}}\, dy = 0. \tag{5.20}$$

We also have some a priori information about the behaviour of λ_1.

Theorem 5.4.1 ([35]) *Let the potential U satisfy the conditions* (**S**), (**G**) *and* (**M**). *Then*

$$\lim_{\varepsilon\to 0}\varepsilon\log\lambda_1 = -v.$$

This means that for any $v' < v$ we have $\lambda_1 = o(e^{-v'/\varepsilon})$ as $\varepsilon \to 0$.

As we see, in the small noise limit λ_1 is the smallest parameter in (5.19). This suggests that it can be possible to look for the solution of (5.19) in the form of a power series in λ_1.

5.4.1 Series solution for Φ_1

We look for the solution of (5.19) in the form of a power series in the exponentially small parameter λ_1, formally

$$\varphi = \sum_{k=0}^{\infty}\varphi_k\lambda_1^k. \tag{5.21}$$

Substituting (5.21) into (5.19) induces ordinary differential equations for φ_k. More precisely, φ_0 satisfies the homogeneous equation

$$\frac{\varepsilon}{2}\varphi_0''(x) - U'(x)\varphi_0'(x) = 0, \quad x\in\mathbb{R} \tag{5.22}$$

and for $k \geq 1$

$$\frac{\varepsilon}{2}\varphi_k''(x) - U'(x)\varphi_k'(x) = -\varphi_{k-1}(x), \quad x\in\mathbb{R} \tag{5.23}$$

The heuristics of Section 5.1 suggests that the first eigenfunction takes constant values outside of some neighbourhood of zero, and there is an internal layer which glues these constant parts together (see Fig. 5.1). Recall also that the first approximation of the eigenfunction (5.6) gave us good asymptotics of the first eigenvalue.

We first show that for arbitrary constants a and b there exists a solution of (5.19) such that $\varphi(-1) = a$, $\varphi(1) = b$, $\varphi\in\mathcal{L}^2(\mathbb{R},\rho^{-1}dx)$. It subsequently turns out that the constants a and b can be chosen in such a way that the orthogonality condition (5.20) holds. This yields the first eigenfunction Φ_1.

In other words we look for the solution of the problem

$$\begin{cases}\dfrac{\varepsilon}{2}\varphi''(x) - U'(x)\varphi'(x) = -\lambda_1\varphi(x), \quad x\in\mathbb{R},\\ \varphi(-1) = a,\ \varphi(1) = b,\\ \varphi\in\mathcal{L}^2(\mathbb{R},\rho^{-1}dx).\end{cases} \tag{5.24}$$

For any $k\geq 0$ let us recursively determine functions h_k, f_k and g_k defined on the intervals $(-\infty,-1]$, $[-1,1]$ and $[1,+\infty)$ respectively such that for $x\in\mathbb{R}$

$$\varphi_k(x) = h_k(x)\mathbf{I}_{(-\infty,-1]}(x) + f_k(x)\mathbf{I}_{[-1,1]}(x) + g_k(x)\mathbf{I}_{[1,+\infty)}(x). \tag{5.25}$$

Then, (5.22) can be rewritten in the form

$$\begin{cases} \frac{\varepsilon}{2} h_0'' - U' h_0' = 0, \\ \text{on } (-\infty, -1], \\ h_0(-1) = a, \\ h_0 \in \mathcal{L}^2((-\infty, -1], \rho^{-1} dx); \end{cases} \begin{cases} \frac{\varepsilon}{2} f_0'' - U' f_0' = 0, \\ \text{on } [-1, 1], \\ f_0(-1) = a, \\ f_0(1) = b; \end{cases} \begin{cases} \frac{\varepsilon}{2} g_0'' - U' g_0' = 0, \\ \text{on } [1, +\infty), \\ g_0(1) = b, \\ g_0 \in \mathcal{L}^2([1, +\infty), \rho^{-1} dx). \end{cases} \tag{5.26}$$

The solution of (5.26) allows us to assemble the first approximation φ_0 which is continuous and lies in $\mathcal{L}^2(\mathbb{R}, \rho^{-1} dx)$. The higher order terms of (5.21) are obtained by solving (5.23) on the three domains which results in the following boundary value problems for $k \geq 1$:

$$\begin{cases} \frac{\varepsilon}{2} h_k'' - U' h_k' = -h_{k-1}, \\ \text{on } (-\infty, -1], \\ h_k(-1) = 0, \\ h_k \in \mathcal{L}^2((-\infty, -1], \rho^{-1} dx); \end{cases} \begin{cases} \frac{\varepsilon}{2} f_k'' - U' f_k' = -f_{k-1}, \\ \text{on } [-1, 1], \\ f_k(-1) = 0, \\ f_k(1) = 0; \end{cases} \begin{cases} \frac{\varepsilon}{2} g_k'' - U' g_k' = -g_{k-1}, \\ \text{on } [1, +\infty), \\ g_k(1) = 0, \\ g_k \in \mathcal{L}^2([1, +\infty), \rho^{-1} dx). \end{cases} \tag{5.27}$$

Lemma 5.4.1 *For any $a, b \in \mathbb{R}$ the problem* (5.26) *has the following solution*

$$\begin{aligned} h_0(x) &= a, \quad x \in (-\infty, -1], \\ f_0(x) &= a + (b-a) \frac{\int_{-1}^{x} e^{\frac{2U(y)}{\varepsilon}} dy}{\int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}} dy}, \quad x \in [-1, 1], \\ g_0(x) &= b, \quad x \in [1, +\infty). \end{aligned}$$

We have $\max_{x \in [-1,1]} |f_0(x)| \leq |b - a|$.

Proof: The statement of the lemma easily follows from the formula for the general solution of equation (5.22) which is

$$\varphi_0(x) = A + B \int_0^x e^{\frac{2U(y)}{\varepsilon}} dy,$$

with constants $A, B \in \mathbb{R}$. On the interval $[-1, 1]$ the constants A and B have to be appropriately chosen for the boundary conditions to hold. On the infinite intervals B must equal zero; otherwise the corresponding solution increases too fast at infinity and does not belong to the corresponding $\mathcal{L}^2$ space. The estimate for $|f_0|$ follows from the monotonicity of f_0. ■

Let $-1 < x_- < 0 < x_+ < 1$ be such that $U(x_-) = -\frac{V}{3}$ and $U(x_+) = -\frac{v}{3}$. The existence of these points follows from the geometry of U.

Lemma 5.4.2 *There exist constants $c > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ the following inequalities hold*

$$\max_{x\in[-1,x_-]} |f_0(x) - a| \le c\sqrt{\varepsilon}|b-a|e^{-\frac{2V}{3\varepsilon}},$$
$$\max_{x\in[x_+,1]} |f_0(x) - b| \le c\sqrt{\varepsilon}|b-a|e^{-\frac{2v}{3\varepsilon}}$$

Proof: Laplace's method (see Appendix) yields for small ε that

$$\int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}}\, dy = \sqrt{\frac{\pi\varepsilon}{\omega_0}}(1+\mathcal{O}(\varepsilon))$$

and

$$\int_{-1}^{x_-} e^{\frac{2U(y)}{\varepsilon}}\, dy = \frac{\varepsilon}{2U'(x_-)} e^{-\frac{2V}{3\varepsilon}}(1+\mathcal{O}(\sqrt{\varepsilon})).$$

By definition of f_0, this results in the following inequality

$$\max_{x\in[-1,x_-]} |f_0(x) - a| \le |b-a|e^{-\frac{2V}{3\varepsilon}} \frac{\varepsilon}{2U'(x_-)}\sqrt{\frac{\omega_0}{\pi\varepsilon}}(1+\mathcal{O}(\sqrt{\varepsilon})) \le c_1\sqrt{\varepsilon}|b-a|e^{-\frac{2V}{3\varepsilon}}$$

for some $c_1 > 0$. A similar inequality holds on the interval $[x_+, 1]$ for some positive constant c_2. Taking $c = \max\{c_1, c_2\}$ completes the proof. ∎

Lemma 5.4.3 *Let w be a bounded continuous function on $\mathbb{R}$. Then there exists $C > 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$*

$$\sup_{x\in(-\infty,-1]} \left| \int_x^{-1} e^{\frac{2U(y)}{\varepsilon}} \int_{-\infty}^{y} e^{-\frac{2U(z)}{\varepsilon}} w(z)\, dz\, dy \right| \le C\sqrt{\varepsilon} \sup_{x\in(-\infty,-1]} |w(x)|, \tag{5.28}$$
$$\sup_{x\in[-1,1]} \left| \int_0^{x} e^{\frac{2U(y)}{\varepsilon}} \int_{0}^{y} e^{-\frac{2U(z)}{\varepsilon}} w(z)\, dz\, dy \right| \le \sup_{x\in[-1,1]} |w(x)|, \tag{5.29}$$
$$\sup_{x\in[1,+\infty)} \left| \int_1^{x} e^{\frac{2U(y)}{\varepsilon}} \int_{y}^{\infty} e^{-\frac{2U(z)}{\varepsilon}} w(z)\, dz\, dy \right| \le C\sqrt{\varepsilon} \sup_{x\in[1,\infty)} |w(x)|. \tag{5.30}$$

Proof: We start with the proof of (5.28). First, we notice that

$$\left| \int_x^{-1} e^{\frac{2U(y)}{\varepsilon}} \int_{-\infty}^{y} e^{-\frac{2U(z)}{\varepsilon}} w(z)\, dz\, dy \right| \le \sup_{x\in(-\infty,-1]} |w(x)| \int_x^{-1} e^{\frac{2U(y)}{\varepsilon}} \int_{-\infty}^{y} e^{-\frac{2U(z)}{\varepsilon}}\, dz\, dy.$$

The function $v(x) = \int_x^{-1} e^{\frac{2U(y)}{\varepsilon}} \int_{-\infty}^{y} e^{-\frac{2U(z)}{\varepsilon}}\, dz\, dy$ is non-negative and decreasing on $[-\infty, -1]$, hence $v(x) \le v(-\infty)$. Moreover, for $\delta > 0$ we have

$$v(-\infty) = \int_{-1-\delta}^{-1} v'(y)\, dy + \int_{-\infty}^{-1-\delta} v'(y)\, dy \le \max_{x\in[-1-\delta,-1]} v(x) + \int_{-\infty}^{-1-\delta} v'(y)\, dy,$$

where $v'(x) = e^{\frac{2U(x)}{\varepsilon}} \int_{-\infty}^{x} e^{-\frac{2U(y)}{\varepsilon}}\, dy$, $x \in (-\infty, -1]$. For $x \in [-1-\delta, -1]$ we have, using Laplace's method for $0 < \varepsilon \le \varepsilon_0$

$$v(x) = v'(-1)(x+1) + \mathcal{O}(x+1) = e^{-\frac{V}{\varepsilon}} e^{\frac{V}{\varepsilon}} \sqrt{\frac{\pi\varepsilon}{\omega_-}}(1+\mathcal{O}(\sqrt{\varepsilon}))(x+1) + \mathcal{O}(x+1) \le c_1\sqrt{\varepsilon}$$

for some $c_1 > 0$.

For x near $-\infty$ we have $U(x) = \frac{x^4}{4}$. Applying de l'Hôspital's rule yields

$$v'(x) = \frac{\int_{-\infty}^{x} e^{-\frac{2U(y)}{\varepsilon}}\, dy}{e^{-\frac{2U(x)}{\varepsilon}}} \approx \frac{e^{-\frac{2U(x)}{\varepsilon}}}{-\frac{2}{\varepsilon}U'(x)e^{-\frac{2U(x)}{\varepsilon}}} = \frac{\varepsilon}{2|x|^3}, \quad \text{as} \quad x \to -\infty,$$

hence $\int_{-\infty}^{-1-\delta} v'(y)\, dy \le c_2\varepsilon$ for some $c_2 > 0$. Therefore, (5.28) is proved with the constant $C_1 = \max\{c_1, c_2\}$.

To show (5.29) we note that $U(y) - U(z) \le 0$ for $-1 \le y \le z \le 0$ and for $0 \le z \le y \le 1$. Hence

$$\begin{aligned}
&\left| \int_0^x e^{\frac{2U(y)}{\varepsilon}} \int_0^y e^{-\frac{2U(z)}{\varepsilon}} w(z)\, dz\, dy \right| \le \\
&\sup_{x\in[-1,1]} |w(x)| \left(\int_{-1}^0 \int_y^0 e^{\frac{2(U(y)-U(z))}{\varepsilon}}\, dz\, dy + \int_0^1 \int_0^y e^{\frac{2(U(y)-U(z))}{\varepsilon}}\, dz\, dy \right) \le \\
&\sup_{x\in[-1,1]} |w(x)| \left(\int_{-1}^0 \int_y^0 dz\, dy + \int_0^1 \int_0^y dz\, dy \right) \le \sup_{x\in[-1,1]} |w(x)|.
\end{aligned}$$

Analogously to (5.28), the inequality (5.30) holds for some positive constant C_2. Taking $C = \max\{C_1, C_2\}$ completes the proof. ■

Lemma 5.4.4 *For $k \ge 1$ the problem* (5.27) *has the following recursively defined solution*

$$h_k(x) = \frac{2}{\varepsilon} \int_x^{-1} e^{\frac{2U(y)}{\varepsilon}} \int_{-\infty}^{y} e^{-\frac{2U(z)}{\varepsilon}} h_{k-1}(z)\, dz\, dy, \quad x \in (-\infty, -1], \tag{5.31}$$

$$\begin{aligned}
f_k(x) = &-\frac{2}{\varepsilon} \int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}} \int_0^y e^{-\frac{2U(z)}{\varepsilon}} f_{k-1}(z)\, dz\, dy \cdot \frac{\int_x^1 e^{\frac{2U(y)}{\varepsilon}}\, dy}{\int_{-1}^1 e^{\frac{2U(y)}{\varepsilon}}\, dy} \\
&+ \frac{2}{\varepsilon} \int_x^1 e^{\frac{2U(y)}{\varepsilon}} \int_0^y e^{-\frac{2U(z)}{\varepsilon}} f_{k-1}(z)\, dz\, dy, \quad x \in [-1, 1],
\end{aligned} \tag{5.32}$$

$$g_k(x) = \frac{2}{\varepsilon} \int_1^x e^{\frac{2U(y)}{\varepsilon}} \int_y^{\infty} e^{-\frac{2U(z)}{\varepsilon}} h_{k-1}(z)\, dz\, dy, \quad x \in [1, +\infty). \tag{5.33}$$

Moreover, for $0 < \varepsilon \le \varepsilon_0$

$$|h_k(x)| \le |a| \left(\frac{2C}{\sqrt{\varepsilon}}\right)^k, \quad x \in (-\infty, -1], \tag{5.34}$$

$$|f_k(x)| \le |b-a| \left(\frac{2}{\varepsilon}\right)^k, \quad x \in [-1, 1], \tag{5.35}$$

$$|g_k(x)| \le |b| \left(\frac{2C}{\sqrt{\varepsilon}}\right)^k, \quad x \in [1, +\infty), \tag{5.36}$$

where ε_0 *and* C *are as in Lemma 5.4.3.*

Proof: The general solution of equation (5.23) is given by

$$\varphi_k(x) = A + B \int_0^x e^{\frac{2U(y)}{\varepsilon}}\, dy - \frac{2}{\varepsilon} \int_0^x e^{\frac{2U(y)}{\varepsilon}} \int_0^y e^{-\frac{2U(z)}{\varepsilon}} \varphi_{k-1}(z)\, dz\, dy,$$

with real constants A, B. On the interval $[-1, 1]$ we determine A and B from the boundary conditions $f_k(\pm 1) = 0$ which results in (5.32).

In case $x \in (-\infty, -1]$, the general solution can be written in the form

$$h_k(x) = A + B \int_x^{-1} e^{\frac{2U(y)}{\varepsilon}}\, dy + \frac{2}{\varepsilon} \int_x^{-1} e^{\frac{2U(y)}{\varepsilon}} \int_{-1}^y e^{-\frac{2U(z)}{\varepsilon}} h_{k-1}(z)\, dz\, dy.$$

It follows from the condition $h_k(-1) = 0$ that $A = 0$. The second constant B is determined by the condition that $h_k \in \mathcal{L}^2((-\infty, -1], \rho^{-1} dx)$. Let us write

$$h_k(x) = \frac{2}{\varepsilon} \int_x^{-1} e^{\frac{2U(y)}{\varepsilon}} \left(\frac{\varepsilon}{2} B + \int_{-1}^y e^{-\frac{2U(z)}{\varepsilon}} h_{k-1}(z)\, dz\right) dy$$

Setting $B = \frac{2}{\varepsilon} \int_{-\infty}^{-1} e^{-\frac{2U(z)}{\varepsilon}} h_{k-1}(z)\, dz$, we obtain the solution (5.31), which is bounded by Lemma 5.4.3.

The case of the function g_k on the interval $[1, \infty)$ is studied analogously.

The estimates (5.34), (5.35) and (5.36) are obtained by applying Lemma 5.4.3 k times and recalling the definition of h_0, f_0, g_0. ■

Now we have enough information to study the closeness of the solution φ of (5.19) to the constants a and b and the first approximation φ_0.

Lemma 5.4.5 *There exist* $\varepsilon_0 > 0$ *and* $A > 0$ *such that for* $0 < \varepsilon \le \varepsilon_0$ *the* *following inequalities hold:*

$$\max_{x\in(-\infty,-1]} |\varphi(x)-a| \le A|a|\frac{\lambda_1}{\sqrt{\varepsilon}},$$
$$\max_{x\in[-1,x_-]} |\varphi(x)-a| \le A|b-a|\sqrt{\varepsilon}e^{-\frac{2V}{3\varepsilon}},$$
$$\max_{x\in[x_-,x_+]} |\varphi(x)-f_0(x)| \le A|b-a|\frac{\lambda_1}{\varepsilon},$$
$$\max_{x\in[x_+,1]} |\varphi(x)-b| \le A|b-a|\sqrt{\varepsilon}e^{-\frac{2v}{3\varepsilon}},$$
$$\max_{x\in[1,\infty)} |\varphi(x)-b| \le A|b|\frac{\lambda_1}{\sqrt{\varepsilon}},$$

Proof: Choose $\varepsilon_0 > 0$ small enough according to the previous lemmas and such that $\max\{\frac{2C}{\sqrt{\varepsilon}}\lambda_1, \frac{2}{\varepsilon}\lambda_1\} \le \frac{1}{2}$ for $0 < \varepsilon \le \varepsilon_0$. This is possible by Theorem 5.4.1.

For $x \le -1$, $0 < \varepsilon \le \varepsilon_0$, we obtain using (5.34)

$$\begin{aligned}|\varphi(x)-a| = |h(x)-h_0(x)| &= \left|\sum_{k=1}^{\infty} h_k(x)\lambda_1^k\right| \le \sum_{k=1}^{\infty} |h_k(x)|\lambda_1^k \\ &\le \sum_{k=1}^{\infty} |a|\left(\frac{2C}{\sqrt{\varepsilon}}\right)^k \lambda_1^k = |a|\frac{\frac{2C}{\sqrt{\varepsilon}}\lambda_1}{1-\frac{2C}{\sqrt{\varepsilon}}\lambda_1} \le A_1|a|\frac{\lambda_1}{\sqrt{\varepsilon}}\end{aligned}$$

for some $A_1 > 0$ independent of ε.

