First exit time of solutions of non-linear SDEs driven by α-stable symmetric Lévy motion

Ilya Pavlyukevich
Humboldt-Universität zu Berlin

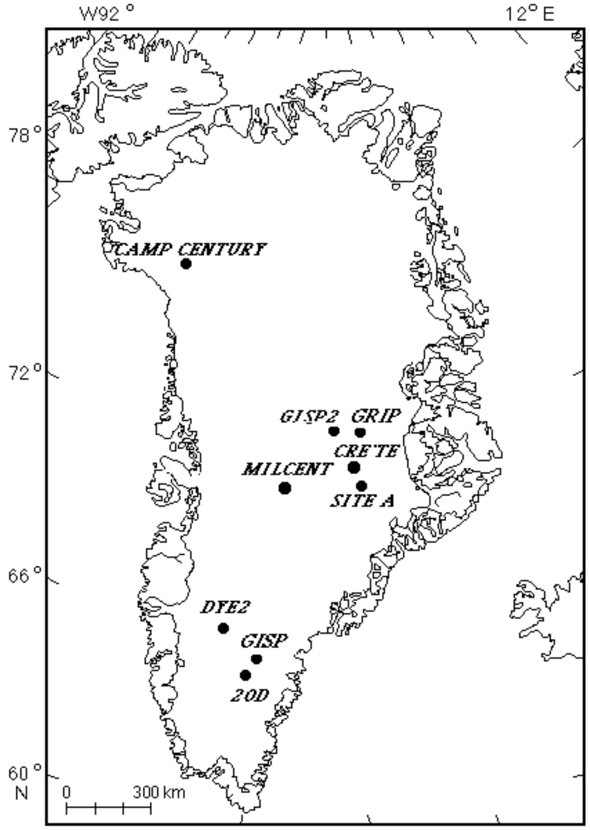
25 March 2004

DFG Research Center
mathematics for key technologies

1. Motivation

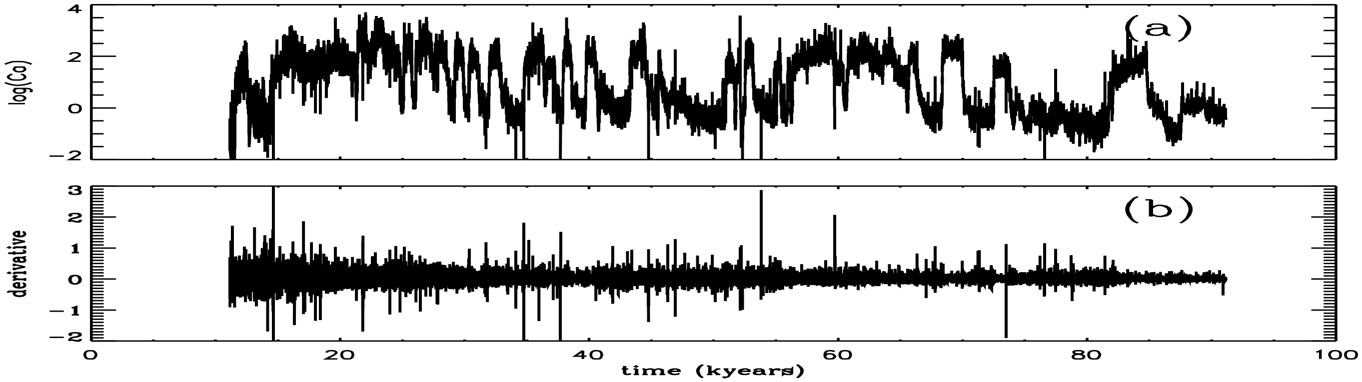
Paleoclimatic records from Greenland ice-core show rapid climate transitions between

- the **cold glacial** periods and
- the **warmer interstadials** (Dansgaard-Oeschger events)



2. Paleo Proxy Data

The calcium (Ca) signal from the GRIP ice-core:
about 80,000 data-points from 11 kyr to 91 kyr before present.



Typical interjump time: between 1000 and 2000 years.
What triggers the transitions?

Langevin equation for the climate dynamics

$$\frac{d}{dt}X(t) = -U'(X(t)) + \text{NOISE}$$

U - double-well potential, wells correspond to the climate states.

P. Ditlevsen (*Geophys. Res. Lett. 1999*): spectral analysis of the data.
NOISE has an α-stable component with $\alpha \approx 1.75$.

