Stochastic Resonance

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Motivation

C. Nicolis; Benzi et al. 80'
Energy-balance model for global Earth temperature: Ice Ages

$$c\frac{dX}{dt} = R_{\text{in}} - R_{\text{out}}$$

Stochastic Differential Equation

$$dX_t = -U'(X_t, \frac{t}{T})\, dt + \sqrt{\varepsilon}\, dW_t$$

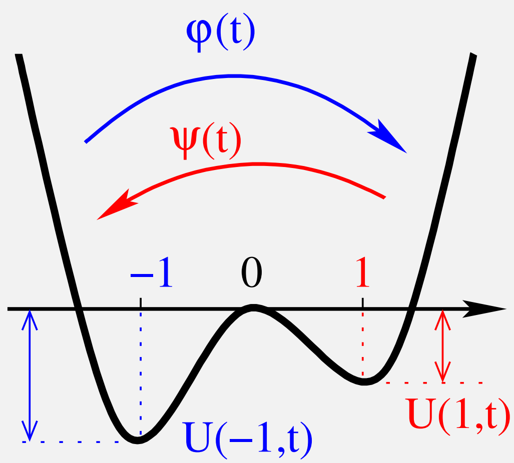
Double-well potential with periodically varying wells' depths
$U'(\pm 1, t) = U'(0, t) = 0$
Period T – large. Noise ε – small.
Different noise intensities lead to different regimes.

Naïve model reduction

Diffusion X, noise $\varepsilon \to 0$

$$dX_t = -U'(X_t, \tfrac{t}{T})\, dt + \sqrt{\varepsilon}\, dW_t$$

Critical points' location does not change with time.
Depths are 1-periodic $|U(-1,t)| = |U(1,t+1)|$
and antisymmetric $|U(-1,t)| = |U(1,t+\frac{1}{2})|$.



2-State Markov Chain Y

$$Q(t) = \begin{pmatrix} -\varphi(\frac{t}{T}) & \varphi(\frac{t}{T}) \\ \psi(\frac{t}{T}) & -\psi(\frac{t}{T}) \end{pmatrix}$$

Kramers' rates

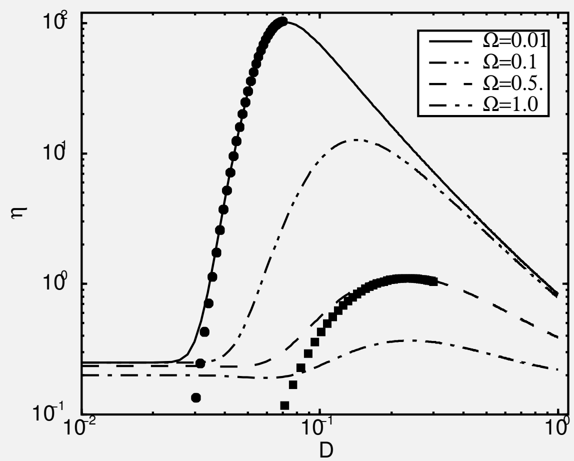
$$\varphi(t) = \frac{\sqrt{|U''(0,t)|U''(-1,t)}}{2\pi} \exp\left(-\frac{2}{\varepsilon}\left(U(0,t) - U(-1,t)\right)\right),$$

$$\psi(t) = \frac{\sqrt{|U''(0,t)|U''(1,t)}}{2\pi} \exp\left(-\frac{2}{\varepsilon}\left(U(0,t) - U(1,t)\right)\right)$$