For $x \in [-1, x_-]$, $0 < \varepsilon \le \varepsilon_0$, we use Lemma 5.4.2, (5.35), and the assumption that $\frac{2}{3} < \frac{v}{V} < 1$ to obtain

$$\begin{aligned}|\varphi(x)-a| \le |f_0(x)-a| + |f(x)-f_0(x)| &\le c|b-a|\sqrt{\varepsilon}e^{-\frac{2V}{3\varepsilon}} + \sum_{k=1}^{\infty}|f_k(x)|\lambda_1^k \\ &\le c|b-a|\sqrt{\varepsilon}e^{-\frac{2V}{3\varepsilon}} + \sum_{k=1}^{\infty}|b-a|\left(\frac{2}{\varepsilon}\right)^k\lambda_1^k \\ &= c|b-a|\sqrt{\varepsilon}e^{-\frac{2V}{3\varepsilon}} + |b-a|\frac{\frac{2}{\varepsilon}\lambda_1}{1-\frac{2}{\varepsilon}\lambda_1} \le A_2|b-a|\sqrt{\varepsilon}e^{-\frac{2V}{3\varepsilon}}\end{aligned}$$

for some $A_2 > 0$ independent of ε.

The estimates on the remaining intervals are obtained analogously. Choosing A to be the maximal constant appearing in the inequalities we complete the proof of the lemma. ■

Let us now determine the constants a and b in the small noise limit so that $\varphi \perp 1$ in $\mathcal{L}^2(\mathbb{R}, \rho^{-1}dx)$. This will provide a good approximation of the first eigenfunction Φ_1. It is clear that φ can be determined up to a constant factor. For definiteness we put $b = 1$ and look for a.

Lemma 5.4.6 *There exists a function $a : \mathbb{R}_+ \to \mathbb{R}$ such that*

$$a = -\sqrt{\frac{\omega_-}{\omega_+}} e^{-\frac{V-v}{\varepsilon}} (1 + \mathcal{O}(\varepsilon))$$

and $\varepsilon_0 > 0$ such that with $b = 1$ the function Φ_1 chosen according to the previous lemmas for $a(\varepsilon)$ and b is orthogonal to $\Phi_0 = 1$ in $\mathcal{L}^2(\mathbb{R}, \rho^{-1} dx)$ for $0 < \varepsilon \le \varepsilon_0$. In sequel we suppress the dependence of $a(\varepsilon)$ on ε and continue to use the symbol a instead.

Proof: Choose ε_0 according to Lemma 5.4.5 and let $0 < \varepsilon \le \varepsilon_0$. In order to determine a we shall solve two inequalities $\int \Phi_1 \Phi_0 \, e^{-\frac{2U}{\varepsilon}} \, dy \ge 0$ and $\int \Phi_1 \Phi_0 \, e^{-\frac{2U}{\varepsilon}} \, dy \le 0$.

We use the inequalities of Lemma 5.4.5 and Laplace's method to obtain for $\varepsilon \le \varepsilon_0$

$$\begin{aligned}
0 \le & \int_{\mathbb{R}} \Phi_1(y) e^{-\frac{2U(y)}{\varepsilon}} \, dy \le a \int_{-\infty}^{x_-} e^{-\frac{2U(y)}{\varepsilon}} \, dy + \int_{x_-}^{x_+} f_0(y) e^{-\frac{2U(y)}{\varepsilon}} \, dy + \int_{x_+}^{\infty} e^{-\frac{2U(y)}{\varepsilon}} \, dy \\
& + A|a| \frac{\lambda_1}{\sqrt{\varepsilon}} \int_{-\infty}^{-1} e^{-\frac{2U(y)}{\varepsilon}} \, dy + A|1-a| \sqrt{\varepsilon} e^{-\frac{2V}{3\varepsilon}} \int_{-1}^{x_-} e^{-\frac{2U(y)}{\varepsilon}} \, dy \\
& + A|1-a| \frac{\lambda_1}{\varepsilon} \int_{x_-}^{x_+} e^{-\frac{2U(y)}{\varepsilon}} \, dy + A|1-a| \sqrt{\varepsilon} e^{-\frac{2v}{3\varepsilon}} \int_{x_+}^{1} e^{-\frac{2U(y)}{\varepsilon}} \, dy \\
& + A \frac{\lambda_1}{\sqrt{\varepsilon}} \int_{1}^{\infty} e^{-\frac{2U(y)}{\varepsilon}} \, dy \\
\le & \, a \sqrt{\frac{\pi\varepsilon}{\omega_-}} e^{\frac{V}{\varepsilon}} (1 + \mathcal{O}(\varepsilon)) + (1 + |a|) \frac{\varepsilon}{2U'(x_-)} e^{\frac{2V}{3\varepsilon}} (1 + \mathcal{O}(\varepsilon)) + \sqrt{\frac{\pi\varepsilon}{\omega_+}} e^{\frac{v}{\varepsilon}} (1 + \mathcal{O}(\varepsilon)) \\
& + A|a| \frac{\lambda_1}{2} \sqrt{\frac{\pi}{\omega_-}} e^{\frac{V}{\varepsilon}} (1 + \mathcal{O}(\sqrt{\varepsilon})) + A(1 + |a|) \frac{\varepsilon}{2} e^{-\frac{2V}{3\varepsilon}} \sqrt{\frac{\pi}{\omega_-}} e^{\frac{V}{\varepsilon}} (1 + \mathcal{O}(\sqrt{\varepsilon})) \\
& + A(1 + |a|) \frac{\lambda_1}{2U'(x_-)} e^{\frac{2V}{3\varepsilon}} (1 + \mathcal{O}(\varepsilon)) + A(1 + |a|) \frac{\varepsilon}{2} e^{-\frac{2v}{3\varepsilon}} \sqrt{\frac{\pi}{\omega_+}} e^{\frac{v}{\varepsilon}} (1 + \mathcal{O}(\sqrt{\varepsilon})) \\
& + A \frac{\lambda_1}{2} \sqrt{\frac{\pi}{\omega_+}} e^{\frac{v}{\varepsilon}} (1 + \mathcal{O}(\sqrt{\varepsilon})).
\end{aligned}$$

One can notice that all terms having $|a|$ as pre-factor are of the order $o(e^{\frac{V}{\varepsilon}})$. The terms without pre-factor $|a|$ except $\sqrt{\frac{\pi\varepsilon}{\omega_+}} e^{\frac{v}{\varepsilon}} (1 + \mathcal{O}(\varepsilon))$ are $o(e^{\frac{v}{\varepsilon}})$. This results in the inequality

$$a \ge -\sqrt{\frac{\omega_-}{\omega_+}} e^{-\frac{V-v}{\varepsilon}} (1 + \mathcal{O}(\varepsilon)).$$

Applying the arguments just given to $-\Phi_1$, one gets the converse inequality

$$a \le -\sqrt{\frac{\omega_-}{\omega_+}} e^{-\frac{V-v}{\varepsilon}} (1 + \mathcal{O}(\varepsilon)).$$

This completes the proof. ■

A good approximation of Φ_1 is described by the following theorem, which is a direct corollary of Lemmas 5.4.5 and 5.4.6.

Theorem 5.4.2 *Let $0 < v - v' < \frac{v}{3}$ and a be given according to Lemma 5.4.6. Let Φ_1 be defined according to the previous lemmas, and*

$$f_0(x) = a + (1-a)\frac{\int_{-1}^{x} e^{\frac{2U(y)}{\varepsilon}}\,dy}{\int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}}\,dy}, \quad x \in [-1, 1].$$

There exist $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ the following inequalities hold

$$\max_{x\in(-\infty,x_-]} |\Phi_1(x) - a| \le e^{-\frac{2V}{3\varepsilon}}, \tag{5.37}$$

$$\max_{x\in[x_-,x_+]} |\Phi_1(x) - f_0(x)| \le e^{-\frac{v'}{\varepsilon}}, \tag{5.38}$$

$$\max_{x\in[x_+,\infty]} |\Phi_1(x) - 1| \le e^{-\frac{2v}{3\varepsilon}}. \tag{5.39}$$

Proof: The statement of the theorem follows immediately from Lemmas 5.4.5 and 5.4.6. For example, recalling the assumption (**M**), for ε small enough

$$\max_{x\in(-\infty,-1]} |\Phi_1(x) - a| \le A|a|\frac{\lambda_1}{\sqrt{\varepsilon}} \le A_1 e^{-\frac{V-v}{\varepsilon}} e^{-\frac{v'}{\varepsilon}} \le e^{-\frac{2V}{3\varepsilon}}.$$

The latter inequality in the previous formula holds for small enough ε because $V - v + v' > \frac{2V}{3}$ for $0 < v - v' < \frac{V}{3}$. Furthermore,

$$\max_{x\in(-1,x_-]} |\Phi_1(x) - a| \le A|1-a|\sqrt{\varepsilon}e^{-\frac{2V}{3\varepsilon}} \le e^{-\frac{2V}{3\varepsilon}}$$

for ε small enough. Combining these two inequalities gives (5.37). The inequalities (5.38) and (5.39) are proved analogously. ■

Corollary 5.4.1 *In the notation of Theorem 5.4.2 the following inequalities hold*

$$a^2 - Be^{-\frac{5V-3v}{3\varepsilon}} \le \max_{x\in(-\infty,x_-]} (\Phi_1(x))^2 \le a^2 + Be^{-\frac{5V-3v}{3\varepsilon}}, \tag{5.40}$$

$$0 \le \max_{x\in[x_-,x_+]} (\Phi_1(x))^2 \le 2, \tag{5.41}$$

$$1 - Be^{-\frac{2v}{3\varepsilon}} \le \max_{x\in[x_+,\infty)} (\Phi_1(x))^2 \le 1 + Be^{-\frac{2v}{3\varepsilon}}. \tag{5.42}$$

Proof: On the interval $(-\infty, x_-]$ we use (5.37) to get

$$a - e^{-\frac{2V}{3\varepsilon}} \le \Phi_1(x) \le a + e^{-\frac{2V}{3\varepsilon}}.$$

Hence, using that $a < 0$ one obtains

$$(a + e^{-\frac{2V}{3\varepsilon}})^2 \le (\Phi_1(x))^2 \le (a - e^{-\frac{2V}{3\varepsilon}})^2,$$

and therefore recalling the asymptotics of a, for some $B > 0$

$$a^2 - Be^{-\frac{3V-2v}{2\varepsilon}} \le (\Phi_1(x))^2 \le a^2 + Be^{-\frac{3V-2v}{2\varepsilon}}.$$

The inequality (5.41) follows from the fact that Φ and f_0 are exponentially close, and $|f_0| \le |1-a| \le \sqrt{2}$.

The inequality (5.42) is obtained analogously to (5.40). ∎

In order to be able to apply the variational principles for eigenvalues we have to study the derivative φ' of the solution of (5.24).

Lemma 5.4.7 *Let φ be a solution of* (5.24). *Then there exist constants $\varepsilon_0 > 0$ and $A' > 0$ such that for $0 < \varepsilon \le \varepsilon_0$*

$$\max_{x\in(-\infty,-1]} |\varphi'(x)| \le A'|a|\frac{\lambda_1}{\varepsilon}, \tag{5.43}$$

$$\max_{x\in[-1,x_-]} |\varphi'(x)| \le A'|b-a|\frac{e^{-\frac{2V}{3\varepsilon}}}{\varepsilon\sqrt{\varepsilon}}, \tag{5.44}$$

$$\max_{x\in[x_-,x_+]} |\varphi'(x) - f_0'(x)| \le A'|b-a|\frac{\lambda_1}{\varepsilon\sqrt{\varepsilon}} \tag{5.45}$$

$$\max_{x\in[x_+,1]} |\varphi'(x)| \le A'|b-a|\frac{e^{-\frac{2v}{3\varepsilon}}}{\varepsilon\sqrt{\varepsilon}}, \tag{5.46}$$

$$\max_{x\in[1,\infty)} |\varphi'(x)| \le A'|b|\frac{\lambda_1}{\varepsilon}. \tag{5.47}$$

Proof: On the interval $(-\infty,-1]$ the equation $\varphi = h = \sum_{k=0}^{\infty} h_k\lambda_1^k$ is valid. This yields, due to uniform convergence and (5.31) that

$$\begin{aligned} h'(x) &= \sum_{k=0}^{\infty} h_k'\lambda_1^k = -\sum_{k=1}^{\infty} \frac{2}{\varepsilon}\lambda_1^k e^{\frac{2U(x)}{\varepsilon}} \int_{-\infty}^{x} e^{-\frac{2U(y)}{\varepsilon}} h_{k-1}(y)\,dy \\ &= -\frac{2}{\varepsilon}\lambda_1 e^{\frac{2U(x)}{\varepsilon}} \int_{-\infty}^{x} e^{-\frac{2U(y)}{\varepsilon}} h(y)\,dy \end{aligned}$$

Using Lemma 5.4.5 implies in (5.43). Indeed for $x \le -1$ and ε small enough

$$|h'(x)| \le \frac{2}{\varepsilon}\lambda_1|a|(1 + A\frac{\lambda_1}{\sqrt{\varepsilon}})e^{\frac{2U(x)}{\varepsilon}} \int_{-\infty}^{x} e^{-\frac{2U(y)}{\varepsilon}}\,dy \le c_1|a|\frac{\lambda_1}{\varepsilon},$$

for some $c_1 > 0$, since $e^{\frac{2U(x)}{\varepsilon}} \int_{-\infty}^{x} e^{-\frac{2U(y)}{\varepsilon}}\,dy$ is uniformly bounded.

On the interval $[-1,1]$ we have $\varphi = f = f_0 + \sum_{k=1}^{\infty} f_k \lambda_1^k$, hence

$$\begin{aligned} f'(x) = f_0'(x) + \sum_{k=1}^{\infty} f_k' \lambda_1^k &= (b-a) \frac{e^{\frac{2U(x)}{\varepsilon}}}{\int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}}\, dy} \\ &- \frac{2}{\varepsilon}\lambda_1 e^{\frac{2U(x)}{\varepsilon}} \int_0^x e^{-\frac{2U(y)}{\varepsilon}} f(y)\, dy \\ &+ \frac{2}{\varepsilon}\lambda_1 e^{\frac{2U(x)}{\varepsilon}} \frac{\int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}} \int_0^y e^{-\frac{2U(z)}{\varepsilon}} f(z)\, dz\, dy}{\int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}}\, dy}, \end{aligned} \tag{5.48}$$

With the help of Laplace's method for small ε we obtain for some constant $c_2 > 0$ that the first summand in (5.48) can be estimated by

$$\max_{x\in[-1,x_-]} |f_0'(x)| \leq |b-a| \frac{e^{-\frac{2V}{3\varepsilon}}}{\sqrt{\frac{\pi\varepsilon}{\omega_0}}(1+\mathcal{O}(\varepsilon))} \leq c_2 |b-a| \frac{e^{-\frac{2V}{3\varepsilon}}}{\sqrt{\varepsilon}}.$$

Due to the monotonicity of U on $[-1,0]$, the second summand in (5.48) can be estimated as

$$\max_{x\in[-1,x_-]} \left| \frac{2}{\varepsilon}\lambda_1 e^{\frac{2U(x)}{\varepsilon}} \int_0^x e^{-\frac{2U(y)}{\varepsilon}} f(y)\, dy \right| \leq c_3 |b-a| \frac{\lambda_1}{\varepsilon} e^{-\frac{2V}{3\varepsilon}},$$

with a constant c_3 independent of ε. Analogously, for some $c_4 > 0$ the Laplace method and (5.29) imply

$$\max_{x\in[-1,x_-]} \left| \frac{2}{\varepsilon}\lambda_1 e^{\frac{2U(x)}{\varepsilon}} \frac{\int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}} \int_0^y e^{\frac{2U(z)}{\varepsilon}} f(z)\, dz\, dy}{\int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}}\, dy} \right| \leq c_4 |b-a| \frac{\lambda_1}{\varepsilon\sqrt{\varepsilon}} e^{-\frac{2V}{3\varepsilon}}.$$

Combining of the previous three inequalities results in (5.44).

The inequality (5.45) follows directly from (5.48). Inequalities (5.46) and (5.47) are proved analogously to (5.44) and (5.43). ■

Recalling that the first eigenfunction Φ_1 is obtained as a solution of (5.24) for $b = 1$ and $a = -\sqrt{\frac{\omega_-}{\omega_+}} e^{-\frac{V-v}{\varepsilon}}(1+\mathcal{O}(\varepsilon))$ we obtain bounds for Φ_1'.

Theorem 5.4.3 *Let $0 < v - v' < \frac{v}{3}$ and a be given according to Lemma 5.4.6. Let Φ_1 be defined according to the previous lemmas, and*

$$f_0(x) = a + (1-a) \frac{\int_{-1}^{x} e^{\frac{2U(y)}{\varepsilon}}\, dy}{\int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}}\, dy}, \quad x \in [-1,1].$$

There exist $\varepsilon_0 > 0$ and $B' > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ the following inequalities hold

$$\max_{x\in(-\infty,x_-]} |\Phi_1'(x)| \le B'\frac{e^{-\frac{2V}{3\varepsilon}}}{\varepsilon\sqrt{\varepsilon}}, \tag{5.49}$$

$$\max_{x\in[x_-,x_+]} |\Phi_1'(x) - f_0'(x)| \le e^{-\frac{v'}{\varepsilon}} \tag{5.50}$$

$$\max_{x\in[x_+,\infty)} |\Phi_1'(x)| \le B'\frac{e^{-\frac{2v}{3\varepsilon}}}{\varepsilon\sqrt{\varepsilon}}, \tag{5.51}$$

$$\tag{5.52}$$

Proof: Inequality (5.49) follows from (5.43) and (5.44). Indeed, for any $0 < v' < v$ there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ we have

$$A'|a|\frac{\lambda_1}{\varepsilon} \le e^{-\frac{V-v}{\varepsilon}} e^{-\frac{v'}{\varepsilon}} \le e^{-\frac{2V}{3\varepsilon}}.$$

For the second inequality we have to choose v' such that $0 < v - v' < \frac{v}{3} < \frac{V}{3}$. Inequality (5.51) is proved analogously. Inequality (5.50) follows directly from (5.45). ∎

Corollary 5.4.2 *In the notation of Theorem 5.4.3 there exists $B_1 > 0$ such that the following holds:*

$$\max_{x\in(-\infty,x_-]} (\Phi_1'(x))^2 \le B_1\frac{e^{-\frac{4V}{3\varepsilon}}}{\varepsilon^3}, \tag{5.53}$$

$$(f_0'(x))^2 - B_1\frac{e^{\frac{2U}{\varepsilon}}e^{-\frac{v'}{\varepsilon}}}{\sqrt{\varepsilon}} \le (\Phi_1'(x))^2 \le (f_0'(x))^2 + B_1\frac{e^{\frac{2U}{\varepsilon}}e^{-\frac{v'}{\varepsilon}}}{\sqrt{\varepsilon}}, \quad x \in [x_-, x_+], \tag{5.54}$$

$$\max_{x\in[x_+,\infty]} (\Phi_1'(x))^2 \le B_1\frac{e^{-\frac{4v}{3\varepsilon}}}{\varepsilon^3}. \tag{5.55}$$

Proof: The inequalities follow directly from Theorem 5.4.3. The formula

$$f_0'(x) = (1-a)\frac{e^{\frac{2U(x)}{\varepsilon}}}{\int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}}\,dy}, \quad x \in [-1,1]$$

and Laplace's method applied to the denominator in the latter formula. ∎

5.4.2 Useful integrals of eigenfunctions

For purposes of the next chapter we now calculate some integrals of Φ_0, Φ_1 and Φ_1'. In what follows, for any function $f : \mathbb{R} \to \mathbb{R}$ we denote $\overline{f}(x) = f(-x)$, $x \in \mathbb{R}$.