3. Object of Study

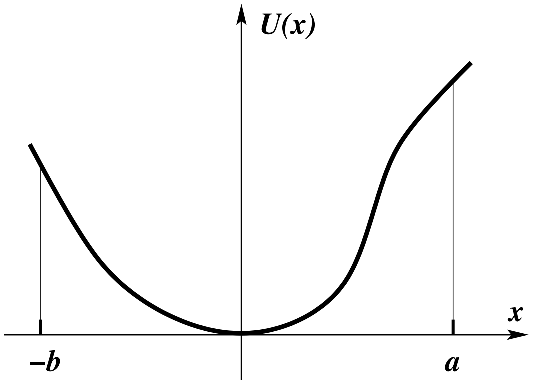
SDE driven by α-stable Lévy process of small intensity:

$$X_t^\varepsilon = x - \int_0^t U'(X_{s-}^\varepsilon)\, ds + \varepsilon L_t, \quad \varepsilon \downarrow 0.$$

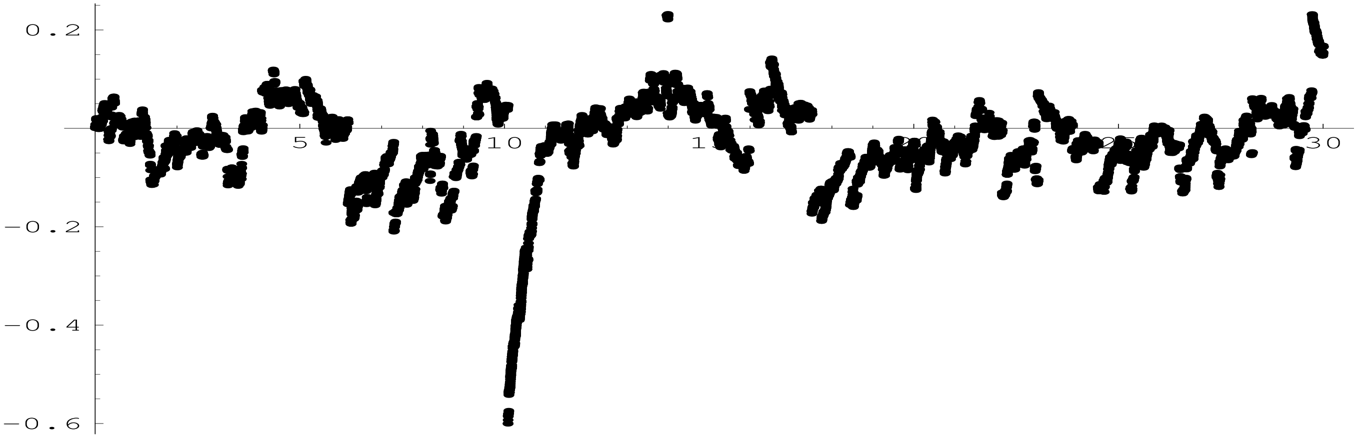
- L — α-stable symmetric Lévy motion, $\alpha \in (0,2)$

Potential $U \in \mathcal{C}^{(3)}(\mathbb{R})$:

- $U'(x)x \geq 0$
- $U'(x) = 0$ iff $x = 0$
- $U''(0) = M > 0$



$$\sigma(\varepsilon) = \inf\{t \geq 0 \,:\, X_t^\varepsilon \notin [-b,a]\}, \quad a,b < \infty$$



4. α–stable Lévy process L

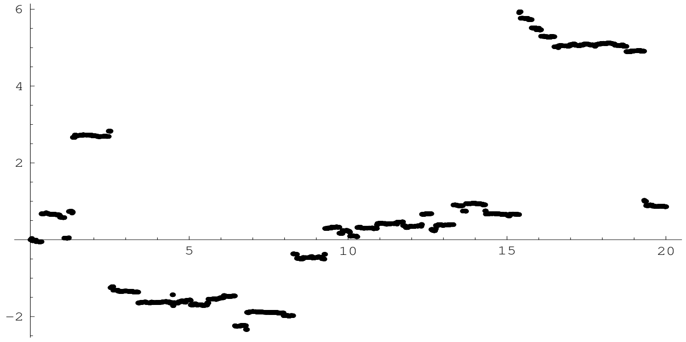
The random process L:

- continuous in probability
- independent stationary increments
- càdlàg paths
- symmetric

