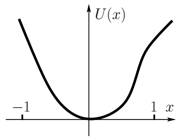
Peter Imkeller*
Ilya Pavlyukevich*
Torsten Wetzel*

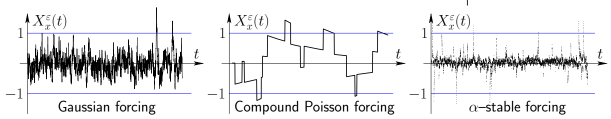
Exit Times of Lévy–Driven Diffusions⋆

1. First exit problem

$$X_x^\varepsilon(t) = x - \int_0^t U'(X_x^\varepsilon(s))\,ds + \varepsilon L(t)$$

$$\sigma_x(\varepsilon) = \inf\{t \geq 0 \,:\, X_x^\varepsilon(t) \notin [-1,1]\}$$





Gaussian forcing | Compound Poisson forcing | α–stable forcing

2. Symmetric Lévy processes

Independent and stationary increments, $L(0) = 0$, stochastically continuous (no fixed jumps), paths are right–continuous and have left limits. Lévy–Hinchin–formula holds for marginal distributions:

$$\mathbf{E}e^{i\lambda L(t)} = \exp\Big[-td\frac{\lambda^2}{2} + t\int_{\mathbb{R}\setminus\{0\}}\big(e^{i\lambda y} - 1 - \frac{i\lambda y}{1+y^2}\big)\nu(dy)\Big].$$

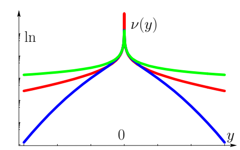
Gaussian variance $d \geq 0$.
Jump measure ν symmetric w.r.t. 0, $\int_{\mathbb{R}\setminus\{0\}}(1\wedge y^2)\nu(dy) < \infty$.

3. Jump measure ν

$\sharp\{s \,:\, (s, L(s)-L(s-)) \in (0,t]\times A\} \stackrel{d}{=}$ **POISSON**$(t\nu(A))$.
$\mathbf{E}\,\sharp\{s \,:\, (s, L(s)-L(s-)) \in (0,t]\times A\} = t\nu(A)$.
Brownian motion: $\nu \equiv 0$.
Symmetric Poisson process: $\nu(y) = \frac{1}{2}(\delta(y-1)+\delta(y+1))$.
Lévy flights: $\nu(y) = |y|^{-1-\alpha}$, $\alpha \in (0,2)$.

4. Weight of big jumps

Weakly tempered Lévy flights:
$\nu(y) = |y|^{-1-\beta_1}(1+y^2)^{-\beta_2/2}$, $\beta_2 > 0$.
Strongly tempered Lévy flights:
$\nu(y) = |y|^{-1-\beta}\exp(-|y|^\alpha)$, $\alpha > 0$.
Subexponential tails $0 < \alpha < 1$.
Superexponential tails $\alpha > 1$.

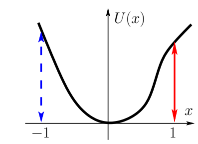


5. Gaussian forcing

L is a standard Brownian motion.
Math. studies: Freidlin & Wentzell ('70s), Day ('82), Bovier *et al.* ('05),
Physical studies: Eyring ('35), Kramers ('40):

$$\mathbf{P}\Big(\frac{\sigma_x(\varepsilon)}{\mathbf{E}\sigma_x(\varepsilon)} > t\Big) \to e^{-t},\ \varepsilon \to 0,$$

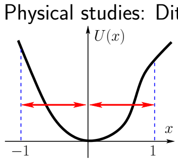
$$\mathbf{E}\sigma_x(\varepsilon) \approx \frac{\varepsilon\sqrt{\pi}}{U'(1)\sqrt{U''(0)}}\exp\Big(\frac{2U(1)}{\varepsilon^2}\Big).$$



To exit, the diffusion overcomes the lowest potential barrier. Unpredictable exponentially long exit times depend on the potential's energy landscape.

6. Forcing with heavy jumps

L has heavy jumps: $\nu(u, +\infty) \approx u^{-r}$, $r > 0$, $u \to +\infty$.
Math. studies: Godovanchuk ('81), Imkeller & Pavlyukevich ('06),
Physical studies: Ditlevsen ('98), Chechkin *et al.* ('05):



$$\mathbf{P}\Big(\frac{\sigma_x(\varepsilon)}{\mathbf{E}\sigma_x(\varepsilon)} > t\Big) \to e^{-t},\ \varepsilon \to 0,$$

$$\mathbf{E}\sigma_x(\varepsilon) \approx \nu\Big(\big[-\tfrac{1}{\varepsilon}, \tfrac{1}{\varepsilon}\big]^c\Big) = \frac{2}{\varepsilon^r}.$$

The diffusion exits with a single big jump. Unpredictable poynomially long exit times depend only on the size of the domain.