Theorem 5.4.4 *In the small noise limit $\varepsilon \to 0$ we have*

$$\|\Phi_0\|^2_{\rho^{-1}} = \int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\,dy = \sqrt{\frac{\pi\varepsilon}{\omega_-}}e^{\frac{V}{\varepsilon}}(1+\mathcal{O}(\varepsilon)), \tag{5.56}$$

$$\|\Phi_1\|^2_{\rho^{-1}} = \int_{\mathbb{R}} (\Phi_1(y))^2 e^{-\frac{2U(y)}{\varepsilon}}\,dy = \sqrt{\frac{\pi\varepsilon}{\omega_+}}e^{\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon)), \tag{5.57}$$

$$\|\Phi_1'\|^2_{\rho^{-1}} = \int_{\mathbb{R}} (\Phi_1'(y))^2 e^{-\frac{2U(y)}{\varepsilon}}\,dy = \sqrt{\frac{\omega_0}{\pi\varepsilon}}(1+\mathcal{O}(\varepsilon)), \tag{5.58}$$

$$\|\overline{\Phi}_1\|^2_{\rho^{-1}} = \int_{\mathbb{R}} (\Phi_1(-y))^2 e^{-\frac{2U(y)}{\varepsilon}}\,dy = \sqrt{\frac{\pi\varepsilon}{\omega_-}}e^{\frac{V}{\varepsilon}}(1+\mathcal{O}(\varepsilon)),$$

$$(\Phi_1,\overline{\Phi}_1)_{\rho^{-1}} = \int_{\mathbb{R}} \Phi_1(y)\Phi_1(-y) e^{-\frac{2U(y)}{\varepsilon}}\,dy = -\sqrt{\frac{\pi\varepsilon}{\omega_+}}e^{\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon)),$$

$$(\Phi_0,\overline{\Phi}_1)_{\rho^{-1}} = \int_{\mathbb{R}} \Phi_1(-y) e^{-\frac{2U(y)}{\varepsilon}}\,dy = \sqrt{\frac{\pi\varepsilon}{\omega_-}}e^{\frac{V}{\varepsilon}}(1+\mathcal{O}(\varepsilon)),$$

$$(x,\Phi_1)_{\rho^{-1}} = \int_{\mathbb{R}} y\Phi_1(y) e^{-\frac{2U(y)}{\varepsilon}}\,dy = 2\sqrt{\frac{\pi\varepsilon}{\omega_+}}e^{\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon)).$$

Proof: The evaluation of these integrals consists in combining Laplace's method, Theorem 5.3.3 concerning the zeroth eigenfunction, and Corollaries 5.4.1 and 5.4.2 which provide bounds for the first eigenfunction and its derivative. We give the complete set of arguments for the first three integrals, the remaining ones being treated similarly.

The norm (5.56) of Φ_0 is calculated by a direct application of Laplace's method. The global minimum of $U(x)$ is attained at $x=-1$ and $U(-1) = -\frac{V}{2}$. Formula (5.56) follows from (A.8) in the Appendix.

To obtain the norm $\|\Phi_1\|^2_{\rho^{-1}}$ we recall Corollary 5.4.1 to get with a suitable constant c_1

$$\begin{aligned}
\|\Phi_1\|^2_{\rho^{-1}} &= \int_{\mathbb{R}} (\Phi_1(y))^2 e^{-\frac{2U(y)}{\varepsilon}}\,dy \\
&\le \left(\frac{\omega_-}{\omega_+}e^{-2\frac{V-v}{\varepsilon}}(1+\mathcal{O}(\varepsilon)) + Be^{-\frac{5V-3v}{3\varepsilon}}\right)\int_{-\infty}^{x_-} e^{-\frac{2U(y)}{\varepsilon}}\,dy \\
&\quad + 2\int_{x_-}^{x_+} e^{-\frac{2U(y)}{\varepsilon}}\,dy + (1+Be^{-\frac{2v}{3\varepsilon}})\int_{x_+}^{\infty} e^{-\frac{2U(y)}{\varepsilon}}\,dy \\
&\le \sqrt{\frac{\pi\varepsilon}{\omega_+}}e^{\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon)) + c_1\left(e^{-\frac{V-2v}{\varepsilon}} + e^{-\frac{2V-3v}{3\varepsilon}} + e^{\frac{2V}{3\varepsilon}} + e^{\frac{v}{3\varepsilon}}\right) \\
&= \sqrt{\frac{\pi\varepsilon}{\omega_+}}e^{\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon)).
\end{aligned}$$

The last line is justified, since the sum of the exponents in the previous line is $o(e^{\frac{v}{\varepsilon}})$. The converse inequality

$$\|\Phi_1\|^2_{\rho^{-1}} = \int_{\mathbb{R}} (\Phi_1(y))^2 e^{-\frac{2U(y)}{\varepsilon}}\,dy \ge \sqrt{\frac{\pi\varepsilon}{\omega_+}}e^{\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon))$$

comes from the estimate of the integral on $[x_+, \infty)$. This proves (5.57)

To obtain the norm $\|\Phi_1'\|^2_{\rho^{-1}}$, we use Corollary 5.4.2, Laplace's method and the fact that $a = -\sqrt{\frac{\omega_-}{\omega_+}} e^{-\frac{V-v}{\varepsilon}}(1+\mathcal{O}(\varepsilon))$. Indeed, with a suitable constant c_2

$$\begin{aligned}
\|\Phi_1'\|^2_{\rho^{-1}} &= \int_{\mathbb{R}} (\Phi_1'(y))^2 e^{-\frac{2U(y)}{\varepsilon}}\, dy \le \frac{B_1}{\varepsilon^3} e^{-\frac{4V}{3\varepsilon}} \int_{-\infty}^{x_-} e^{-\frac{2U(y)}{\varepsilon}}\, dy \\
&+ \frac{(1-a)^2}{\left(\int_{-1}^{1} e^{\frac{2U(y)}{\varepsilon}}\, dy\right)^2} \int_{x_-}^{x_+} e^{\frac{2U(y)}{\varepsilon}}\, dy + \frac{B_1}{\sqrt{\varepsilon}} e^{-\frac{v'}{\varepsilon}} \int_{x_-}^{x_+} dy \\
&+ \frac{B_1}{\varepsilon^3} e^{-\frac{4v}{3\varepsilon}} \int_{x_+}^{\infty} e^{-\frac{2U(y)}{\varepsilon}}\, dy \\
&\le \sqrt{\frac{\omega_0}{\pi\varepsilon}}(1+\mathcal{O}(\varepsilon)) + \frac{c_2}{\varepsilon^3}\left(e^{-\frac{V}{3\varepsilon}} + e^{-\frac{V-v}{\varepsilon}} + e^{-\frac{v'}{\varepsilon}} + e^{-\frac{v}{3\varepsilon}}\right) \\
&= \sqrt{\frac{\omega_0}{\pi\varepsilon}}(1+\mathcal{O}(\varepsilon)).
\end{aligned}$$

The converse inequality is more easily obtained by evaluating the integral only on $[x_-, x_+]$, and (5.58) follows. ■

5.4.3 The accurate asymptotics for λ_1

In this section we refine the result of Theorem 5.4.1 for the case of a double-well potential.

Theorem 5.4.5 *In the small noise limit $\varepsilon \to 0$ the first eigenvalue of $-L$ satisfies*

$$\lambda_1 = \frac{\sqrt{\omega_0\omega_+}}{2\pi} e^{-\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon)). \tag{5.59}$$

Proof: According to the variational principle formulated in Theorem 5.3.1

$$\lambda_1 = \frac{(-L\Phi_1, \Phi_1)_{\rho^{-1}}}{(\Phi_1, \Phi_1)_{\rho^{-1}}} = \frac{\varepsilon}{2} \frac{\|\Phi_1'\|^2_{\rho^{-1}}}{\|\Phi_1\|^2_{\rho^{-1}}}.$$

The norms of Φ_1 and Φ_1' were obtained in Theorem 5.4.4. Applying (5.57) and (5.58) results in

$$\lambda_1 = \frac{\varepsilon}{2} \frac{\|\Phi_1'\|^2_{\rho^{-1}}}{\|\Phi_1\|^2_{\rho^{-1}}} = \frac{\varepsilon}{2} \sqrt{\frac{\omega_0}{\pi\varepsilon}} \sqrt{\frac{\omega_+}{\pi\varepsilon}} e^{-\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon)) = \frac{\sqrt{\omega_0\omega_+}}{2\pi} e^{-\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon)).$$

■

5.5 The spectral gap between first and second eigenvalues of L

Theorem 5.5.1 *There exists a constant $M > 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ the second eigenvalue of the operator $-L$ satisfies*

$$\lambda_2 \ge M. \tag{5.60}$$

Proof: Let $\varepsilon > 0$. To prove the theorem we combine Courant's minimax principle and the method used in [12] in the proof of the analogous result for the first eigenvalue for the corresponding problem on a closed interval. Indeed, it follows from Theorem 5.3.2 that for any two functions $v, w \in \mathcal{L}^2(\mathbb{R}, \rho^{-1}dx)$ (sometimes called *constraints*) the following inequality holds:

$$\lambda_2 \ge \inf_{\substack{f\in\mathcal{D}_L \\ f\perp v,w \\ f\neq 0}} \frac{\varepsilon}{2}\frac{\|f'\|^2_{\rho^{-1}}}{\|f\|^2_{\rho^{-1}}}, \tag{5.61}$$

We choose v and w in $C_0^\infty(\mathbb{R})$ as approximations of the Dirac δ-distributions δ_{-1} and δ_1 in the distributional sense. Then (5.61) implies the following inequality in which $f \in C_0^\infty(\mathbb{R})$ appear with the constraint $f(\pm 1) = 0$ (see Remark 5.3.1).

$$\lambda_2 \ge \inf_{\substack{f\in\mathcal{D}_L \\ f(-1)=0, f(1)=0 \\ f\neq 0}} \frac{\varepsilon}{2}\frac{\|f'\|^2_{\rho^{-1}}}{\|f\|^2_{\rho^{-1}}} = \inf_{\substack{f\in C_0^\infty(\mathbb{R}) \\ f(-1)=0, f(1)=0 \\ f\neq 0}} \frac{\varepsilon}{2}\frac{\|f'\|^2_{\rho^{-1}}}{\|f\|^2_{\rho^{-1}}}.$$

Take an arbitrary function $f \in C_0^\infty(\mathbb{R})$, such that $f(\pm 1) = 0$ and estimate its norm $\|f\|^2_{\rho^{-1}}$.

Consider the half-line $[1, \infty)$. The potential U increases on this interval, and $U' \ge 0$ there. Then for $x \ge 1$

$$\begin{aligned} 0 \le (f(x))^2 e^{-\frac{2U(x)}{\varepsilon}} &= 2\int_1^x (fe^{-\frac{U}{\varepsilon}})(fe^{-\frac{U}{\varepsilon}})'\,dy \\ &= 2\int_1^x ff'e^{-\frac{2U}{\varepsilon}}\,dy - \frac{2}{\varepsilon}\int_1^x f^2U'e^{-\frac{2U}{\varepsilon}}\,dy. \end{aligned} \tag{5.62}$$

The inequality of Cauchy-Buniakovski-Schwarz implies that for $x \ge 1$

$$2\int_1^x ff'e^{-\frac{2U}{\varepsilon}}\,dy \le 2\left[\int_1^x f^2e^{-\frac{2U}{\varepsilon}}\,dy \int_1^x (f')^2e^{-\frac{2U}{\varepsilon}}\,dy\right]^{\frac{1}{2}} \le 2\|f\|_{\rho^{-1}}\cdot\|f'\|_{\rho^{-1}}.$$

From (5.62) we get for $x \ge 1$

$$\frac{2}{\varepsilon}\int_1^x f^2U'e^{-\frac{2U}{\varepsilon}}\,dy \le 2\int_1^x ff'e^{-\frac{2U}{\varepsilon}}\,dy \le 2\|f\|_{\rho^{-1}}\cdot\|f'\|_{\rho^{-1}},$$

and

$$(f(x))^2 e^{-\frac{2U(x)}{\varepsilon}} \le 2\int_1^x f f' e^{-\frac{2U}{\varepsilon}}\,dy \le 2\|f\|_{\rho^{-1}}\cdot\|f'\|_{\rho^{-1}}. \tag{5.63}$$

Let us fix some $\delta > 0$ to be specified later. Then we estimate

$$\begin{aligned} &\frac{2\delta}{\sqrt{\varepsilon}}\int_{\{U'\ge\delta\sqrt{\varepsilon}\}} f^2 e^{-\frac{2U}{\varepsilon}}\,dy = \frac{2}{\varepsilon}\int_{\{U'\ge\delta\sqrt{\varepsilon}\}} f^2\delta\sqrt{\varepsilon}e^{-\frac{2U}{\varepsilon}}\,dy \\ &\le \frac{2}{\varepsilon}\int_{\{U'\ge\delta\sqrt{\varepsilon}\}} f^2 U' e^{-\frac{2U}{\varepsilon}}\,dy \le \frac{2}{\varepsilon}\int_1^\infty f^2 U' e^{-\frac{2U}{\varepsilon}}\,dy \le 2\|f\|_{\rho^{-1}}\cdot\|f'\|_{\rho^{-1}}. \end{aligned} \tag{5.64}$$

On the other hand, using (5.63) and $U'(1)=0$, $U''(1)=\omega_+>0$, we obtain

$$\int_{\{U'\le\delta\sqrt{\varepsilon}\}\cap[1,\infty)} f^2 e^{-\frac{2U}{\varepsilon}}\,dy \le 2\|f\|_{\rho^{-1}}\cdot\|f'\|_{\rho^{-1}}\cdot\frac{\delta\sqrt{\varepsilon}}{\omega_+}(1+\mathcal{O}(\sqrt{\varepsilon})). \tag{5.65}$$

Combining (5.64) and (5.65) results in

$$\int_1^\infty f^2 e^{-\frac{2U}{\varepsilon}}\,dy \le 2\sqrt{\varepsilon}\|f\|_{\rho^{-1}}\cdot\|f'\|_{\rho^{-1}}\left(\frac{1}{2\delta}+\frac{\delta(1+\mathcal{O}(\sqrt{\varepsilon}))}{\omega_+}\right).$$

Acting analogously on the intervals $(-\infty,-1]$, $[-1,0]$, and $[0,1]$ we obtain

$$\begin{aligned} &\|f\|^2_{\rho^{-1}} = \int_{-\infty}^\infty f^2 e^{-\frac{2U}{\varepsilon}}\,dy \\ &\le 2\sqrt{\varepsilon}\|f\|_{\rho^{-1}}\cdot\|f'\|_{\rho^{-1}}\left(\frac{1}{\delta}+\delta(\frac{1}{\omega_+}+\frac{1}{\omega_0}+\frac{1}{\omega_-})(1+\mathcal{O}(\sqrt{\varepsilon}))\right). \end{aligned}$$

Minimizing the expression in parenthesises in the last formula in δ for ε small enough we find that it is bounded below by some $A>0$. Hence finally, there exists $\varepsilon_0>0$ such that for $0<\varepsilon\le\varepsilon_0$

$$\lambda_2 \ge \frac{\varepsilon}{2}\frac{\|f'\|^2_{\rho^{-1}}}{(2A\sqrt{\varepsilon}\|f'\|_{\rho^{-1}})^2} = M > 0,$$

where the constant M does not depend on ε. ■

5.6 Concluding remarks

We have studied the spectral properties of the generator of a small-noise diffusion in $\mathbb{R}$. In this section we briefly discuss the possibility to extend the results to the d-dimensional setting, $d\in\mathbb{N}$.

Denote by

$$Hf(x) = \frac{\varepsilon}{2}\Delta f(x) - \nabla U(x)\nabla f(x), \quad x\in\mathbb{R}^d,\ f\in\mathcal{C}_0^\infty(\mathbb{R}^d),$$

the generator of the d-dimensional diffusion induced by the SDE

$$dX_t = -\nabla U(X_t) + \sqrt{\varepsilon} dW_t,$$

where W is a standard d-dimensional Wiener process. The potential $U : \mathbb{R}^d \to \mathbb{R}$ is an infinitely differentiable function, and $U(x) \to +\infty$ as $|x| \to \infty$ with polynomial growth.

We are interested in the eigenvalues and eigenfunctions of H in the small noise limit $\varepsilon \to 0$, especially in the existence of a spectral gap between its eigenvalues.

To get criteria for the discreteness of the spectrum of H let us consider the d-dimensional Schrödinger operator

$$hf(x) = \Delta f(x) - v(x), \quad x \in \mathbb{R}^d, \ f \in \mathcal{C}_0^\infty(\mathbb{R}^d),$$

with the Schrödinger potential

$$v(x) = \frac{(\nabla U(x))^2}{\varepsilon^2} - \frac{\Delta U(x)}{\varepsilon}, \quad x \in \mathbb{R}^d. \tag{5.66}$$

The diffusion and Schrödinger operators are related by the formula

$$hf = \frac{2}{\varepsilon} e^{-\frac{U}{\varepsilon}} H(e^{\frac{U}{\varepsilon}} f)$$

Then the operator h defined on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is essentially-self adjoint if a condition similar to the corresponding condition in the one-dimensional case holds (see Theorem 5.2.1).

Theorem 5.6.1 ([6]) *Let the Schrödinger potential v satisfy the condition*

$$v(x) \geq -Q(|x|), \quad x \in \mathbb{R}^d,$$

where $Q(r)$ is an increasing positive continuous function on $[0, +\infty)$ such that

$$\int_0^\infty \frac{dr}{\sqrt{Q(2r)}} = \infty.$$

Then $(h, \mathcal{C}_0^\infty(\mathbb{R}^d))$ is essentially self-adjoint.

The condition of the theorem is obviously satisfied for v defined by (5.66).

The diffusion generator $(H, \mathcal{C}_0^\infty(\mathbb{R}^d))$ is then an essentially self-adjoint and non-positive operator in $\mathcal{L}^2(\mathbb{R}^d, e^{-\frac{2U}{\varepsilon}} dx)$. The increase of the potential U at infinity implies discreteness of the spectrum and the existence of an orthonormal system of eigenfunctions in $\mathcal{L}^2(\mathbb{R}^d, e^{-\frac{2U}{\varepsilon}} dx)$. Moreover, the variational principles entail that the zeroth eigenvalue of H is zero, and the corresponding eigenfunction is constant. The spectrum of H is non-positive.

We have seen that in the one-dimensional case of a double-well potential the first eigenvalue is exponentially small in ε, and the second eigenvalue is bounded away from zero by a constant which does not depend on ε. It turns out that a similar result holds in $\mathbb{R}^d$.

Theorem 5.6.2 ([37]) *Let the potential U have exactly n non-degenerate minima in $\mathbb{R}^d$. Then the first n eigenvalues $\mu_0(\varepsilon), \mu_1(\varepsilon), \ldots, \mu_{n-1}(\varepsilon)$ of H satisfy $\mu_0(\varepsilon) = 0$, $\mu_1(\varepsilon), \ldots, \mu_{n-1}(\varepsilon)$ are exponentially small in the small noise limit $\varepsilon \to 0$, and $|\mu_n(\varepsilon)|$ is bounded below by a positive constant not depending on ε.*

If we denote by Ω_i, $1 \leq i \leq n$, the domains of attraction of the dynamical system $\dot{x} = -\nabla U(x)$, then it turns out that the eigenfunctions corresponding to the exponentially small eigenvalues of H are exponentially close in the $\mathcal{L}^2(\mathbb{R}^d, e^{-\frac{2U}{\varepsilon}}dx)$-norm to constants on Ω_i.

However, the precise asymptotics of the exponentially small eigenvalues including the pre-factors is an open question. This problem was studied formally in [53, 44], especially in the case of a double-well potential in $\mathbb{R}^2$.