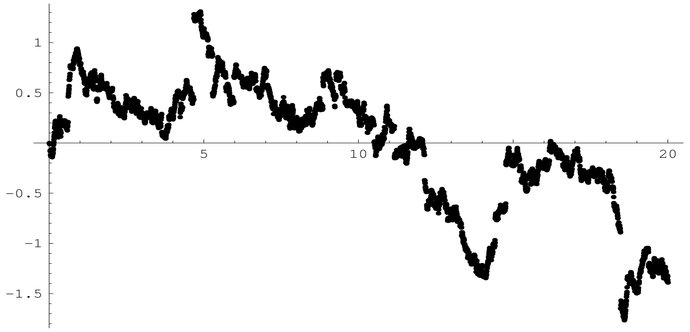
Characteristic function

$$\mathbf{E}e^{i\lambda L_t} = \exp\left\{ t\int_{\mathbb{R}\backslash\{0\}} (e^{i\lambda y} - 1 - i\lambda y\,\mathbb{I}\{|y|<1\})\frac{dy}{|y|^{1+\alpha}} \right\}, \quad \alpha \in (0,2)$$

The Lévy measure of L: $\nu(dy) = \frac{dy}{|y|^{1+\alpha}}$, $\nu(\mathbb{R}) = \infty$ for $\alpha \in (0,2)$.



$\alpha = 0.75$



$\alpha = 1.75$

5. Main result

Theorem 1. *There exist positive constants* ε_0, γ, δ, *and* $C > 0$ *such that for* $0 < \varepsilon \le \varepsilon_0$ *the following asymptotics holds*

$$\begin{aligned}\exp\left\{-u\frac{\varepsilon^\alpha}{\alpha}\left[\frac{1}{a^\alpha}+\frac{1}{b^\alpha}\right](1+C\varepsilon^\delta)\right\}(1-C\varepsilon^\delta)&\\ \le \mathbf{P}_x(\sigma(\varepsilon)>u)&\\ \le \exp\left\{-u\frac{\varepsilon^\alpha}{\alpha}\left[\frac{1}{a^\alpha}+\frac{1}{b^\alpha}\right](1-C\varepsilon^\delta)\right\}(1+C\varepsilon^\delta)&\end{aligned}$$

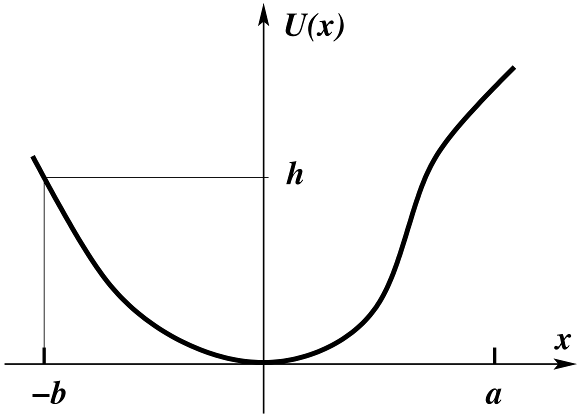
uniformly for all $x \in [-b+\varepsilon^\gamma, a-\varepsilon^\gamma]$ *and* $u \ge 0$*.*

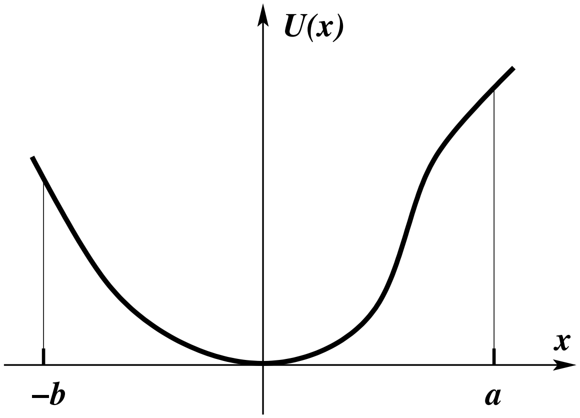
Theorem 2. *There exist positive constants* ε_0, γ *and* δ *such that for* $0 < \varepsilon \le \varepsilon_0$ *the following asymptotics holds*

$$\mathbf{E}_x\sigma(\varepsilon) = \frac{\alpha}{\varepsilon^\alpha}\left[\frac{1}{a^\alpha}+\frac{1}{b^\alpha}\right]^{-1}(1+\mathcal{O}(\varepsilon^\delta))$$

uniformly for all $x \in [-b+\varepsilon^\gamma, a-\varepsilon^\gamma]$*.*

6. Comparison with Gaussian Case





$$X_t^\varepsilon = x - \int_0^t U'(X_s^\varepsilon)\,ds + \varepsilon W_t$$

$$X_t^\varepsilon = x - \int_0^t U'(X_{s-}^\varepsilon)\,ds + \varepsilon L_t$$

Freidlin–Wentsel:

$$\mathbf{P}_x(e^{(2h-\delta)/\varepsilon^2} < \sigma < e^{(2h+\delta)/\varepsilon^2}) \to 1$$

$$\mathbf{P}_x(\tfrac{1}{\varepsilon^{\alpha-\delta}} < \sigma < \tfrac{1}{\varepsilon^{\alpha+\delta}}) \to 1$$