Questions

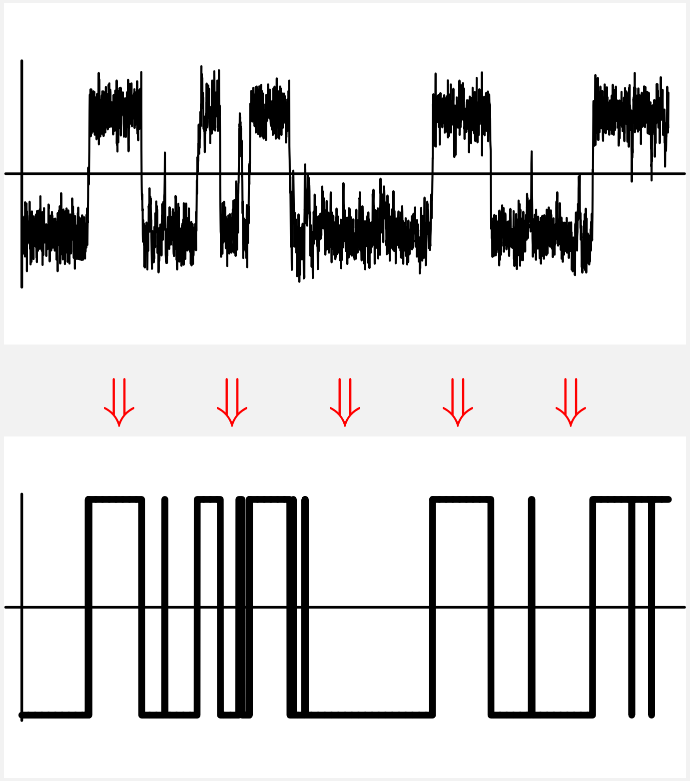
Periodicity

- How to measure **periodicity** of random output?
- How to choose the **optimal** intensity ε to obtain the best quality?



Reduction

- Do the **reduction** to the two-state process preserve the optimal tuning?



Measures of Quality

Consider exponentially long periods $T = \exp\left(\frac{\rho}{\varepsilon}\right)$, $\varepsilon \to 0$.

Energy approach: Spectral Power Amplification

$$\eta(\varepsilon, T) = \left| \int_0^1 \mathbf{E}_\mu(X_{Ts}) \cdot e^{2\pi i s}\, ds \right|^2 \to \max$$

Meaning: energy carried by the averaged trajectories on the frequency of external perturbation.

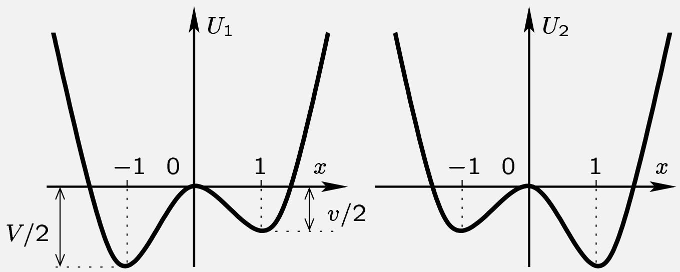
Path-wise approach: Transition Probability

$$M(\varepsilon, T) = \mathbf{P}\left(X_{T\cdot} \begin{array}{c}\text{changes the well}\\ \text{during the time-window}\end{array} [a_\rho - h, a_\rho + h] \right)$$

Meaning: probability to change the well during a h-window of a certain time instant a_ρ.

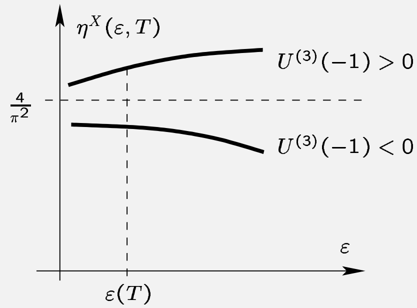
Spectral power Amplification

Diffusion X in ‘flipping’ potential:



$$\frac{4}{\pi^2} \cdot \left(\frac{\int y e^{-2U/\varepsilon}\, dy}{\int e^{-2U/\varepsilon}\, dy} \right)^2 \cdot \frac{\lambda_1^2 T^2}{4\pi^2 + \lambda_1^2 T^2}$$

SPA has no local maximum. It depends on the fine geometry of the potential.