Suppose $U : \mathbb{R}^2 \to \mathbb{R}$ has two local minima, say at x_1 and x_2, and a saddle point at x_0. We assume that $U(x_1) < U(x_2)$, and denote by Ω_i the domain of attraction of x_i, $i = 1, 2$ for the dynamical system $\dot{x} = -\nabla U(x)$. Denote by Γ the curve separating the fundamental domains, i.e. $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. Γ is parametrized by its arc length s, such that $s = 0$ at the minimum point x_0 of U on Γ. Let η denote the normal to Γ.

Then the first eigenvalue of H is formally found to be asymptotically equal to

$$\mu_1 \approx -\frac{1}{2\pi}\sqrt{\det\left(\frac{\partial^2 U(x)}{\partial x^i \partial x^j}\right)(x_2)}\sqrt{\frac{\left|\frac{\partial^2 U}{\partial \eta^2}(x_0)\right|}{\frac{\partial^2 U}{\partial s^2}(0)}}e^{-\frac{2(U(x_2)-U(x_1))}{2\varepsilon}}$$

in the small noise limit.

Along the lines of our analysis, it might be possible to obtain the first eigenfunction of H in terms of its series expansion in $\mu_1(\varepsilon)$.

Chapter 6

Stochastic Resonance in Diffusions

In this final chapter we return to the diffusion in a time-periodic double-well potential. Our aim is to compare the spectral power amplification coefficient of the diffusion with its counterpart for the dynamically adapted continuous-time Markov chain on the two-point space composed of the metastable states of the diffusion. The invariant density of the diffusion satisfies the forward Kolmogorov (Fokker-Planck) equation, which is in this case a parabolic partial differential equation with antisymmetric boundary conditions. The invariant density is described by some type of Fourier expansion along the discrete spectrum of the diffusion's infinitesimal generator. It is analysed with the fine asymptotic results of Chapter 5. The noise-independent spectral gap between the first and the second eigenvalues of the infinitesimal generator implies that only the terms of the invariant density expansion which correspond to the first two eigenvalues play a significant role in the induced expansion of the SPA coefficient. Theorem 6.3.1 contains the asymptotics of the SPA coefficient, if the noise parameter ε runs through the intervals $[\frac{v+\delta}{\log T}, \frac{2V}{\log T}]$, $\delta > 0$. In the large period limit $T \to \infty$ these intervals shrink in the natural scale on the one hand. Freidlin [22] on the other hand suggests that these are the relevant resonance intervals to look for. This impression is in particular supported by the observation made in Chapter 4 where we prove that the Markov chain SPA coefficient has a local maximum at $\varepsilon \sim \frac{v+V}{2\log T}$. Surprisingly, it turns out that for the diffusion the SPA tuning curve is either decreasing or increasing on the resonance intervals. This means that the reduction to a Markov chain on the metastable states, however naturally it main retain the dynamical properties of the diffusion, does not preserve optimal tuning effects, at least not for the physicists' favourite measure of quality. The reason for this is hidden in the significance of many small random oscillations of the diffusion in the potential valley bottoms where it spends most of the time. If we cut off these fluctuations by identifying the valley bottoms with the minima themselves, we obtain a modified SPA coefficient which exactly shows the same resonance effects as the Markov chain in the large period limit (small noise limit).

6.1 Diffusion with time-periodic drift and its invariant density

We return now to the main subject of this thesis and consider a family of diffusion processes $X^{\varepsilon,T} = (X_t^{\varepsilon,T})_{t\geq 0}$ given by the real-valued stochastic differential equation

$$dX_t^{\varepsilon,T} = -U'(X_t^{\varepsilon,T}, \frac{1}{2T})\,dt + \sqrt{\varepsilon}\,dW_t, \quad \varepsilon, T > 0, \tag{6.1}$$

where W_t is a standard 1-dimensional Wiener process on some complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

We have already introduced the potential function $U(\cdot,\cdot)$ in Chapter 1, see Fig. 1.3. It is periodic in time, i.e. $U(\cdot,t) = U(\cdot, t+1)$. We also assume that it is a step-function in time alternating between two spatially antisymmetric states, i.e.

$$U(x,t) = \begin{cases} U(x), & t \in [k, k+\frac{1}{2}), \\ U(-x), & t \in [k+\frac{1}{2}, k+1), \quad k = 0, 1, \ldots \end{cases} \tag{6.2}$$

We repeat the standing assumptions about the potential function U:

(**S**) $U \in \mathcal{C}^\infty(\mathbb{R})$;

(**G**) there exists $R > 0$ such that $U(x) = x^4/4$ for $|x| \geq R$;

(**M**) U has exactly two local minima at $x = \pm 1$ and one local maximum at $x = 0$; moreover,
$$U(-1) = -\frac{V}{2}, \quad U(0) = 0, \quad U(1) = -\frac{v}{2}, \quad \frac{2}{3} < \frac{v}{V} < 1;$$
the extrema are non-degenerate, i.e.
$$U''(\pm 1) = \omega_\pm > 0, \quad U''(0) = -\omega_0 < 0.$$

The process $X^{\varepsilon,T}$ has continuous trajectories. The fast increase of the potential at infinity guarantees that the process is positively recurrent. This means that for any initial point, the mean time to enter any interval on $\mathbb{R}$ is finite. For details see [36], [23].

The results of Chapter 5 yield that under conditions formulated above the infinitesimal generator L associated with time-invariant potential function U considered as an operator in $\mathcal{L}^2(\mathbb{R}, e^{-\frac{2U}{\varepsilon}}dx)$ has a discrete spectrum and an orthogonal system of eigenfunctions.

Note that $X^{\varepsilon,T}$ is time-homogeneous during the half-periods $[kT, (k+1)T)$, $k \geq 0$. Moreover, during the intervals $[2Tk, 2Tk+T)$, $k \geq 0$, it is a solution of

$$dX_t^{\varepsilon,T} = -U'(X_t^{\varepsilon,T})\,dt + \sqrt{\varepsilon}\,dW_t,$$

and during the intervals $[2Tk+T, 2T(k+1))$, $k \geq 0$, a solution of

$$dX_t^{\varepsilon,T} = -\overline{U}'(X_t^{\varepsilon,T})\,dt + \sqrt{\varepsilon}\,dW_t,$$

where $\overline{U}(x) = U(-x)$, $x \in \mathbb{R}$.

Consider the time-homogeneous diffusion given by the SDE

$$dX_t^\varepsilon = -U'(X_t^\varepsilon)\,dt + \sqrt{\varepsilon}\,dW_t. \tag{6.3}$$

The process X^ε has a unique invariant measure m_0^ε, on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$, i.e. the probability measure such that

$$m_0^\varepsilon(A) = \int_{\mathbb{R}} P^\varepsilon(t,x,A) m_0^\varepsilon(dx),$$

where $P^\varepsilon(t,x,A)$, $t \geq 0$, $x \in \mathbb{R}$, $A \in \mathcal{B}(\mathbb{R})$ is the transition function of X^ε. The measure m_0^ε has a density μ_0^ε, which is the unique positive solution of the forward Kolmogorov (Fokker-Planck) equation

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d^2}{dx^2}\mu_0^\varepsilon(x) + \frac{d}{dx}(U'(x)\mu_0^\varepsilon(x)) = 0,\\
&\mu_0^\varepsilon(x) > 0, \quad \int_{\mathbb{R}} \mu_0^\varepsilon(x)\,dx = 1, \quad x \in \mathbb{R}.
\end{aligned}$$

In case when the diffusion drift is a gradient of some potential, as for example

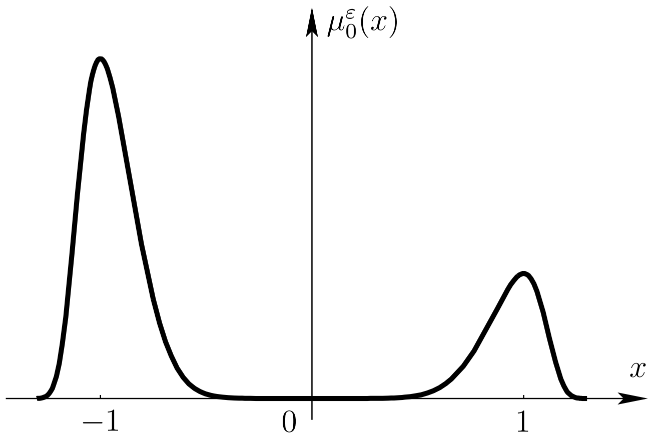

Fig. 6.1: The invariant density of time homogeneous diffusion X^ε.

in (6.3), μ_0^ε is given explicitly by the formula

$$\mu_0^\varepsilon(x) = c_\varepsilon e^{-\frac{2U(x)}{\varepsilon}}, \quad x \in \mathbb{R},$$

with $c_\varepsilon = \int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\,dy$. A typical form of the invariant density μ_0^ε of a time-homogeneous diffusion X^ε in a double-well potential is shown in Fig. 6.1.

Moreover, for any initial conditions, the law of X_t^ε converges to an invariant measure m_0^ε as $t \to \infty$. This means that asymptotically the process 'forgets' about its initial conditions.

In case the process is defined by the SDE (6.1) there is no invariant measure in the usual sense as the process in not time-homogeneous. On the other hand, the time-periodicity of the drift $-U'(\cdot,t)$ suggests that the law of $X_t^{\varepsilon,T}$ should converge to some time-periodic law.

For convenience we rescale time $t \mapsto \frac{t}{2T}$. To give a rigorous mathematical meaning to this we consider a new two-dimensional process

$$\mathbf{X}_t^{\varepsilon,T} = (X_{2Tt}^{\varepsilon,T}, t \pmod 1), \quad t \geq 0.$$

The process $\mathbf{X}^{\varepsilon,T}$ takes values on the cylinder $\mathbb{R} \times S^1$, and is a *time-homogeneous* Markov process. This means that it has an invariant measure $\mathbf{m}^{\varepsilon,T}$ on $\mathcal{B}(\mathbb{R} \times S^1)$.

The measure $\mathbf{m}^{\varepsilon,T}$ has a density $\mu^{\varepsilon,T}$, which is a unique positive solution of the forward Kolmogorov equation for the process $\mathbf{X}^{\varepsilon,T}$, namely,

$$A_{\varepsilon,T}^* \mu^{\varepsilon,T}(x,\theta) = 0, \quad (x,\theta) \in \mathbb{R} \times (0,1), \tag{6.4}$$

with a continuity condition

$$\mu^{\varepsilon,T}(\cdot,0) = \mu^{\varepsilon,T}(\cdot,1) \tag{6.5}$$

and such that $\int_0^1 \int_{\mathbb{R}} \mu^{\varepsilon,T}(x,\theta)\, dx d\theta = 1$. Note that $\mu^{\varepsilon,T}(\cdot,\theta)$ determines the law of the r.v. $X_\theta^{\varepsilon,T}$, and therefore $\int_{\mathbb{R}} \mu^{\varepsilon,T}(x,\theta)\, dx = 1$ for any $\theta \in [0,1]$.

The operator

$$A_{\varepsilon,T}^* f = \frac{\varepsilon}{2} \frac{\partial^2}{\partial x^2} f + \frac{\partial}{\partial x}\left(f \frac{\partial}{\partial x} U \right) - \frac{1}{2T} \frac{\partial}{\partial \theta} f, \quad f \in C_0^\infty(\mathbb{R} \times S^1),$$

is the formal ajoint of the infinitesimal generator of $\mathbf{X}^{\varepsilon,T}$

$$A_{\varepsilon,T} f = \frac{\varepsilon}{2} \frac{\partial^2}{\partial x^2} f - \frac{\partial}{\partial x} f \frac{\partial}{\partial x} U + \frac{1}{2T} \frac{\partial}{\partial \theta} f, \quad f \in C_0^\infty(\mathbb{R} \times S^1).$$

Taking (6.2) into account we rewrite (6.4) and (6.5) in the form

$$\begin{cases} \dfrac{\varepsilon}{2} \dfrac{\partial^2}{\partial x^2} \mu^{\varepsilon,T} + \dfrac{\partial}{\partial x}\left(\mu^{\varepsilon,T} U' \right) = \dfrac{1}{2T} \dfrac{\partial}{\partial \theta} \mu^{\varepsilon,T}, \text{ on } \mathbb{R} \times (0, \frac{1}{2}), \\ \dfrac{\varepsilon}{2} \dfrac{\partial^2}{\partial x^2} \mu^{\varepsilon,T}(x,\theta) + \dfrac{\partial}{\partial x}\left(\mu^{\varepsilon,T} \overline{U}' \right) = \dfrac{1}{2T} \dfrac{\partial}{\partial \theta} \mu^{\varepsilon,T}, \text{ on } \mathbb{R} \times (\frac{1}{2}, 1), \\ \mu^{\varepsilon,T} \text{ is positive and continuous on } \mathbb{R} \times [0,1], \\ \displaystyle\int_{\mathbb{R}} \mu^{\varepsilon,T}(x,\theta)\, dx = 1, \ \theta \in [0,1]. \end{cases} \tag{6.6}$$

From the spatial antisymmetry (6.2) we can immediately deduce a similar antisymmetry property for $\mu^{\varepsilon,T}$.

Proposition 6.1.1 *For $x \in \mathbb{R}$ and $\theta \in (0, \frac{1}{2})$ we have*

$$\mu^{\varepsilon,T}(x,\theta) = \mu^{\varepsilon,T}(-x, \theta + \tfrac{1}{2}).$$

This proposition is analogous to Proposition 4.1.1 concerning the invariant law of the Markov chain.

Proposition 6.1.1 together with (6.6) show that it is enough to find the invariant density on the first half-period, i.e. in the strip $\mathbb{R} \times [0, \frac{1}{2}]$, and in this strip $\mu^{\varepsilon,T}$ is a solution of the following boundary-value problem

$$\begin{cases} \dfrac{\varepsilon}{2}\dfrac{\partial^2}{\partial x^2}\mu^{\varepsilon,T} + \dfrac{\partial}{\partial x}(\mu^{\varepsilon,T}U') = \dfrac{1}{2T}\dfrac{\partial}{\partial\theta}\mu^{\varepsilon,T} \text{ on } \mathbb{R} \times (0, \frac{1}{2}), \\ \mu^{\varepsilon,T}(\cdot, 0) = \mu^{\varepsilon,T}(-\cdot, \frac{1}{2}), \\ \mu^{\varepsilon,T} > 0, \quad \text{on } \mathbb{R} \times [0, \frac{1}{2}], \\ \displaystyle\int_{\mathbb{R}} \mu^{\varepsilon,T}(x,\theta)\, dx = 1, \;\; \theta \in [0, \tfrac{1}{2}]. \end{cases} \tag{6.7}$$

A typical form of the invariant density $\mu^{\varepsilon,T}$ as a function of x and θ is shown in Fig. 6.2. In the following section we describe $\mu^{\varepsilon,T}$ in the small noise limit $\varepsilon \to 0$ in a Fourier type expansion.

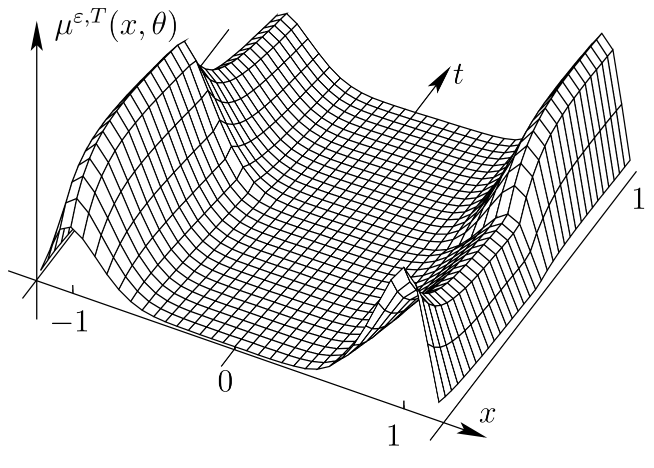

Fig. 6.2: The invariant density of time-inhomogeneous diffusion $X^{\varepsilon,T}$.

6.2 Asymptotic expansion of the invariant density

To solve (6.7), we proceed by separation of variables to an eigenvalue problem for the infinitesimal generator of a time-homogeneous diffusion. Assume that a solution $\mu^{\varepsilon,T}$ of the partial differential equation in (6.7) allows a factorization

$$\mu^{\varepsilon,T}(x,\theta) = \Psi^{\varepsilon,T}(x)\Theta^{\varepsilon,T}(\theta), \quad x \in \mathbb{R}, \theta \in (0, \tfrac{1}{2})$$

Then it follows from (6.7) that

$$\frac{\varepsilon}{2}\frac{(\Psi^{\varepsilon,T})''}{\Psi^{\varepsilon,T}} + U'\frac{(\Psi^{\varepsilon,T})'}{\Psi^{\varepsilon,T}} + U'' = \frac{1}{2T}\frac{\dot{\Theta}^{\varepsilon,T}}{\Theta^{\varepsilon,T}} = -\lambda,$$

where $f' = \frac{df}{dx}$ and $\dot{g} = \frac{dg}{d\theta}$. The constant λ does not depend on x and θ and is an eigenvalue of the differential operator $-L^*$,

$$L^* f = \frac{\varepsilon}{2}f'' + U'f' + U''f, \quad f \in \mathcal{C}_0^\infty(\mathbb{R}).$$

In the previous chapter we have shown that the operator L^* is the formal ajoint of the infinitesimal generator of the time-homogeneous diffusion with potential U. It has a discrete spectrum $\{-\lambda_k\}_{k\geq 0}$ such that $0 = \lambda_0 < \lambda_1 < M < \lambda_2 < \cdots$. The first eigenvalue of L^* is exponentially small in ε and is equal to $\lambda_1 = \frac{\sqrt{\omega_0\omega_+}}{2\pi}e^{-\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon))$. The ε-independent constant M determines the spectral gap between the first and the second eigenvalues of L^*.

The normalized eigenfunctions $\{\frac{\Psi_k}{\|\Psi_k\|_\rho}\}_{k\geq 0}$ provide a complete orthonormal system in $\mathcal{L}^2(\mathbb{R}, \rho dx)$ with $\rho = e^{\frac{2U}{\varepsilon}}$, i.e.

$$\int_{\mathbb{R}} \frac{\Psi_k(y)}{\|\Psi_k\|_\rho}\frac{\Psi_j(y)}{\|\Psi_j\|_\rho}e^{\frac{2U(y)}{\varepsilon}}\, dy = \delta_{kj}, \quad k, j = 0, 1, \ldots \tag{6.8}$$

Recall that the operator L has the same spectrum $\{-\lambda_k\}_{k\geq 0}$, and its eigenfunctions $\{\Phi_k\}_{k\geq 0}$ are related to those of L^* by $\Psi_k = e^{-\frac{2U}{\varepsilon}}\Phi_k$, $k \geq 0$.

Let $\mu^{\varepsilon,T}$ be the unique solution of (6.7). Consider $\mu^{\varepsilon,T}(\cdot, 0)$ and expand it into the Fourier series with respect to the system $\{\Psi_k\}_{k\geq 0}$

$$\mu^{\varepsilon,T}(\cdot, 0) = \sum_{k=0}^{\infty} a_k^{\varepsilon,T}\frac{\Psi_k}{\|\Psi_k\|_\rho}, \tag{6.9}$$

where the Fourier coefficients are determined by the inner products

$$a_k^{\varepsilon,T} = \left(\frac{\Psi_k}{\|\Psi_k\|_\rho}, \mu^{\varepsilon,T}(\cdot, 0)\right)_\rho, \quad k \geq 0.$$

Parseval's equality states that

$$\|\mu^{\varepsilon,T}(\cdot, 0)\|_\rho^2 = \sum_{k=0}^{\infty}(a_k^{\varepsilon,T})^2.$$

From the existence and uniqueness theorem for parabolic partial differential equations [19, 51] we conclude the following

Proposition 6.2.1 *$\mu^{\varepsilon,T}$, $x \in \mathbb{R}$, $s \in [0, 1/2]$, can be represented by the series*

$$\mu^{\varepsilon,T}(x,s) = \sum_{k=0}^{\infty} a_k^{\varepsilon,T} \frac{\Psi_k(x)}{\|\Psi_k\|_\rho} \exp\left(-2T\lambda_k\, s\right), \quad x \in \mathbb{R}, s \in [0, \tfrac{1}{2}]. \tag{6.10}$$

The coefficients $a_k^{\varepsilon,T}$ in the representation (6.9) can be expressed in terms of the function $\mu^{\varepsilon,T}(\cdot, 0)$. Next we find the first coefficient in closed form.