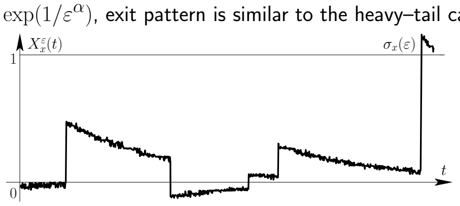
7. Forcings with sub-exponential jumps

Sub-exponentially light tails: $\nu(u,\infty) \approx \exp(-u^\alpha)$, $0 < \alpha < 1$.

$$\exp(-C_\varepsilon^{1-\delta}t) \leq \mathbf{P}(\sigma_x(\varepsilon) > t) \leq \exp(-C_\varepsilon^{1+\delta}t),\ \delta > 0,$$

$C_\varepsilon = \nu\Big(\big[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\big]^c\Big) = 2\exp(-1/\varepsilon^\alpha)$.

$\mathbf{E}\sigma_x(\varepsilon) \propto \exp(1/\varepsilon^\alpha)$, exit pattern is similar to the heavy–tail case.



Exit via one big jump.

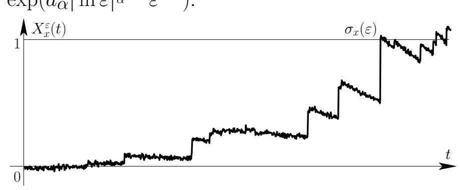
8. Forcing with super-exponential jumps

Super-exponentially light tails: $\nu(u,\infty) \approx \exp(-u^\alpha)$, $\alpha > 1$.

$$\exp(-D_\varepsilon^{1-\delta}t) \leq \mathbf{P}(\sigma_x(\varepsilon) > t) \leq \exp(-D_\varepsilon^{1+\delta}t),\ \delta > 0,$$

$D_\varepsilon = \exp(-d_\alpha|\ln\varepsilon|^{\frac{1}{\alpha}-1}\varepsilon^{-1})$, $d_\alpha = \alpha(\alpha-1)^{\frac{1}{\alpha}-1}$,

$\mathbf{E}\sigma_x(\varepsilon) \propto \exp(d_\alpha|\ln\varepsilon|^{\frac{1}{\alpha}-1}\varepsilon^{-1})$.



Exit via 'climbing' in one direction.

9. Phase transition at $\alpha = 1$

Important part of the proof: minimisation problem

$$\sum_{i=1}^n x_i^\alpha \to \min \quad \text{under constraints} \quad \varepsilon\sum_{i=1}^n x_i = 1,\ x_i \geq 0.$$

Different solutions due to convexity/concavity of $x \mapsto x^\alpha$:

$$\min = \begin{cases} \frac{1}{\varepsilon^\alpha} & \text{for } x_i = \frac{1}{\varepsilon}, x_j = 0, i \neq j; \\ n\Big(\frac{1}{n\varepsilon}\Big)^\alpha & \text{for } x_i = \frac{1}{n\varepsilon}. \end{cases}$$

10. Jumps are always faster than diffusion

The limit case $\alpha = +\infty$ corresponds to the case of bounded jumps.
All Lévy non-Gaussian forcings of the form $\varepsilon L(t)$ induce exit times

$$\mathbf{E}\sigma_x(\varepsilon) \lesssim \exp\big(\tfrac{c}{\varepsilon|\ln\varepsilon|}\big) \ll \exp\big(\tfrac{c}{\varepsilon^2}\big).$$

No forcing of the type εL can fill the gap between non-Gaussian and Gaussian time scales. Lévy forcings with ε–dependent jump measures ν_ε, e.g. a symmetric Poisson process with $\nu_\varepsilon(y) = \frac{1}{2\varepsilon}(\delta(y-\varepsilon)+\delta(y+\varepsilon))$, can lead to Gaussian–like asymptotics.

Tails $\nu(u,+\infty) \approx$	$\frac{1}{u^r}$, $r > 0$	$\exp(-u^\alpha)$			Gaussian
		$\alpha \in (0,1)$	$\alpha \in (1,+\infty)$	bounded jumps, $\alpha = +\infty$	
$\mathbf{E}\sigma_x(\varepsilon) \propto$	$\frac{c}{\varepsilon^r}$	$\exp\Big(\frac{c}{\varepsilon^\alpha}\Big)$	$\exp\Big(\frac{c}{\varepsilon\lvert\ln\varepsilon\rvert^{1-\frac{1}{\alpha}}}\Big)$	$\exp\Big(\frac{c}{\varepsilon\lvert\ln\varepsilon\rvert}\Big)$	$\exp\Big(\frac{c}{\varepsilon^2}\Big)$

* Humboldt–Universität zu Berlin, Institut für Mathematik, Rudower Chaussee 25, 12489 Berlin, Germany, imkeller@Z, pavljuke@Z, wetzel@Z, Z=mathematik.hu-berlin.de.

⋆ P. Imkeller, I. Pavlyukevich and T. Wetzel, First exit times for Lévy–driven diffusions with exponentially light jumps, arXiv:0711.0982.