Eyring-Kramers law (Williams, Bovier):

$$\mathbf{E}_x\sigma \approx \frac{\varepsilon\sqrt{\pi}}{|U'(-b)|\sqrt{U''(0)}}e^{2h/\varepsilon^2}$$

$$\mathbf{E}_x\sigma \approx \tfrac{\alpha}{\varepsilon^\alpha}\big[\tfrac{1}{a^\alpha} + \tfrac{1}{b^\alpha}\big]^{-1}$$

Exponential exit (Day, Bovier)

$$\mathbf{P}_x(\tfrac{\sigma}{\mathbf{E}_x\sigma} > u) \sim \exp(-u)$$

$$\mathbf{P}_x(\tfrac{\sigma}{\mathbf{E}_x\sigma} > u) \sim \exp(-u)$$

Diffusion ‘climbs up and out’

Lévy motion driven SDE ‘jumps out’

7. Sketch of the Proof

$$L_t = \xi_t^\varepsilon + \eta_t^\varepsilon$$

- $\nu_\xi^\varepsilon(\,\cdot\,) = \nu(\,\cdot\, \cap \{0 < |y| \le \frac{1}{\sqrt{\varepsilon}}\})$
$\nu_\xi^\varepsilon(\mathbb{R}) = \infty$

- $|$Jumps of $\xi^\varepsilon| \le \frac{1}{\sqrt{\varepsilon}}$

- $\nu_\eta^\varepsilon(\,\cdot\,) = \nu(\,\cdot\, \cap \{|y| > \frac{1}{\sqrt{\varepsilon}}\})$
Finite Lévy measure
$\nu_\eta^\varepsilon(\mathbb{R}) = \beta_\varepsilon = \frac{2}{\alpha}\varepsilon^{\alpha/2}$

- η^ε is a compound Poisson process

- τ_k its arrival times
Inter-jump times
$T_k = \tau_k - \tau_{k-1} \sim \exp(\beta_\varepsilon)$

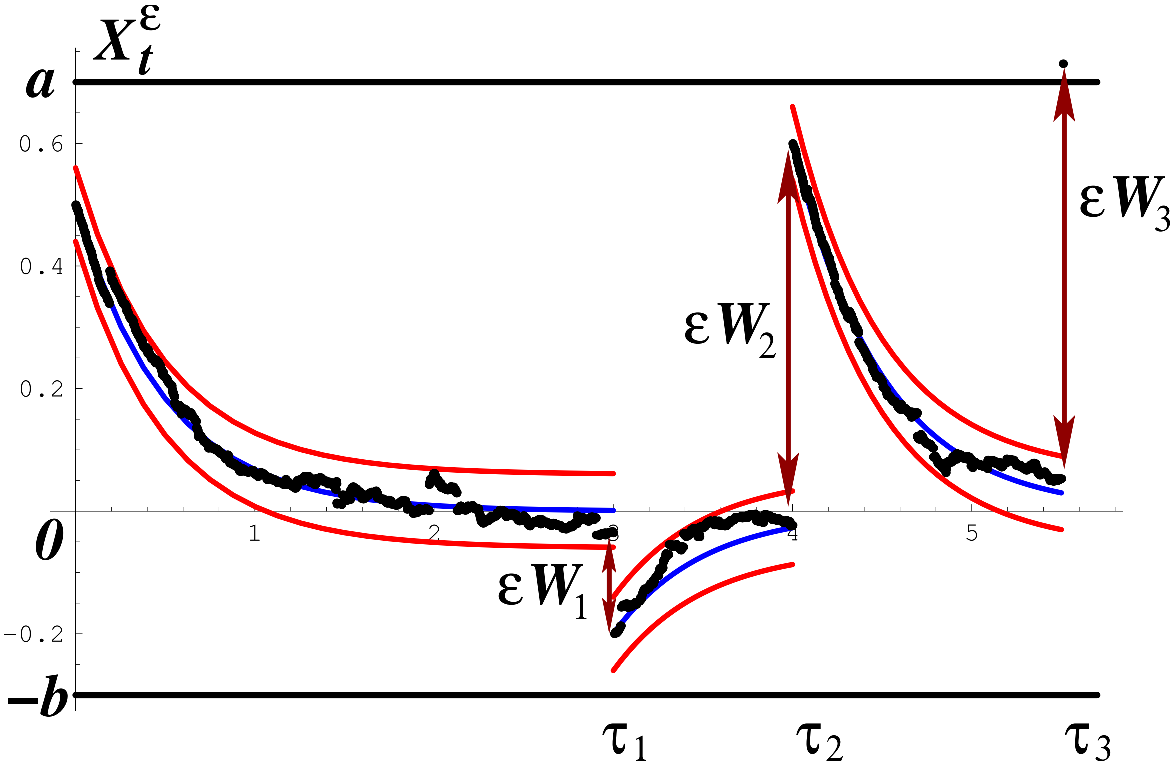
- Jumps $W_k \sim \beta_\varepsilon^{-1}\nu_\eta^\varepsilon(\cdot)$