Proposition 6.2.2 *For $\varepsilon > 0$ and $T > 0$*

$$a_0^{\varepsilon,T} = a_0^{\varepsilon} = \|\Psi_0\|_\rho^{-1} = \left(\int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\, dy\right)^{-1/2}. \tag{6.11}$$

Proof: Using the condition that $\int_{\mathbb{R}} \mu^{\varepsilon,T}(x,s)\, dx = 1$ for any s, and $\Psi_0 = e^{-\frac{2U}{\varepsilon}}$ we find

$$\begin{aligned} 1 &= \int_{\mathbb{R}} \mu^{\varepsilon,T}(x,0) e^{\frac{2U(x)}{\varepsilon}} \frac{\Psi_0(x)}{\|\Psi_0\|_\rho} \|\Psi_0\|_\rho\, dx = \|\Psi_0\|_\rho \left(\mu^{\varepsilon,T}(\cdot, 0), \frac{\Psi_0}{\|\Psi_0\|_\rho}\right)_\rho \\ &= a_0^{\varepsilon,T} \|\Psi_0\|_\rho = a_0^{\varepsilon,T} \left(\int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\, dy\right)^{1/2}. \end{aligned}$$

■

Corollary 6.2.1 *Let $T > 0$ and $s \geq 0$. There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$*

$$\|\mu^{\varepsilon,T}(\cdot, s)\|_\rho \geq a_0^{\varepsilon} \geq C\varepsilon^{-1/4} e^{-\frac{V}{2\varepsilon}}$$

Proof: The proof consists in evaluating the integral from (6.11) by Laplace's method, see (A.8). ■

The second coefficient in the expansion (6.10) cannot be described explicitly. But the following proposition gives an approximation the goodness of which is guaranteed by the spectral gap.

Proposition 6.2.3 *Let M be the constant according to Theorem 5.5.1 marking the spectral gap. There exist $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ and $T > 0$ we have*

$$a_1^{\varepsilon,T} = \frac{1}{\|\Psi_0\|_\rho} \frac{\left(\frac{\overline{\Psi}_0}{\|\Psi_0\|_\rho}, \frac{\Psi_1}{\|\Psi_1\|_\rho}\right)_\rho + r(\varepsilon, T)}{1 - e^{-\lambda_1 T} \left(\frac{\overline{\Psi}_1}{\|\Psi_1\|_\rho}, \frac{\Psi_1}{\|\Psi_1\|_\rho}\right)_\rho}, \tag{6.12}$$

where

$$|r(\varepsilon, T)| \leq e^{-MT} \|\mu^{\varepsilon,T}(\cdot, 0)\|_\rho \cdot \left\| \frac{\overline{\Psi}_1}{\|\Psi_1\|_\rho} e^{\frac{2(\overline{U} - U)}{\varepsilon}} \right\|_\rho. \tag{6.13}$$

Proof: From (6.10) and the boundary condition $\mu^{\varepsilon,T}(x,0) = \mu^{\varepsilon,T}(-x,\frac{1}{2})$, $x \in \mathbb{R}$, we obtain

$$\sum_{k=0}^{\infty} a_k^{\varepsilon,T} \frac{\Psi_k}{\|\Psi_k\|_\rho} = \sum_{k=0}^{\infty} a_k^{\varepsilon,T} e^{-\lambda_k T} \frac{\overline{\Psi}_k}{\|\Psi_k\|_\rho}. \tag{6.14}$$

Multiplying both sides of (6.14) by $e^{\frac{2U}{\varepsilon}} \frac{\Psi_1}{\|\Psi_1\|_\rho}$, integrating, and using (6.8) gives

$$a_1^{\varepsilon,T} = a_0^\varepsilon \left(\frac{\overline{\Psi}_0}{\|\Psi_0\|_\rho}, \frac{\Psi_1}{\|\Psi_1\|_\rho} \right)_\rho + a_1^{\varepsilon,T} e^{-\lambda_1 T} \left(\frac{\overline{\Psi}_1}{\|\Psi_1\|_\rho}, \frac{\Psi_1}{\|\Psi_1\|_\rho} \right)_\rho + r(\varepsilon, T),$$

where

$$r(\varepsilon,T) = \sum_{k=2}^{\infty} a_k^{\varepsilon,T} e^{-\lambda_k T} \left(\frac{\overline{\Psi}_k}{\|\Psi_k\|_\rho}, \frac{\Psi_1}{\|\Psi_1\|_\rho} \right)_\rho .$$

To estimate the remainder term r, we now use the spectral gap result from Theorem 5.5.1. We may choose ε_0 such that for $\varepsilon \le \varepsilon_0$ the third eigenvalue $\lambda_2 \ge M > 0$. Hence

$$\begin{aligned} |r(\varepsilon,T)| &\le e^{-MT} \sum_{k=2}^{\infty} |a_k^{\varepsilon,T}| \left| \left(\frac{\overline{\Psi}_k}{\|\Psi_k\|_\rho}, \frac{\Psi_1}{\|\Psi_1\|_\rho} \right)_\rho \right| \\ &\le e^{-MT} \left[\sum_{k=2}^{\infty} (a_k^{\varepsilon,T})^2 \right]^{1/2} \left[\sum_{k=2}^{\infty} \left(\frac{\overline{\Psi}_k}{\|\Psi_k\|_\rho}, \frac{\Psi_1}{\|\Psi_1\|_\rho} \right)_\rho^2 \right]^{1/2} . \end{aligned}$$

The inner products can be rewritten in the following form:

$$\begin{aligned} \left(\frac{\overline{\Psi}_k}{\|\Psi_k\|_\rho}, \frac{\Psi_1}{\|\Psi_1\|_\rho} \right)_\rho &= \int_{\mathbb{R}} \frac{\overline{\Psi}_k}{\|\Psi_k\|_\rho} \frac{\Psi_1}{\|\Psi_1\|_\rho} e^{\frac{2U}{\varepsilon}} \, dx \\ &= \int_{\mathbb{R}} \frac{\Psi_k}{\|\Psi_k\|_\rho} \frac{\overline{\Psi}_1}{\|\Psi_1\|_\rho} e^{\frac{2\overline{U}}{\varepsilon}} e^{-\frac{2U}{\varepsilon}} e^{\frac{2U}{\varepsilon}} \, dx \\ &= \left(\frac{\overline{\Psi}_1}{\|\Psi_1\|_\rho} e^{\frac{2(\overline{U}-U)}{\varepsilon}}, \frac{\Psi_k}{\|\Psi_k\|_\rho} \right)_\rho \end{aligned}$$

Two applications of Parseval's equality complete the estimate (6.13). ■

The only function not even approximately known which appears in the estimate (6.13) is $\mu^{\varepsilon,T}(\cdot,0)$. The following proposition provides an upper bound for its ρ-norm in terms of the ρ-norm of the explicit function $\overline{\rho}^{-1} = e^{-\frac{2\overline{U}}{\varepsilon}}$. Denote $c = \int_{\mathbb{R}} \rho^{-1}(y)\, dy = (a_0^\varepsilon)^{-2} \approx \sqrt{\frac{\pi\varepsilon}{\omega_-}} e^{\frac{V}{\varepsilon}}$ (see Theorem 5.4.4). Then, the function $\overline{\rho}^{-1}/c$ is the invariant density of the time-homogeneous diffusion in the potential $\overline{U}(x) = U(-x)$, $x \in \mathbb{R}$. The spectral gap between $\lambda_0 = 0$ and λ_1 of the corresponding infinitesimal generator implies that the law of the diffusion converges to the invariant law exponentially fast and determines the rate of convergence. This indicates that the density $\mu^{\varepsilon,T}(\cdot,0) = \mu^{\varepsilon,T}(\cdot,1)$ should be close to $\overline{\rho}^{-1}/c$ if T is large enough. One can obtain the following estimate for the norm.

Proposition 6.2.4 *For any $\delta > 0$ there exist $T_0, \varepsilon_0 > 0$ such that for $\varepsilon \in [\frac{v+\delta}{\log T}, \varepsilon_0]$*

$$\|\mu^{\varepsilon,T}(\cdot,0)\|_\rho \leq 6\|\overline{\rho}^{-1}/c\|_\rho. \tag{6.15}$$

Proof: Combining the triangle inequality with the inequality $(a+b)^2 \leq 2(a^2+b^2)$, $a, b \in \mathbb{R}$, gives

$$\|\mu^{\varepsilon,T}(\cdot,0)\|_\rho^2 \leq 2\|\mu^{\varepsilon,T}(\cdot,0) - \frac{\overline{\rho}^{-1}}{c}\|_\rho^2 + 2\|\frac{\overline{\rho}^{-1}}{c}\|_\rho^2. \tag{6.16}$$

For $n \in \mathbb{N}$, denote by P_n^Ψ the orthogonal projector on the orthogonal complement of the span of the first n eigenfunctions $\Psi_0, \ldots, \Psi_{n-1}$. Let us estimate the first summand in (6.16). Using the boundary condition in (6.7) we obtain

$$\begin{aligned}
\|\mu^{\varepsilon,T}(\cdot,0) - \frac{\overline{\rho}^{-1}}{c}\|_\rho^2 &= \|\overline{\mu}^{\varepsilon,T}(\cdot,\tfrac{1}{2}) - \frac{\overline{\rho}^{-1}}{c}\|_\rho^2 = \|\mu^{\varepsilon,T}(\cdot,\tfrac{1}{2}) - \frac{\rho^{-1}}{c}\|_{\overline{\rho}}^2 \\
&= \int_{\mathbb{R}} \left(P_1^\Psi \mu^{\varepsilon,T}(\cdot,\tfrac{1}{2})\right)^2 \overline{\rho}\, dy = \int_{\mathbb{R}} \left|P_1^\Psi \mu^{\varepsilon,T}(\cdot,\tfrac{1}{2})\right| \cdot \left|\mu^{\varepsilon,T}(\cdot,\tfrac{1}{2}) - \frac{\rho^{-1}}{c}\right| \overline{\rho}\, dy \\
&\leq \int_{\mathbb{R}} \left|P_1^\Psi \mu^{\varepsilon,T}(\cdot,\tfrac{1}{2})\right| \mu^{\varepsilon,T}(\cdot,\tfrac{1}{2})\overline{\rho}\, dy + \int_{\mathbb{R}} \left|P_1^\Psi \mu^{\varepsilon,T}(\cdot,\tfrac{1}{2})\right| \frac{\rho^{-1}}{c}\overline{\rho}\, dy \\
&\leq \int_{\mathbb{R}} \left|P_1^\Psi \mu^{\varepsilon,T}(\cdot,\tfrac{1}{2})\right| \mu^{\varepsilon,T}(\cdot,\tfrac{1}{2})\overline{\rho}\rho^{-1}\, \rho\, dy + \int_{\mathbb{R}} \left|P_1^\Psi \mu^{\varepsilon,T}(\cdot,\tfrac{1}{2})\right| \frac{\rho^{-2}}{c}\overline{\rho}\, \rho\, dy \\
&\leq \|P_1^\Psi \mu^{\varepsilon,T}(\cdot,\tfrac{1}{2})\|_\rho \left[\int_{\mathbb{R}} \left(\mu^{\varepsilon,T}(\cdot,\tfrac{1}{2})\right)^2 (\overline{\rho}\rho^{-1})^2 \rho\, dy\right]^{\frac{1}{2}} + \\
&\quad + \|P_1^\Psi \mu^{\varepsilon,T}(\cdot,\tfrac{1}{2})\|_\rho \left[\int_{\mathbb{R}} \left(\frac{\overline{\rho}\rho^{-2}}{c}\right)^2 \rho\, dy\right]^{\frac{1}{2}} \\
&\leq e^{-\lambda_1 T} \max\{\overline{\rho}\rho^{-1}\} \cdot \|\mu^{\varepsilon,T}(\cdot,0)\|_\rho^2 + \frac{e^{-\lambda_1 T}}{c} \left[\int_{\mathbb{R}} \overline{\rho}^2 \rho^{-3}\, dy\right]^{\frac{1}{2}} \|\mu^{\varepsilon,T}(\cdot,0)\|_\rho.
\end{aligned}$$

Taking into account the latter inequality and (6.16) we obtain a quadratic inequality for $\|\mu^{\varepsilon,T}(\cdot,0)\|_\rho$

$$\begin{aligned}
&(1 - 2e^{-\lambda_1 T} \max\{\overline{\rho}\rho^{-1}\})\|\mu^{\varepsilon,T}(\cdot,0)\|_\rho^2 \\
&\qquad - 2\frac{e^{-\lambda_1 T}}{c} \left[\int_{\mathbb{R}} \overline{\rho}^2 \rho^{-3}\, dy\right]^{\frac{1}{2}} \|\mu^{\varepsilon,T}(\cdot,0)\|_\rho - 2\|\frac{\overline{\rho}^{-1}}{c}\|_\rho^2 \leq 0.
\end{aligned} \tag{6.17}$$

Let us estimate the coefficients of (6.17) and thus find an upper bound for $\|\mu^{\varepsilon,T}(\cdot,0)\|_\rho$.

For any $\delta > 0$, let $\frac{v+\delta}{\log T} \leq \varepsilon \leq \varepsilon_0$, where ε_0 is is given by Theorem 5.5.1. Then for some $C > 0$, recalling Theorem 5.4.5

$$e^{-\lambda_1 T} \leq \exp\{-\frac{\sqrt{\omega_0 \omega_+}}{2\pi} T^{-\frac{v}{v+\delta}} T(1 + \mathcal{O}(\frac{1}{\log T}))\} \leq \exp\{-CT^{\frac{\delta}{v+\delta}}\}. \tag{6.18}$$

This expression tends to 0 exponentially fast as $T \to \infty$. Moreover,

$$\max\{\overline{\rho}\rho^{-1}\} = \exp\{\frac{2}{\varepsilon}\max(U(-x) - U(x))\} = \exp\{\frac{\alpha}{\varepsilon}\},$$

where $\alpha = 2\max\{U(-x) - U(x)\} \geq V - v > 0$ is a finite number, defined by the potential U. For $\varepsilon \geq \frac{v+\delta}{\log T}$ we obtain

$$\max\{\overline{\rho}\rho^{-1}\} \leq T^{\frac{\alpha}{v+\delta}}. \tag{6.19}$$

Similarly, let $\beta' = 2\max\{2U(-x) - 3U(x)\} \geq 3V - 2v > 0$. Then, using Laplace's method, see (A.8), and recalling (**G**) we can estimate for some $C_1 > 0$

$$\frac{1}{c}\left[\int_{\mathbb{R}} \overline{\rho}^2 \rho^{-3}\, dy\right]^{\frac{1}{2}} \leq C_1 \sqrt{\varepsilon} e^{\frac{\beta' - V}{\varepsilon}} \leq T^{\frac{\beta}{v+\delta}} \tag{6.20}$$

for $\varepsilon \geq \frac{v+\delta}{\log T}$ and some $\beta > \beta' - V \geq 2(V - v) > 0$.

Since $\max\{U(x) - 2U(-x)\} \geq V - \frac{v}{2} > 0$, the free term of (6.17) is estimated in $\frac{v+\delta}{\log T} \leq \varepsilon \leq \varepsilon_0$ by

$$\left\|\frac{\overline{\rho}^{-1}}{c}\right\|_{\rho}^2 = \frac{1}{c^2}\int_{\mathbb{R}} e^{\frac{2}{\varepsilon}(U(y) - 2U(-y))}\, dy \geq C_2 \log T T^{-\frac{v}{v+\delta}} \tag{6.21}$$

for some positive constant C_2. This estimate means that the norm $\|\overline{\rho}^{-1}/c\|_\rho$ decays in T not faster than polynomially.

Using (6.16), (6.18), (6.19) and (6.20) implies that (6.17) holds if the following inequality holds:

$$\|\mu^{\varepsilon,T}(\cdot,0)\|_\rho^2 (1 - 2e^{-CT^{\frac{\delta}{v+\delta}}} T^{\frac{\alpha}{v+\delta}}) - 2e^{-CT^{\frac{\delta}{v+\delta}}} T^{\frac{\beta}{v+\delta}} \|\mu^{\varepsilon,T}(\cdot,0)\|_\rho - 2\left\|\frac{\overline{\rho}^{-1}}{c}\right\|_\rho^2 \leq 0.$$

Consequently,

$$\begin{aligned}
\|\mu^{\varepsilon,T}(\cdot,0)\|_\rho &\leq \frac{e^{-CT^{\frac{\delta}{v+\delta}}} T^{\frac{\beta}{v+\delta}} + \sqrt{(e^{-CT^{\frac{\delta}{v+\delta}}} T^{\frac{\beta}{v+\delta}})^2 + 2\|\frac{\overline{\rho}^{-1}}{c}\|_\rho^2 (1 - 2e^{-CT^{\frac{\delta}{v+\delta}}} T^{\frac{\alpha}{v+\delta}})}}{1 - 2e^{-CT^{\frac{\delta}{v+\delta}}} T^{\frac{\alpha}{v+\delta}}} \\
&\leq 2\left(\|\frac{\overline{\rho}^{-1}}{c}\|_\rho + \sqrt{\|\frac{\overline{\rho}^{-1}}{c}\|_\rho^2 + 2\|\frac{\overline{\rho}^{-1}}{c}\|_\rho^2}\right) \\
&\leq 6\|\frac{\overline{\rho}^{-1}}{c}\|_\rho
\end{aligned}$$

for $T \geq T_0$, where T_0 is the minimal value for which $\frac{v+\delta}{\log T} \leq \varepsilon_0$, and

$$1 - 2e^{-CT^{\frac{\delta}{v+\delta}}} T^{\frac{\alpha}{v+\delta}} \geq \frac{1}{2}$$

and

$$e^{-CT^{\frac{\delta}{v+\delta}}} T^{\frac{\beta}{v+\delta}} \leq C_1 \log T T^{-\frac{v}{v+\delta}} \leq \|\frac{\overline{\rho}^{-1}}{c}\|_\rho.$$

■

6.3 Spectral power amplification

This and the following sections are devoted to the problem of stochastic resonance for the diffusion (6.1). As a measure of quality of tuning, we shall consider the spectral power amplification coefficient for $X^{\varepsilon,T}$, just as for the two-state reduction in Chapter 4. Of course, to make the two-state chain a consistent model of the reduced diffusion dynamics, we now have to adapt the pre-factors p and q to recover the true asymptotics of Kramers' times hidden in the precise description of λ_1. In this setting, our original plan was to prove that the resonance point obtained in Chapter 4 for the SPA coefficient of the Markov chain determines and thus characterizes a resonance point for the diffusion, in the small noise limit. To our surprise, this turns out not to be the case, as we shall now make precise.

The *spectral power amplification* coefficient is similarly to the two-state case defined by

$$\eta^X(\varepsilon,T) = \left| \int_0^1 \mathbf{E}_\mu X^{\varepsilon,T}_{2Ts} e^{2\pi i s}\, ds \right|^2 \tag{6.22}$$

It will be compared with the analogous coefficient (4.9) of the Markov chain. Hereby we take the average with respect to the invariant law of $X^{\varepsilon,T}$ the density of which is $\mu^{\varepsilon,T}$.