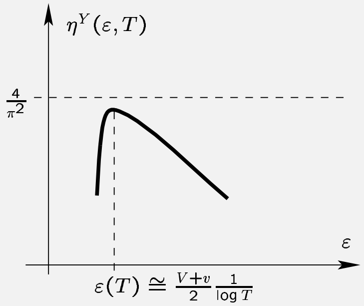


Corresponding Markov chain

$$\varphi = \frac{\sqrt{U''(-1)|U''(0)|}}{2\pi}\, e^{-V/\varepsilon}$$
$$\psi = \frac{\sqrt{U''(1)|U''(0)|}}{2\pi}\, e^{-v/\varepsilon}$$

$$\frac{4}{\pi^2} \cdot \frac{(\varphi - \psi)^2 T^2}{4\pi^2 + (\varphi + \psi)^2 T^2}$$

The uniquely determined optimal tuning is given by the coordinate of the maximum.



Reason: intra-well fluctuations destroy the tuning.

Transition Probability — Robust Quality Measure

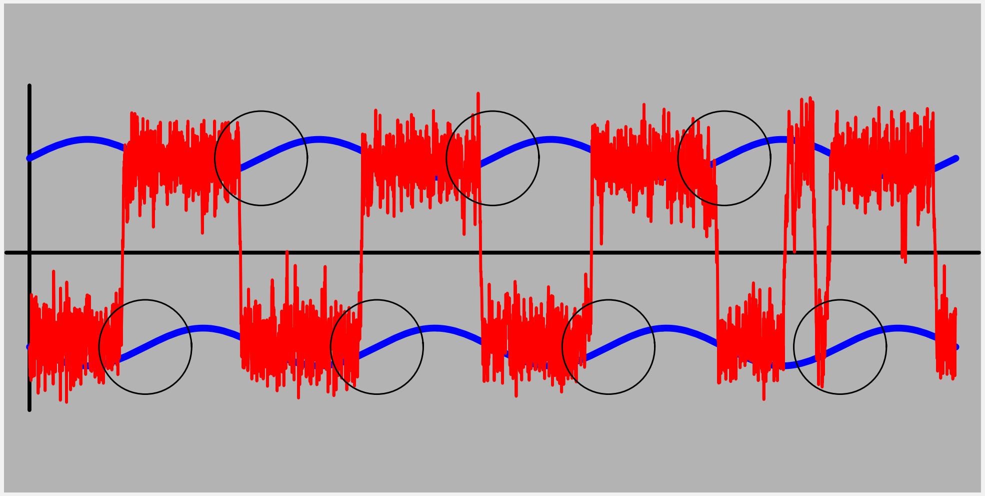
For the diffusion **AND** the corresponding Markov chain:

The family of time scales:
$T = \exp(\rho/\varepsilon)$

The wells' depths:
$|U(-1,t)| = \frac{V+v}{2} + \frac{V-v}{2}\sin(2\pi t)$
$|U(+1,t)| = \frac{V+v}{2} - \frac{V-v}{2}\sin(2\pi t)$

Then the **most regular** behaviour of the processes is obtained at the time scale $T = e^{\frac{V+v}{2\varepsilon}}$.

Transitions of the scaled processes occur at time windows $[\frac{k}{2} - h, \frac{k}{2} + h]$, $k = 1, 2, \dots$, i.e. when the wells are almost equally deep.



Regular: The probability of this behaviour is maximal over all time scales $T = \exp(\rho/\varepsilon)$. It is exponentially close to 1 as $\varepsilon \to 0$.

Moral

- **BE CAUTIOUS** when reducing!

- The role of **SMALL FLUCTUATIONS** must not be underestimated!

- Choose **APPROPRIATE** quality measures!

References

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3. S. Herrmann, P. Imkeller, *Barrier crossings characterize stochastic resonance.* Stoch. Dyn. 2, No. 3, 413-436 (2002).
4. S. Herrmann, P. Imkeller *The exit problem for diffusions with time periodic drift and stochastic resonance.* to appear in J. Appl. Probab.

Sources

Title page Picture:

`http://www.vs-durach.de/projekte/landschaft/Eiszeit Mammut.JPG`

Page 3 Picture:

`V.S. Anishchenko, A.B. Neiman, F. Moss, and L. Schimansky-Geier,`
`Usp. Fiz. Nauk. 169, 7 (1999) [Sov. Phys. Usp. 42, 7 (1999)].`

Page 7 Picture:

`http://www.cote.azur.fr/fondsecran/fondsecrantri/nice/1024/1024nice1xl.jpg`