T_k, W_k and ξ^ε are independent

On $[\tau_{k-1}, \tau_k)$, X^ε is driven by $\varepsilon\xi^\varepsilon$

Deterministic trajectory $Y_t(x) = x - \int_0^t U'(Y_s)\,ds$

8. Sketch of the Proof II



$\mathbf{E}T_k = \frac{1}{\beta_\varepsilon} = \frac{\alpha}{2}\varepsilon^{-\alpha/2} \to \infty$ polynomially fast.

Between the jumps of η^ε, X^ε is driven by $\varepsilon\xi^\varepsilon$.

$\mathbf{P}_x(\sup_{[\tau_{k-1},\tau_k)} |X_t^\varepsilon - Y_t(X^\varepsilon_{\tau_{k-1}})| \geq \varepsilon^\gamma)$ is small

Relaxation time $R(\varepsilon) = \sup\{t > 0 \,:\, |Y_t(x)| \leq \varepsilon^\gamma, x \in [-b, a]\} = \mathcal{O}(|\ln \varepsilon|)$

$R(\varepsilon) \ll \mathbf{E}T_k \Rightarrow$ at $t = \tau_k$, X^ε jumps from a neighbourhood of 0 at εW_k.

9. Sketch of the Proof III

With high probability, X^ε exits $I = [-b, a]$ at arrival times of η^ε, $t = \tau_k$.

$\tau_k = T_1 + T_2 + \cdots + T_k \sim$ Gamma(β_ε, k)

$\mathbf{P}(\tau_k \in [t, t+dt]) = \beta_\varepsilon e^{-\beta_\varepsilon t} \frac{(\beta_\varepsilon t)^{k-1}}{(k-1)!} dt$.

Then for $u \geq 0$,

$$\begin{aligned}
\mathbf{P}_x(\sigma(\varepsilon) > u) &\approx \sum_{k=1}^{\infty} \mathbf{P}(\tau_k > u) \cdot \mathbf{P}_x(\sigma(\varepsilon) = \tau_k) \\
&= \sum_{k=1}^{\infty} \mathbf{P}(\tau_k > u) \cdot \mathbf{P}(\varepsilon W_1 \in I, \ldots, \varepsilon W_{k-1} \in I, \varepsilon W_k \notin I) \\
&= \sum_{k=1}^{\infty} \int_u^\infty \beta_\varepsilon e^{-\beta_\varepsilon t} \frac{(\beta_\varepsilon t)^{k-1}}{(k-1)!} \, dt \cdot (1 - \mathbf{P}(\varepsilon W_1 \notin I))^{k-1} \cdot \mathbf{P}(\varepsilon W_1 \notin I) \\
&= \beta_\varepsilon \mathbf{P}(\varepsilon W_1 \notin I) \int_u^\infty e^{-\beta t} \sum_{k=1}^{\infty} \frac{(\beta_\varepsilon t)^{k-1} (1 - \mathbf{P}(\varepsilon W_1 \notin I))^{k-1}}{(k-1)!} \, dt \\
&= \beta_\varepsilon \mathbf{P}(\varepsilon W_1 \notin I) \int_u^\infty e^{-\beta_\varepsilon t} e^{\beta_\varepsilon t (1 - \mathbf{P}(\varepsilon W_1 \notin I))} \, dt = e^{-u \beta_\varepsilon \mathbf{P}(\varepsilon W_1 \notin I)} = \exp\{-u \frac{\varepsilon^\alpha}{\alpha} \left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha} \right] \}
\end{aligned}$$