The SPA coefficient describes the energy of the averaged trajectory $X^{\varepsilon,T}$ carried by the spectral component of period $2T$, i.e. the period of the 'input signal'. For the sake of brevity, in this definition we omit the constant factor $\frac{\pi^2}{(V-v)^2}$ used in (4.9).

First, let us rewrite and simplify (6.22). We get

$$\begin{aligned}
\eta^X(\varepsilon,T) &= \left| \int_0^1 \mathbf{E}_\mu X^{\varepsilon,T}_{2Ts} e^{2\pi i s}\, ds \right|^2 \\
&= \left| \int_0^{1/2} e^{2\pi i s} \int_{\mathbb{R}} x\mu^{\varepsilon,T}(x,s)\, dx\, ds + \int_{1/2}^1 e^{2\pi i s} \int_{\mathbb{R}} x\mu^{\varepsilon,T}(x,s)\, dx\, ds \right|^2 \\
&= \left| \int_0^{1/2} e^{2\pi i s} \int_{\mathbb{R}} x\mu^{\varepsilon,T}(x,s)\, dx\, ds + e^{\pi i} \int_0^{1/2} e^{2\pi i s} \int_{\mathbb{R}} x\mu^{\varepsilon,T}(x,s+\frac{1}{2})\, dx\, ds \right|^2 \\
&= \left| \int_0^{1/2} e^{2\pi i s} \int_{\mathbb{R}} x\mu^{\varepsilon,T}(x,s)\, dx\, ds - \int_0^{1/2} e^{2\pi i s} \int_{\mathbb{R}} x\mu^{\varepsilon,T}(-x,s)\, dx\, ds \right|^2 \\
&= 4 \left| \int_0^{1/2} e^{2\pi i s} \int_{\mathbb{R}} x\mu^{\varepsilon,T}(x,s)\, dx\, ds \right|^2 = 4 \left| S^X(\varepsilon,T) \right|^2 .
\end{aligned}$$

Using (6.10) we find

$$\begin{aligned} S^X(\varepsilon, T) = \int_0^{1/2} e^{2\pi i s} \int_{\mathbb{R}} x\mu^{\varepsilon,T}(x,s)\, dx\, ds = \\ a_0^{\varepsilon} \int_0^{1/2} e^{2\pi i s}\, ds \int_{\mathbb{R}} x \frac{\Psi_0(x)}{\|\Psi_0\|_\rho}\, dx + \\ a_1^{\varepsilon,T} \int_0^{1/2} e^{2\pi i s} e^{-2\lambda_1 T s}\, ds \int_{\mathbb{R}} x \frac{\Psi_1(x)}{\|\Psi_1\|_\rho}\, dx + r_1(\varepsilon, T), \end{aligned} \tag{6.23}$$

where

$$r_1(\varepsilon, T) = \int_0^{1/2} e^{2\pi i s} \int_{\mathbb{R}} x \sum_{k=2}^{\infty} a_k^{\varepsilon,T} \frac{\Psi_k(x)}{\|\Psi_k\|_\rho} \exp\left(-2T\lambda_k\, s\right) dx\, ds. \tag{6.24}$$

In the sequel, we shall occasionally use the symbol x to also denote the identity function on $\mathbb{R}$, $x \mapsto x$.

Proposition 6.3.1 *There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ and $T > 0$ we have*

$$|r_1(\varepsilon, T)| \leq \frac{1}{2TM} \|\mu^{\varepsilon,T}(\cdot, 0)\|_\rho \cdot \|xe^{-\frac{2U}{\varepsilon}}\|_\rho, \tag{6.25}$$

with M from Theorem 5.5.1.

Proof: Indeed, using Theorem 5.5.1 for the fourth inequality, we deduce

$$\begin{aligned} |r_1(\varepsilon, T)| &\leq \sum_{k=2}^{\infty} |a_k^{\varepsilon,T}| \int_0^{1/2} e^{-2T\lambda_k s}\, ds \left| \int_{\mathbb{R}} x \frac{\Psi_k(x)}{\|\Psi_k\|_\rho}\, dx \right| \\ &\leq \sum_{k=2}^{\infty} |a_k^{\varepsilon,T}| \left| \int_{\mathbb{R}} x \frac{\Psi_k(x)}{\|\Psi_k\|_\rho}\, dx \right| \frac{1 - \exp\left(-\lambda_k T\right)}{2\lambda_k T} \\ &\leq \frac{1}{2\lambda_2 T} \sum_{k=2}^{\infty} |a_k^{\varepsilon,T}| \left| \int_{\mathbb{R}} x \frac{\Psi_k(x)}{\|\Psi_k\|_\rho}\, dx \right| \\ &\leq \frac{1}{2MT} \left[\sum_{k=2}^{\infty} (a_k^{\varepsilon,T})^2 \sum_{k=2}^{\infty} \left(xe^{-\frac{2U}{\varepsilon}}, \frac{\Psi_k}{\|\Psi_k\|_\rho} \right)_\rho^2 \right]^{1/2}. \end{aligned}$$

It remains to apply Parseval's equality. ■

We next determine the leading term of $S^X(\varepsilon, T)$. We use formula (6.23), and recall that the coefficient a_0^{ε} was determined in Proposition 6.2.2, the coefficient $a_1^{\varepsilon,T}$ was found in Proposition 6.2.3.

Lemma 6.3.1 *For any $\delta > 0$ there exist $T_0, \varepsilon_0 > 0$ such that for $\varepsilon \in [\frac{v+\delta}{\log T}, \varepsilon_0]$ and $T \geq T_0$*

$$\begin{aligned} S^X(\varepsilon, T) &= \frac{i}{\pi} \frac{\int_{\mathbb{R}} y\Psi_0(y)\,dy}{\|\Psi_0\|_\rho^2} \\ &- \frac{1+e^{-\lambda_1 T}}{2(\pi i - \lambda_1 T)} \frac{\int_{\mathbb{R}} y\Psi_1(y)\,dy}{\|\Psi_0\|_\rho^2} \frac{(\overline{\Psi}_0, \Psi_1)_\rho}{\|\Psi_1\|_\rho^2 - e^{-\lambda_1 T}(\overline{\Psi}_1, \Psi_1)_\rho} + r_2(\varepsilon, T), \end{aligned} \tag{6.26}$$

where

$$|r_2(\varepsilon, T)| \leq \frac{6}{MT} \left(\int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\,dy \right)^{-1} \|e^{-\frac{2\overline{U}}{\varepsilon}}\|_\rho \cdot \|xe^{-\frac{2U}{\varepsilon}}\|_\rho, \tag{6.27}$$

and M is given by Theorem 5.5.1.

Proof: The leading term of (6.26) is obtained from (6.23) by integration in $s \in [0, \frac{1}{2}]$. Let us estimate the error term $r_2(\varepsilon, T)$ which is composed of $r_1(\varepsilon, T)$ and $r(\varepsilon, T)$ from formula (6.12). Using Proposition 6.3.1 and Proposition 6.2.3, we get

$$\begin{aligned} |r_2(\varepsilon, T)| &\leq |r_1(\varepsilon, T)| \\ &+ \frac{|r(\varepsilon, T)|}{\|\Psi_0\|_\rho |1 - e^{-\lambda_1 T}(\frac{\overline{\Psi}_1}{\|\Psi_1\|_\rho}, \frac{\Psi_1}{\|\Psi_1\|_\rho})_\rho|} \left| \frac{1+e^{-\lambda_1 T}}{2(\pi i - \lambda_1 T)} \int_{\mathbb{R}} y \frac{\Psi_1(y)}{\|\Psi_1\|_\rho}\,dy \right| \\ &\leq \frac{1}{2MT} \|\mu^{\varepsilon,T}(\cdot, 0)\|_\rho \cdot \|xe^{-\frac{2U}{\varepsilon}}\|_\rho \\ &+ \frac{e^{-MT}}{\|\Psi_0\|_\rho} \frac{\|\mu^{\varepsilon,T}(\cdot,0)\|_\rho \cdot \|\overline{\Psi}_1 e^{2\frac{\overline{U}-U}{\varepsilon}}\|_\rho}{|\|\Psi_1\|_\rho^2 - e^{-\lambda_1 T}(\overline{\Psi}_1, \Psi_1)_\rho|} \left| \int_{\mathbb{R}} y\Psi_1(y)\,dy \right| \\ &= \frac{1}{2MT} \|\mu^{\varepsilon,T}(\cdot, 0)\|_\rho \\ &\times \left(\|xe^{-\frac{2U}{\varepsilon}}\|_\rho + \frac{2MTe^{-MT}}{\|\Psi_0\|_\rho} \frac{\|\overline{\Psi}_1 e^{2\frac{\overline{U}-U}{\varepsilon}}\|_\rho \cdot |\int_{\mathbb{R}} y\Psi_1(y)\,dy|}{|\|\Psi_1\|_\rho^2 - e^{-\lambda_1 T}(\overline{\Psi}_1, \Psi_1)_\rho|} \right). \end{aligned} \tag{6.28}$$

Note that for T large enough Laplace's method yields

$$C_1 \leq \max_{\varepsilon \in [\frac{v+\delta}{\log T}, \varepsilon_0]} \|xe^{-\frac{2U}{\varepsilon}}\|_\rho = \max_{\varepsilon \in [\frac{v+\delta}{\log T}, \varepsilon_0]} \left(\int_{\mathbb{R}} y^2 e^{-\frac{2U(y)}{\varepsilon}}\,dy \right)^{\frac{1}{2}} \leq C_2 T^{\frac{V}{2(v+\delta)}}$$

for some positive constants C_1 and C_2. This means that $\|xe^{-\frac{2U}{\varepsilon}}\|_\rho$ is bounded on $[\frac{v+\delta}{\log T}, \varepsilon_0]$ polynomially in T. Next we show that due to the factor e^{-MT} the second summand in the parenthesis in (6.28) is exponentially small in T.

Recall that $\Psi_k = \Phi_k \rho^{-1}$, hence $\|\Psi_k\|_\rho = \|\Phi_k\|_{\rho^{-1}}$, $k \geq 0$, and note that $\|\overline{\Phi}_1\|_{\rho^{-1}} = \|\overline{\Psi}_1 e^{2\frac{\overline{U}-U}{\varepsilon}}\|_\rho$. Then using Theorem 5.4.4 we estimate in $\varepsilon \in [\frac{v+\delta}{\log T}, \varepsilon_0]$

for T large enough and with universal constant C eventually changing from line to line

$$\begin{aligned}
\frac{2MTe^{-MT}}{\|\Psi_0\|_\rho}&\frac{\|\overline{\Psi}_1 e^{2\frac{\overline{U}-U}{\varepsilon}}\|_\rho\cdot|\int_{\mathbb{R}} y\Psi_1(y)\,dy|}{|\|\Psi_1\|_\rho^2-e^{-\lambda_1 T}(\overline{\Psi}_1,\Psi_1)_\rho|}\\
&=\frac{2MTe^{-MT}}{\|\Phi_0\|_{\rho^{-1}}}\frac{\|\overline{\Phi}_1\|_{\rho^{-1}}\cdot|\int_{\mathbb{R}} y\Phi_1(y)e^{-\frac{2U(y)}{\varepsilon}}\,dy|}{|\|\Phi_1\|_{\rho^{-1}}^2-e^{-\lambda_1 T}(\overline{\Phi}_1,\Phi_1)_{\rho^{-1}}|} \qquad (6.29)\\
&\le CTe^{-MT}\frac{\sqrt[4]{\varepsilon}e^{\frac{V}{2\varepsilon}}\cdot\sqrt{\varepsilon}e^{\frac{v}{\varepsilon}}}{\sqrt[4]{\varepsilon}e^{\frac{V}{2\varepsilon}}\sqrt{\varepsilon}e^{\frac{v}{\varepsilon}}(1+e^{-\lambda_1 T})}\\
&\le CTe^{-MT}\le\|xe^{-\frac{2U}{\varepsilon}}\|_\rho.
\end{aligned}$$

Applying the previous formula and the inequality $\|\mu^{\varepsilon,T}(\cdot,0)\|_\rho\le 6\|\overline{\rho}^{-1}/c\|_\rho$ obtained in Proposition 6.2.4 to (6.28) completes the proof. ∎

Next we find the value of the spectral power amplification coefficient $\eta^X(\varepsilon,T)$ in the interval $[\frac{v+\delta}{\log T},\varepsilon_0]$ for $\delta>0$ and large T. For abbreviation of the leading term, let us set

$$\begin{aligned}
b_0=b_0^\varepsilon&=\frac{\int_{\mathbb{R}} y\Psi_0(y)\,dy}{\|\Psi_0\|_\rho^2}=\frac{\int_{\mathbb{R}} ye^{-\frac{2U(y)}{\varepsilon}}\,dy}{\int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\,dy},\\
b_1=b_1^{\varepsilon,T}&=-\frac{1+e^{-\lambda_1 T}}{2}\frac{\int_{\mathbb{R}} y\Psi_1(y)\,dy}{\|\Psi_0\|_\rho^2}\frac{(\overline{\Psi}_0,\Psi_1)_\rho}{\|\Psi_1\|_\rho^2-e^{-\lambda_1 T}(\overline{\Psi}_1,\Psi_1)_\rho}.
\end{aligned}$$

Then (6.26) can be rewritten in the form

$$S^X(\varepsilon,T)=\frac{i}{\pi}b_0+\frac{1}{\pi i-\lambda_1 T}b_1+r_2(\varepsilon,T),$$

and therefore

$$\eta^X(\varepsilon,T)=4|S^X(\varepsilon,T)|^2=\frac{4}{\pi^2}\frac{b_0^2(\lambda_1 T)^2}{\pi^2+(\lambda_1 T)^2}+4\frac{(b_1-b_0)^2}{\pi^2+(\lambda_1 T)^2}+r_3(\varepsilon,T), \qquad (6.30)$$

where

$$r_3=4|r_2|^2+8\,\mathrm{Re}\,(s^X r_2^*), \qquad (6.31)$$

'$*$' denotes the complex conjugate and $s^X=\frac{i}{\pi}b_0+\frac{1}{\pi i-\lambda_1 T}b_1$ is the leading term of S^X. We see that η^X is represented as a sum of three terms. Let us recall its Markov chain counterpart η^Y determined in Proposition 4.2.1. Up to the constant factor $\frac{\pi^2}{(V-v)^2}$ we have

$$\eta^Y(\varepsilon,T)=\frac{4}{\pi^2}\frac{T^2(\varphi-\psi)^2}{\pi^2+(\varphi+\psi)^2T^2}. \qquad (6.32)$$

It is clearly seen, that the first leading term of (6.30) is similar to (6.32). The correspondence were exact if $\lambda_1\approx\psi\pm\varphi$ and $b_0\approx 1$.

In the following Lemma we find asymptotic estimates for b_0 and b_1.

Lemma 6.3.2 *There is $\varepsilon_0 > 0$ such that for $\varepsilon \le \varepsilon_0$ we have*

$$b_0 = -1 - \frac{U^{(3)}(-1)}{4\omega_-^2}\varepsilon + \mathcal{O}(\varepsilon^2), \tag{6.33}$$

$$b_1 = -1 + \mathcal{O}(\varepsilon), \tag{6.34}$$

and consequently

$$b_0^2 = 1 + \frac{U^{(3)}(-1)}{2\omega_-^2}\varepsilon + \mathcal{O}(\varepsilon^2), \tag{6.35}$$

$$(b_1 - b_0)^2 = \mathcal{O}(\varepsilon^2). \tag{6.36}$$

Proof: We use Laplace's method to obtain the asymptotic expansions for the integrals $\int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\,dy$ and $\int_{\mathbb{R}} ye^{-\frac{2U(y)}{\varepsilon}}\,dy$; compare the formulae (A.7) and (A.9) in the Appendix. We get

$$\int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\,dy = \sqrt{\frac{\pi\varepsilon}{\omega_-}}e^{\frac{V}{\varepsilon}}\left[1 + \frac{\varepsilon}{16\omega_-^3}\left(\frac{5U^{(3)}(-1)^2}{3} - \omega_- U^{(4)}(-1)\right) + \mathcal{O}(\varepsilon^2)\right],$$

$$\int_{\mathbb{R}} ye^{-\frac{2U(y)}{\varepsilon}}\,dy = -\sqrt{\frac{\pi\varepsilon}{\omega_-}}e^{\frac{V}{\varepsilon}}$$
$$\times\left[1 + \frac{U^{(3)}(-1)}{4\omega_-^2}\varepsilon + \frac{\varepsilon}{16\omega_-^3}\left(\frac{5U^{(3)}(-1)^2}{3} - \omega_- U^{(4)}(-1)\right) + \mathcal{O}(\varepsilon^2)\right].$$

The relationships (6.33) and (6.35) follow from these formulae and the asymptotic expansion rule $\frac{1+a_1\varepsilon+\mathcal{O}(\varepsilon^2)}{1+a_2\varepsilon+\mathcal{O}(\varepsilon^2)} = 1 + (a_1 - a_2)\varepsilon + \mathcal{O}(\varepsilon^2)$, $a_1, a_2 \in \mathbb{R}$.

The estimate for b_1 is obtained analogously to Lemma 6.3.1 with the help of Theorem 5.4.4. More precisely, we get

$$\begin{aligned}
b_1 &= -\frac{1+e^{-\lambda_1 T}}{2}\frac{\int_{\mathbb{R}} y\Psi_1(y)\,dy}{\|\Psi_0\|_\rho^2}\frac{(\overline{\Psi}_0, \Psi_1)_\rho}{\|\Psi_1\|_\rho^2 - e^{-\lambda_1 T}(\overline{\Psi}_1, \Psi_1)_\rho} \\
&= -\frac{1+e^{-\lambda_1 T}}{2}\frac{(x, \Phi_1)_{\rho^{-1}}}{\|\Phi_0\|_{\rho^{-1}}^2}\frac{(\Phi_0, \overline{\Phi}_1)_{\rho^{-1}}}{\|\Phi_1\|_{\rho^{-1}}^2 - e^{-\lambda_1 T}(\overline{\Phi}_1, \Phi_1)_{\rho^{-1}}} \\
&= -\frac{2\left(1+e^{-\lambda_1 T}\right)\cdot\sqrt{\frac{\pi\varepsilon}{\omega_+}}e^{\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon))\cdot\sqrt{\frac{\pi\varepsilon}{\omega_-}}e^{\frac{V}{\varepsilon}}(1+\mathcal{O}(\varepsilon))}{2\sqrt{\frac{\pi\varepsilon}{\omega_-}}e^{\frac{V}{\varepsilon}}(1+\mathcal{O}(\varepsilon))\cdot\left(\sqrt{\frac{\pi\varepsilon}{\omega_+}}e^{\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon)) + e^{-\lambda_1^\varepsilon T}\sqrt{\frac{\pi\varepsilon}{\omega_+}}e^{\frac{v}{\varepsilon}}(1+\mathcal{O}(\varepsilon))\right)} \\
&= -1 + \mathcal{O}(\varepsilon).
\end{aligned}$$

The combination of (6.33) and (6.34) leads to (6.36). ∎

Let us compare the first term in (6.30) with η^Y. Lemma 6.3.2 states that $b_0 \approx 1$ as $\varepsilon \to 0$. Recall that

$$\lambda_1 \approx \frac{\sqrt{\omega_0\omega_+}}{2\pi}e^{-\frac{v}{\varepsilon}},$$

and that the infinitesimal probabilities in Chapter 4 were defined by $\varphi = pe^{-\frac{V}{\varepsilon}}$ and $\psi = pe^{-\frac{v}{\varepsilon}}$, with $p, q > 0$.

We now choose p and q so that Kramers' times for the diffusion and the reducing Markov chain coincide not only to exponential order, but in addition up to the leading subexponential pre-factors. If we set

$$p = \frac{\sqrt{\omega_0 \omega_-}}{2\pi}, \qquad q = \frac{\sqrt{\omega_0 \omega_+}}{2\pi} \tag{6.37}$$

we also get

$$\psi + \varphi \approx \lambda_1, \qquad \psi - \varphi \approx \lambda_1$$

to the leading subexponential pre-factor. This moreover implies

$$b_0 \frac{(\lambda_1 T)^2}{\pi^2 + (\lambda_1 T)^2} \approx \frac{(\psi - \varphi)^2 T^2}{\pi^2 + (\psi + \varphi)^2 T^2}.$$

This correspondence is a sure temptation for suspecting that provided the remainder terms in (6.30) are small enough, the SPA coefficients of the diffusion and the Markov chain are close. This also strongly suggests that one could be able to relate asymptotically the optimal tuning rates for the diffusion and the reducing Markov chain.

This argument is in fact very common in the physics literature [49, 26, 2] and is used to pass to a simpler two-state framework for the investigation of various dynamucal properties of the diffusion in the small noise limit, especially in the context of stochastic resonance. This approach will now be shown to possibly have drastic side effects. In determining eigenvalues and Kramers' times we always had to take into account a multiplicative correction term of the type $1+\mathcal{O}(\varepsilon)$. The errors of order $\mathcal{O}(\varepsilon)$ in all the formulae we have derived reflect small random oscillations of the diffusion near the metastable states of the potential. Neglecting these terms means neglecting the 'diffusive' nature of the diffusion. Although these oscillations are small, they occur with a high probability. This leads very subtle drag effects in the potential wells' bottoms disturbing or even destroying the tuning picture the two-state reduction presents.

In Chapter 4 the optimal tuning rate for the Markov chain $Y^{\varepsilon,T}$ in the sense of the SPA coefficient was determined by

$$\varepsilon = \frac{V+v}{2\log T}\left(1 + \frac{\log\left(\frac{\pi}{\sqrt{2pq}}\sqrt{\frac{v}{V-v}}\right)}{\log T} + \mathcal{O}\left(\frac{1}{\log^2 T}\right)\right)$$

If the diffusion's tuning properties corresponding to the SPA coefficient were retained by the reducing Markov chain, we would expect a local maximum of the SPA coefficient $\eta^X(\cdot, T)$ at some point $\varepsilon_T \approx \frac{V+v}{2\log T}$. Let us consider $\alpha \mapsto \eta^X(\frac{\alpha}{\log T}, T)$ for large T on the interval

$$[v+\delta, \Delta], \quad \text{for some} \quad \delta > 0,\ \Delta > v + \delta.$$

On the ε-scale this corresponds to shrinking intervals $[\frac{v+\delta}{\log T}, \frac{\Delta}{\log T}]$. We shall investigate, whether $\alpha \mapsto \eta^X(\frac{\alpha}{\log T}, T)$ posesses a local maximum for large T.

Theorem 6.3.1 *Let $0 < \delta < \frac{v}{3}$ and $\Delta > v + \delta$ be fixed. Let $\max_{x\in\mathbb{R}}\{U(x) - 2U(-x)\} < V + v$. Then there exists $T_0 > 0$ such that for $T > T_0$, $\alpha \in [v+\delta, \Delta]$ we have*

$$\eta^X(\frac{\alpha}{\log T}, T) = \frac{4}{\pi^2}\left(1 + \frac{U^{(3)}(-1)}{2\omega_-^2}\frac{\alpha}{\log T}\right) + \mathcal{O}\left(\frac{1}{\log^2 T}\right).$$

Remark 6.3.1 Geometrically, the condition $\max_{x\in\mathbb{R}}\{U(x) - 2U(-x)\} < V + v$ may be seen to express the fact that the potential is not too asymmetric outside of the wells.

Proof: The proof consists in expanding of (6.30) as $T \to \infty$ and estimating the remainder terms.

First, we note that in order to apply Lemma 6.3.1 the noise parameter ε must satisfy $\varepsilon \in [\frac{v+\delta}{\log T}, \varepsilon_0]$ for some $\varepsilon_0 > 0$. It is clear that to verify this condition it is enough to take $T_0 > e^{\Delta/\varepsilon_0}$.

Consider $\eta^X(\frac{\alpha}{\log T}, T)$ for $T > 0$. The factor b_0 in the leading term of η^X is expanded with the help of Lemma 6.3.2.

On the interval $[\frac{v+\delta}{\log T}, \frac{\Delta}{\log T}]$ with some constant $C > 0$ we estimate

$$(\lambda_1 T)^2 \geq C(e^{-\frac{v}{\varepsilon}}T)^2 \geq C(T^{1-\frac{v}{v+\delta}})^2 = CT^{\frac{2\delta}{v+\delta}}. \tag{6.38}$$

This results in

$$\frac{(\lambda_1 T)^2}{\pi^2 + (\lambda_1 T)^2} = 1 - \frac{\pi^2}{\pi^2 + (\lambda_1 T)^2} = 1 + \mathcal{O}\left(\frac{1}{(\lambda_1 T)^2}\right) = 1 + \mathcal{O}\left(\frac{1}{T^{\frac{2\delta}{v+\delta}}}\right).$$

Hence, we obtain the expansion of the leading term of (6.30)

$$\frac{4}{\pi^2}\frac{b_0^2(\lambda_1 T)^2}{\pi^2 + (\lambda_1 T)^2} = \frac{4}{\pi^2}\left(1 + \frac{U^{(3)}(-1)}{2\omega_-^2}\frac{\alpha}{\log T}\right) + \mathcal{O}\left(\frac{1}{\log^2 T}\right).$$

It is left to estimate the terms $\frac{4(b_1-b_0)^2}{\pi^2+(\lambda_1 T)^2}$ and $|r_3|$. Analogously to (6.38), the first one is of the order $T^{-\frac{2\delta}{v+\delta}}\log^{-2} T$ on $\alpha \in [v+\delta, \Delta]$ as $T \to \infty$ since $(b_1 - b_0)^2 = \mathcal{O}(\frac{1}{\log^2 T})$.

Consider the third term given by (6.31):

$$|r_3(\varepsilon, T)| \leq 8|s^X(\varepsilon, T)||r_2(\varepsilon, T)| + 4|r_2(\varepsilon, T)|^2,$$

where

$$|s^X(\varepsilon, T)| = \left(\frac{1}{\pi^2}\frac{b_0^2(\lambda_1 T)^2}{\pi^2 + (\lambda_1 T)^2} + \frac{4(b_1 - b_0)^2}{\pi^2 + (\lambda_1 T)^2}\right)^{\frac{1}{2}} \leq \frac{1}{2}$$

for $T \geq T_0$, $\frac{v+\delta}{\log T} \leq \varepsilon \leq \frac{\Delta}{\log T}$.

Let us estimate $|r_2(\varepsilon, T)|$. Lemma 6.3.1 states that

$$|r_2(\varepsilon, T)| \leq \frac{6}{MT} \frac{\left[\int_{\mathbb{R}} e^{-\frac{2}{\varepsilon}(2U(-y)-U(y))}\, dy\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} y e^{-\frac{2U(y)}{\varepsilon}}\, dy\right]^{\frac{1}{2}}}{\int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\, dy}.$$

Assume that $\max_{x\in\mathbb{R}}\{U(x) - 2U(-x)\} < \kappa$, where κ is a positive number. Obviously, $\kappa > V - \frac{v}{2} > 0$, since

$$\max_{x\in\mathbb{R}}\{U(x) - 2U(-x)\} \geq U(1) - 2U(-1) = V - \frac{v}{2}.$$

Then, using Laplace's method we obtain for some $C > 0$ independent of $\varepsilon \leq \varepsilon_0$

$$|r_2(\varepsilon, T)| \leq C \frac{e^{\frac{\kappa}{2\varepsilon}} e^{\frac{V}{2\varepsilon}}}{T e^{\frac{V}{\varepsilon}}} = C \frac{e^{\frac{\kappa - V}{2\varepsilon}}}{T}.$$

If we choose $\kappa = V + v$ and $\delta \in (0, \frac{v}{3})$, then the inequality $\frac{v+2\delta}{2(v+\delta)} > \frac{2\delta}{v+\delta}$ entails that

$$\max_{\alpha\in[v+\delta,\Delta]} \left| r_2(\frac{\alpha}{\log T}, T) \right| \leq C \frac{1}{T^{1-\frac{v}{2(v+\delta)}}} = o\left(\frac{1}{T^{\frac{2\delta}{v+\delta}}}\right), \quad \delta \in (0, \tfrac{v}{3}).$$

Thus, the remainder terms in (6.30) are polynomially small and of the order $T^{-\frac{2\delta}{v+\delta}} \log^{-2} T$. This completes the proof. ■

As we see, the form of the tuning curve for the SPA coefficient crucially depends on the sign of $U^{(3)}(-1)$. If $U^{(3)}(-1) > 0$, the tuning curve increases in α and does not have a local maximum on the interval $[\frac{v+\delta}{\log T}, \frac{\Delta}{\log T}]$. If $U^{(3)}(-1) < 0$ then η^X decreases and does not have a local maximum either.

Moreover, depending on the sign of $U^{(3)}(-1)$, the resonance curve is either greater or less than $\frac{4}{\pi^2}$. However, we can state, that the SPA coefficient $\eta^X(\varepsilon, T(\varepsilon)) = 1 + \mathcal{O}(\varepsilon)$ for $T(\varepsilon) = e^{\frac{\alpha}{\varepsilon}}$, $\alpha \in [v + \delta, \Delta]$, which is near the maximal value of η^Y. This means, that amplification occurs, but an optimal tuning rate cannot be determined. Especially, the Markov chain behaviour is not shared by the diffusion, and the optimal tuning rate by the chain is not asymptotically equal to an optimal tuning rate for the diffusion, in which sense ever the latter may exist.

The observed subtle drag effect may be interpreted in the following way. If $U^{(3)}(-1) < 0$, the potential in the deep well slightly leans towards the shallow one. Therefore in the ε-window considered increasing the noise intensity tends to reduce the averaged overall amplitude of the motion.

In case $U^{(3)}(-1) > 0$, the outward leaning of the potential in its global maximum increases the averaged overall amplitude of the random motion with increasing noise intensity.

6.4 Optimal tuning for the modified SPA coefficient.

Stochastic resonance is an inter-well and not an intra-well effect. Given our experience gained in the previous section, we now suppress oscillations near the potential minima and take into account only big hoppings between the wells. In this modified setting, we shall now show that the behaviour of the reducing Markov chain is correctly retained asymptotically in the small noise limit.

In order to *cut off* small random fluctuations near -1 and 1 we define a function

$$g(x) = \begin{cases} x, & x \in (-\infty, x_1] \cup [x_2, y_1] \cup [y_2, \infty), \\ -1, & x \in [x_1, x_2], \\ 1, & x \in [y_1, y_2], \end{cases}$$

where $x_1 < -1 < x_2 < 0$ are such that $U(x_1) = U(x_2) = -\frac{V}{4}$, and $0 < y_1 < 1 < y_2$ are such that $U(y_1) = U(y_2) = -\frac{v}{4}$.

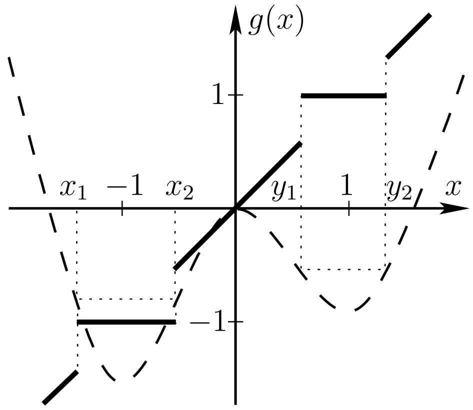

Fig. 6.3: Function g designed to cut off small fluctuations.

For $\varepsilon, T > 0$ we consider the modified SPA coefficient

$$\widetilde{\eta}^X(\varepsilon, T) = \left| \int_0^1 \mathbf{E}_\mu g(X^{\varepsilon,T}_{2Ts}) e^{2\pi i s}\, ds \right|^2.$$

By inspection of the steps in the calculations of Section 6.3, replacing x with $g(x)$ if necessary, we obtain a formula for $\widetilde{\eta}^X$ which is analogous to (6.30):

$$\widetilde{\eta}^X(\varepsilon, T) = \frac{4}{\pi^2} \frac{\widetilde{b}_0^2 (\lambda_1 T)^2}{\pi^2 + (\lambda_1 T)^2} + 4 \frac{(\widetilde{b}_1 - \widetilde{b}_0)^2}{\pi^2 + (\lambda_1 T)^2} + \widetilde{r}_3(\varepsilon, T), \tag{6.39}$$

where

$$\begin{aligned}
\widetilde{b}_0 &= \frac{\int_{\mathbb{R}} g(y) e^{-\frac{2U(y)}{\varepsilon}}\, dy}{\int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\, dy}, \\
\widetilde{b}_1 &= -\frac{1+e^{-\lambda_1 T}}{2} \frac{\int_{\mathbb{R}} g(y)\Psi_1(y)\, dy}{\|\Psi_0\|_\rho^2} \frac{(\overline{\Psi}_0, \Psi_1)_\rho}{\|\Psi_1\|_\rho^2 - e^{-\lambda_1 T}(\overline{\Psi}_1, \Psi_1)_\rho}, \\
\widetilde{r}_3(\varepsilon, T) &= 4|\widetilde{r}_2(\varepsilon, T)|^2 + 8\,\mathrm{Re}\,(\widetilde{s}^X(\varepsilon, T)\widetilde{r}_2^*(\varepsilon, T)), \\
\widetilde{s}^X(\varepsilon, T) &= \frac{i}{\pi}\widetilde{b}_0 + \frac{1}{\pi i - \lambda_1 T}\widetilde{b}_1, \\
|\widetilde{r}_2(\varepsilon, T)| &\le \frac{6}{MT}\left(\int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\, dy\right)^{-1} \|e^{-\frac{2\overline{U}}{\varepsilon}}\|_\rho \cdot \|g e^{-\frac{2U}{\varepsilon}}\|_\rho.
\end{aligned}$$

It turns out that due to different behaviour of the factor $\widetilde{b}_0$ has quite different asymptotics from b_0. Hence the modified SPA coefficient $\widetilde{\eta}^X(\varepsilon, T)$ has a local maximum close to the corresponding one for the Markov chain.

Lemma 6.4.1 *There is $\varepsilon_0 > 0$ such that for $\varepsilon \le \varepsilon_0$, $T > 0$ we have*

$$\widetilde{b}_0 = -1 + 2\sqrt{\frac{\omega_-}{\omega_+}} e^{-\frac{V-v}{\varepsilon}}(1 + \mathcal{O}(\varepsilon)) \ge -1, \tag{6.40}$$

$$\widetilde{b}_1 = -1 + \mathcal{O}(\varepsilon), \tag{6.41}$$

and, consequently,

$$\widetilde{b}_0^2 = 1 - 4\sqrt{\frac{\omega_-}{\omega_+}} e^{-\frac{V-v}{\varepsilon}}(1 + \mathcal{O}(\varepsilon)) \le 1 \tag{6.42}$$

$$(\widetilde{b}_1 - \widetilde{b}_0)^2 = \mathcal{O}(\varepsilon^2). \tag{6.43}$$

Proof: Consider $\widetilde{b}_0$ for small ε and use Laplace's method to obtain (6.40):

$$\begin{aligned}
\widetilde{b}_0 &= \frac{\int_{\mathbb{R}} g(y) e^{-\frac{2U(y)}{\varepsilon}}\, dy}{\int_{\mathbb{R}} e^{-\frac{2U(y)}{\varepsilon}}\, dy} \\
&= \frac{\left(-\int_{x_1}^{x_2} + \int_{y_1}^{y_2}\right) e^{-\frac{2U(y)}{\varepsilon}}\, dy + \left(\int_{-\infty}^{x_1} + \int_{x_2}^{y_1} + \int_{y_2}^{\infty}\right) y e^{-\frac{2U(y)}{\varepsilon}}\, dy}{\left(\int_{x_1}^{x_2} + \int_{y_1}^{y_2}\right) e^{-\frac{2U(y)}{\varepsilon}}\, dy + \left(\int_{-\infty}^{x_1} + \int_{x_2}^{y_1} + \int_{y_2}^{\infty}\right) e^{-\frac{2U(y)}{\varepsilon}}\, dy} \\
&= -\frac{\left(\int_{x_1}^{x_2} + \int_{y_1}^{y_2}\right) e^{-\frac{2U(y)}{\varepsilon}}\, dy - 2\int_{y_1}^{y_2} e^{-\frac{2U(y)}{\varepsilon}}\, dy + \mathcal{O}(e^{\frac{V}{2\varepsilon}})}{\left(\int_{x_1}^{x_2} + \int_{y_1}^{y_2}\right) e^{-\frac{2U(y)}{\varepsilon}}\, dy + \mathcal{O}(e^{\frac{V}{2\varepsilon}})} \\
&= -1 + \frac{2\sqrt{\frac{\pi\varepsilon}{\omega_+}} e^{\frac{v}{\varepsilon}}(1 + \mathcal{O}(\varepsilon))}{\sqrt{\frac{\pi\varepsilon}{\omega_-}} e^{\frac{V}{\varepsilon}}(1 + \mathcal{O}(\varepsilon))} = -1 + 2\sqrt{\frac{\omega_-}{\omega_+}} e^{-\frac{V-v}{\varepsilon}}(1 + \mathcal{O}(\varepsilon)).
\end{aligned}$$

For the third equation we used our hypothesis concerning the cutoff levels the inequality $V - v < \frac{V}{2}$ which follows from (**M**).

Formulae (6.41) and (6.43) are obtained analogously to Lemma 6.3.2. Expression (6.42) follows directly from (6.40). ■

Theorem 6.4.1 *Let* $\max_{x\in\mathbb{R}}\{U(x) - 2U(-x)\} < V + v$ *and* $0 < \delta < \frac{v}{3}$. *Then for any* $1 < \gamma < \frac{V+v}{2(v+\delta)}$ *there exists* $T(\gamma)$ *such that for* $T > T(\gamma)$ *the modified SPA coefficient* $\varepsilon \mapsto \widetilde{\eta}^X(\varepsilon, T)$ *has a local maximum on* $[\gamma^{-1}\frac{V+v}{2\log T}, \gamma\frac{V+v}{2\log T}]$. *The optimal tuning rate* $\varepsilon(T)$ *is exponentially equivalent to* $\frac{V+v}{2\log T}$ *in the limit* $T \to \infty$.

Proof: To show that $\varepsilon \mapsto \widetilde{\eta}^X(\varepsilon, T)$ given by (6.39) has a local maximum we consider it at the three points

$$\varepsilon_1(T) = \frac{V+v}{2\log T}, \qquad \varepsilon_2(T) = \gamma\frac{V+v}{2\log T}, \text{ and } \varepsilon_3(T) = \gamma^{-1}\frac{V+v}{2\log T}. \tag{6.44}$$

Since $\gamma > 1$, we have $\varepsilon_3(T) < \varepsilon_1(T) < \varepsilon_2(T)$ (see Fig. 6.4).

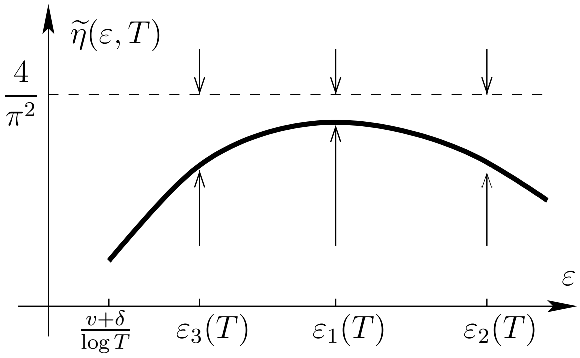

Fig. 6.4: The modified SPA coefficient $\widetilde{\eta}^X(\varepsilon, T)$ at the points $\varepsilon_i(T)$, $i = 1, 2, 3$, defined in (6.44).

Consider the leading term of $\widetilde{\eta}^X(\varepsilon_1(T), T)$ for $T \to \infty$. The factor $\widetilde{b}_0^2$ is given by Lemma 6.4.1. Expanding the factor $\frac{(\lambda_1 T)^2}{\pi^2 + (\lambda_1 T)^2}$ as $T \to \infty$ gives:

$$\begin{aligned}
\frac{4}{\pi^2}\frac{\widetilde{b}_0^2(\lambda_1 T)^2}{\pi^2 + (\lambda_1 T)^2} &= \frac{4}{\pi^2}\left(1 - 4\sqrt{\frac{\omega_-}{\omega_+}}T^{-2\frac{V-v}{V+v}}(1 + \mathcal{O}(\log^{-1} T))\right) \\
&\times \left(1 - \frac{4\pi^4}{\omega_0\omega_+}T^{-2\frac{V-v}{V+v}}(1 + \mathcal{O}(\log^{-1} T))\right) \\
&= \frac{4}{\pi^2} - \frac{16}{\pi^2}\left(\sqrt{\frac{\omega_-}{\omega_+}} + \frac{\pi^4}{\omega_0\omega_+}\right)T^{-2\frac{V-v}{V+v}}\left(1 + \mathcal{O}(\log^{-1} T)\right).
\end{aligned}$$

Analogously to Theorem 6.3.1 one shows that the remainder terms in (6.39) are of the order $T^{-2\frac{V-v}{V+v}}\log^{-2}T$. This yields that

$$\widetilde{\eta}^X(\varepsilon_1(T),T) = \frac{4}{\pi^2} - \frac{16}{\pi^2}\left(\sqrt{\frac{\omega_-}{\omega_+}} + \frac{\pi^4}{\omega_0\omega_+}\right)\frac{1}{T^{2\frac{V-v}{V+v}}}\left(1+\mathcal{O}\left(\frac{1}{\log T}\right)\right).$$

Analogously, we consider the modified SPA coefficient at $\varepsilon_2(T)$ and $\varepsilon_3(T)$ to obtain

$$\widetilde{\eta}^X(\varepsilon_2(T),T) = \frac{4}{\pi^2} - \frac{16}{\pi^2}\sqrt{\frac{\omega_-}{\omega_+}}\frac{1}{T^{2\frac{V-v}{\gamma(V+v)}}}\left(1+\mathcal{O}\left(\frac{1}{\log T}\right)\right),$$

and

$$\widetilde{\eta}^X(\varepsilon_3(T),T) = \frac{4}{\pi^2} - \frac{16\pi^2}{\omega_0\omega_+}\frac{1}{T^{2\frac{V+v-2\gamma v}{V+v}}}\left(1+\mathcal{O}\left(\frac{1}{\log T}\right)\right).$$

This entails that for $T(\gamma)$ large enough and $T > T(\gamma)$ we have

$$\begin{aligned}&\widetilde{\eta}^X(\varepsilon_1(T),T) > \widetilde{\eta}^X(\varepsilon_2(T),T) \qquad \text{and}\\ &\widetilde{\eta}^X(\varepsilon_1(T),T) > \widetilde{\eta}^X(\varepsilon_3(T),T)\end{aligned}$$

since $V - v > \frac{1}{\gamma}(V-v)$ and $V - v > V + v - 2\gamma v$ for $\gamma > 1$.

Since the differences between the values taken are of higher order than the bounds for the remainder terms, this completes the proof. ■

Let us finish with some remarks concerning the dependence of the optimal tuning rate on the geometry of the potential U. We have seen in Chapter 4 that for some values of the pre-factors p and q and half-period T the tuning curve vanishes at certain noise levels or is monotonically increasing, see Proposition 4.2.2. We do not observe such a phenomenon in the present setting since we consider the SPA coefficient of the diffusion in the small noise and large period limits. Recall that the Markov chain SPA coefficient vanishes at $\widehat{\varepsilon} = (V-v)/\log\left(\frac{p}{q}\right)$ which is a positive number independent of ε and T. Of course, it can happen that η^X or $\widetilde{\eta}^X$ vanishes for some noise intensity. However, our approach describes neither this effect nor monotonicity of $\widetilde{\eta}^X$. The reason is this: in the small noise limit considered here we are outside of the domains of parameter space for which this behaviour is exhibited.

Appendix A

Laplace's method

In this Appendix Laplace's method of asymptotic evaluation of integrals depending on parameter is explained. In our exposition we follow [20] and [50].

Consider the integral

$$I(\varepsilon) = \int_a^b e^{-\frac{2U(x)}{\varepsilon}} w(x)\, dx, \tag{A.1}$$

in which $a, b \in [-\infty, +\infty]$, U and w are smooth functions on $\mathbb{R}$, $\varepsilon > 0$. The following powerful method for approximating $I(\varepsilon)$, $\varepsilon \to 0$, goes back to Laplace [39]. According to Laplace, the major contribution to the value of the integral arises from the immediate vicinity of those points of the interval $[a, b]$ at which U assumes its smallest value. Let the minimum of U occur, say, at $x = x_0$. If ε is small, the graph of the integrand has a very sharp peak at x_0. It suggests that the overwhelming contribution to the integral comes from the neighbourhood of x_0. Accordingly, we replace U and w by the leading terms in their series expansions in $x - x_0$, and then extend the integration limits to $\pm\infty$. The evaluation of the resulting integral yields the required approximation.

We consider two major cases. Suppose first that a is finite, $x_0 = a$, $U'(a) > 0$ and $w(a) \neq 0$. Then Laplace's estimation reads as follows

$$\begin{aligned} I(\varepsilon) &= \int_a^b e^{-\frac{2U(x)}{\varepsilon}} w(x)\, dx \doteqdot \int_a^b e^{-\frac{2}{\varepsilon}(U(a)+(x-a)U'(a))} w(a)\, dx \\ &\doteqdot w(a) e^{-\frac{2U(a)}{\varepsilon}} \int_a^\infty e^{-\frac{2}{\varepsilon}(x-a)U'(a)}\, dx = \frac{\varepsilon w(a) e^{-\frac{2U(a)}{\varepsilon}}}{2U'(a)}. \end{aligned}$$

The second major case arises when U has a simple minimum at an interior point x_0 of (a, b) and $w(x_0) \neq 0$. Then

$$\begin{aligned} I(\varepsilon) &= \int_a^b e^{-\frac{2U(x)}{\varepsilon}} w(x)\, dx \doteqdot \int_a^b e^{-\frac{2}{\varepsilon}(U(x_0)+\frac{1}{2}(x-x_0)^2 U''(x_0))} w(x_0)\, dx \\ &\doteqdot w(x_0) e^{-\frac{2U(x_0)}{\varepsilon}} \int_{-\infty}^\infty e^{-\frac{U''(x_0)}{\varepsilon}(x-x_0)^2}\, dx = w(x_0) e^{-\frac{2U(x_0)}{\varepsilon}} \sqrt{\frac{\pi\varepsilon}{U''(x_0)}}. \end{aligned}$$

If U has a finite number of minima, we may break up the integral (A.1) into a finite number of integrals so that in each interval U reaches its minimum at one of the end-points and at no other point. Accordingly, we shall assume that U reaches its minimum at $x = a$ and that $U(x) > U(a)$, $a < x \leq b$. Now we precisely formulate the theorem about Laplace's approximation, see [50, Chapters 7,9].

Theorem A.0.2 *Let $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$, $a < b$. Let $U : \mathbb{R} \to \mathbb{R}$ be differentiable, and $w : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ be measurable.*

Suppose in addition that

i) the minimum of U is attained only at a;

ii) U' and w are continuous in a neighbourhood of a;

iii) as $x \downarrow a$,

$$U(x) = U(a)+P(x-a)^{\mu}+\mathcal{O}((x-a)^{\mu+1}), \quad w(x) = Q(x-a)^{\lambda-1}+\mathcal{O}((x-a)^{\lambda}),$$

and the first of these relations is differentiable. Here P, μ and λ are positive constants, and Q is a real or complex constant.

iv)

$$I(\varepsilon) = \int_a^b e^{-\frac{2U(x)}{\varepsilon}} w(x)\, dx,$$

converges absolutely throughout its range for all sufficiently small ε.

Then

$$I(\varepsilon) = \frac{Q}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\left(\frac{\varepsilon}{2P}\right)^{\frac{\lambda}{\mu}} e^{-\frac{2U(a)}{\varepsilon}}(1 + \mathcal{O}(\varepsilon^{\frac{1}{\mu}}))$$

If the asymptotic expansions in ascending powers of $x - a$ exist for U and w, the expansion of the integral $I(\varepsilon)$ can be also obtained. Although there is no general formula for this expansion, we determine its first three terms.

Theorem A.0.3 *Let conditions (i), (ii) and (iv) of Theorem A.0.2 be satisfied and the expansions*

$$U(x) = U(a) + \sum_{s=0}^{n-1} p_s(x-a)^{\mu+s} + \mathcal{O}\left((x-a)^{\mu+n}\right),$$

$$w(x) = \sum_{s=0}^{n-1} q_s(x-a)^{\lambda-1+s} + \mathcal{O}\left((x-a)^{\lambda+n}\right)$$

hold. Suppose that $p_0 \neq 0$, $q_0 \neq 0$. Then

$$I(\varepsilon) = e^{-\frac{2U(a)}{\varepsilon}}\left[\sum_{s=0}^{n-1}\Gamma\left(\frac{\lambda+s}{\mu}\right)a_s\left(\frac{\varepsilon}{2}\right)^{\frac{\lambda+s}{\mu}} + \mathcal{O}(\varepsilon^{\frac{\lambda+n}{\mu}})\right], \tag{A.2}$$

where

$$
\begin{aligned}
a_0 &= \frac{q_0}{\mu p_0^{\lambda/\mu}}, \\
a_1 &= \left\{ \frac{q_1}{\mu} - \frac{(\lambda+1)p_1 q_0}{\mu^2 p_0} \right\} \frac{1}{p_0^{(\lambda+1)/\mu}}, \\
a_2 &= \left[\frac{q_2}{\mu} - \frac{(\lambda+2)p_1 q_1}{\mu^2 p_0} + \{(\lambda+\mu+2)p_1^2 - 2\mu p_0 p_2\} \frac{(\lambda+2)q_0}{2\mu^3 p_0^2} \right] \frac{1}{p_0^{(\lambda+2)/\mu}}.
\end{aligned} \tag{A.3}
$$

Let us apply Theorems A.0.2 and A.0.3 to the double-well potential U from Chapters 5 and 6 to find the asymptotics of the integral $\int_{\mathbb{R}} e^{-\frac{2U(x)}{\varepsilon}}\, dx$ for $\varepsilon \to 0$.

The function U is supposed to be infinitely differentiable and to possess a unique global minimum at -1 such that $U(-1) = -\frac{V}{2}$. We break the interval $(-\infty, +\infty)$ into two intervals $(-\infty, -1]$ and $[-1, +\infty)$, and note that

$$
\begin{aligned}
\int_{\mathbb{R}} e^{-\frac{2U(x)}{\varepsilon}}\, dx &= \int_{-\infty}^{-1} e^{-\frac{2U(x)}{\varepsilon}}\, dx + \int_{-1}^{\infty} e^{-\frac{2U(x)}{\varepsilon}}\, dx \\
&= \int_{-1}^{+\infty} e^{-\frac{2U(x)}{\varepsilon}}\, dx + \int_{1}^{+\infty} e^{-\frac{2\overline{U}(x)}{\varepsilon}}\, dx,
\end{aligned} \tag{A.4}
$$

where $\overline{U}(x) = U(-x)$, $x \in \mathbb{R}$. Both integrals in the last line of (A.4) satisfy the conditions of Theorem A.0.3. To determine the coefficients p_k, $k = 0, 1, 2$, we expand U near -1 and $\overline{U}$ near 1 to get

$$
U(x) = -\frac{V}{2} + \frac{\omega_-}{2}(x+1)^2 + \frac{U^{(3)}(-1)}{6}(x+1)^3 + \frac{U^{(4)}(-1)}{24}(x+1)^4 + \mathcal{O}((x+1)^5), \tag{A.5}
$$

$$
\overline{U}(x) = -\frac{V}{2} + \frac{\omega_-}{2}(x-1)^2 - \frac{U^{(3)}(-1)}{6}(x-1)^3 + \frac{U^{(4)}(-1)}{24}(x-1)^4 + \mathcal{O}((x-1)^5). \tag{A.6}
$$

Thus, $\mu = 2$, $\lambda = 1$, $q_0 = 1$ and $q_k = 0$, $k \geq 1$. A direct application of (A.3) and (A.2) yields

$$
\begin{aligned}
&\int_{-1}^{+\infty} e^{-\frac{2U(x)}{\varepsilon}}\, dx = e^{\frac{V}{\varepsilon}} \frac{1}{2} \sqrt{\frac{\pi\varepsilon}{\omega_-}} \\
&\times \left[1 - \frac{U^{(3)}(-1)}{3\omega_-^{3/2}\sqrt{\pi}} \sqrt{\varepsilon} + \frac{1}{16\omega_-^3} \left(\frac{5U^{(3)}(-1)^2}{3} - \omega_- U^{(4)}(-1) \right) \varepsilon + \mathcal{O}(\varepsilon^{3/2}) \right] \\
&\int_{1}^{+\infty} e^{-\frac{2\overline{U}(x)}{\varepsilon}}\, dx = e^{\frac{V}{\varepsilon}} \frac{1}{2} \sqrt{\frac{\pi\varepsilon}{\omega_-}} \\
&\times \left[1 + \frac{U^{(3)}(-1)}{3\omega_-^{3/2}\sqrt{\pi}} \sqrt{\varepsilon} + \frac{1}{16\omega_-^3} \left(\frac{5U^{(3)}(-1)^2}{3} - \omega_- U^{(4)}(-1) \right) \varepsilon + \mathcal{O}(\varepsilon^{3/2}) \right],
\end{aligned}
$$

and, consequently,

$$\int_{\mathbb{R}} e^{-\frac{2U(x)}{\varepsilon}}\,dx = e^{\frac{V}{\varepsilon}}\sqrt{\frac{\pi\varepsilon}{\omega_-}}\left[1+\frac{1}{16\omega_-^3}\left(\frac{5U^{(3)}(-1)^2}{3}-\omega_- U^{(4)}(-1)\right)\varepsilon+\mathcal{O}(\varepsilon^{3/2})\right]. \tag{A.7}$$

The error term $\mathcal{O}(\varepsilon^{3/2})$ is in fact of order ε^2, since due to the infinite differentiability of U all terms a_k with odd indices k vanish. This variant of the asymptotics is used in Lemma 6.3.2. The less exact asymptotics

$$\int_{\mathbb{R}} e^{-\frac{2U(x)}{\varepsilon}}\,dx = e^{\frac{V}{\varepsilon}}\sqrt{\frac{\pi\varepsilon}{\omega_-}}(1+\mathcal{O}(\varepsilon)). \tag{A.8}$$

is also used.

Analogously, one evaluates the integral

$$\int_{\mathbb{R}} x\,e^{-\frac{2U(x)}{\varepsilon}}\,dx = \int_{-1}^{\infty} x\,e^{-\frac{2U(x)}{\varepsilon}}\,dx - \int_{1}^{\infty} x\,e^{-\frac{2\overline{U}(x)}{\varepsilon}}\,dx =$$

$$-\,e^{\frac{V}{\varepsilon}}\sqrt{\frac{\pi\varepsilon}{\omega_-}}\left[1+\left\{\frac{U^{(3)}(-1)}{4\omega_-^2}+\frac{1}{16\omega_-^3}\left(\frac{5U^{(3)}(-1)^2}{3}-\omega_- U^{(4)}(-1)\right)\right\}\varepsilon+\mathcal{O}(\varepsilon^2)\right]. \tag{A.9}$$

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List of Notations

$\mathbb{R}$	real numbers
$\mathbb{R}_+$	$\{x \in \mathbb{R} : x \geq 0\}$ non-negative real numbers
Λ	Lebesgue measure on $\mathbb{R}$
$\mathcal{C}(\mathbb{R})$	continuous functions on $\mathbb{R}$
$\mathcal{C}_0^\infty(\mathbb{R})$	infinitely differentiable functions on $\mathbb{R}$ with compact support
$\mathcal{C}^\infty(\mathbb{R})$	infinitely differentiable functions on $\mathbb{R}$
$\mathcal{L}^2(\mathbb{R})$	square-integrable functions on $\mathbb{R}$
$\mathcal{L}^2(\mathbb{R}, \rho\, dx)$	weighted $\mathcal{L}^2$ space
$(\cdot, \cdot)$	inner product in $\mathcal{L}^2(\mathbb{R})$
$\lVert \cdot \rVert$	norm in $\mathcal{L}^2(\mathbb{R})$
$(\cdot, \cdot)_\rho$	inner product in $\mathcal{L}^2(\mathbb{R}, \rho\, dx)$
$\lVert \cdot \rVert_\rho$	norm in $\mathcal{L}^2(\mathbb{R}, \rho\, dx)$
$\approx$	$a(\varepsilon) \approx b(\varepsilon)$ iff $\frac{a(\varepsilon)}{b(\varepsilon)} = 1 + \mathcal{O}(\varepsilon)$
ε	noise parameter, pp. 3, 14, 21, 32, 52, 82
L	infinitesimal generator of time-homogeneous diffusion in a double-well potential $U(x)$, pp. 52, 58, 86
L^*	formal adjoint of L, pp. 53, 59, 86
$\rho = e^{\frac{2U}{\varepsilon}}$	weight function, pp. 54, 86
$\{-\lambda_k, \Phi_k\}_{k\geq 0}$	eigenvalues and eigenfunctions of L in $\mathcal{L}^2(\mathbb{R}, \rho^{-1})$, p. 61
$\{-\lambda_k, \Psi_k\}_{k\geq 0}$	eigenvalues and eigenfunctions of L^* in $\mathcal{L}^2(\mathbb{R}, \rho)$, pp. 53, 86
T	half-period of deterministic component, pp. 2, 16, 32, 82
$U(x)$	double-well potential, pp. 4, 15, 54, 82
$\omega_\pm, \omega_0$	second derivarives of $U(x)$ at ± 1 and 0, pp. 54, 76, 95
$U(x,t)$	space-time antisymmetric double-well potential, pp. 3, 16, 82
$X^{\varepsilon,T}$	one-dimensional diffusion with time-periodic drift, pp. 4, 16, 82
$\mathbf{X}^{\varepsilon,T}$	two-dimensional time-homogeneous process, p. 84
$Y^{\varepsilon,T}$	one-dimensional Markov chain with continuous time and $2T$-periodic infinitesimal probabilities, p. 32
$\mathbf{Y}^{\varepsilon,T}$	two-dimensional time-homogeneous Markov chain with continuous time, p. 32
Z_m	one-dimensional Markov chain with discrete time and $2m$-periodic transition probabilities, p. 20
$\mathbf{Z}_m$	two-dimensional time-homogeneous Markov chain with discrete time, p. 21
$\mu^{\varepsilon,T}$	invariant density of $X^{\varepsilon,T}$, p. 84
$\nu^\pm_{\varepsilon,T}(\theta)$	invariant measure of $Y^{\varepsilon,T}$, p. 33
$\pi^\pm_m(k)$	invariant measure of Z_m, p. 22

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