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Lévy-Noise Induced Exit[⋆]

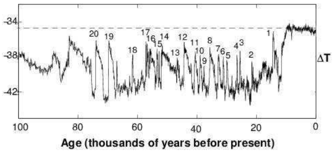
1. Motivation. Paleoclimatic Data

Greenland ice core data allow to reconstruct Earth's climate up to 200,000 yrs. before present.

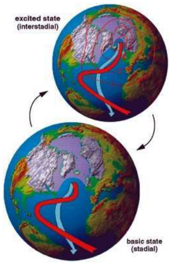
Paleo data proxies (oxigen isotopes, Ca) indicate at least 20 abrupt and large-amplitude shifts — **Dansgaard–Oeschger Events**

2. Dansgaard–Oeschger Events

- Rapid warming by 5-10 °C: a few decades
- Plateau phase with slow cooling: several centuries
- drop to cold stadial conditions: a few decades



Mean waiting time 1470 yrs.



From: Ganopolski, A. and Rahmstorf, S. (*Phys. Rev. Let. 88(3), 2002*)

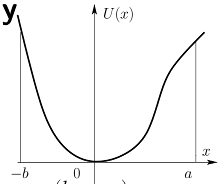
3. Spectral Analysis of Data

Langevin equation for climate dynamics

$$\dot X(t) = -U'(X(t)) + \mathbf{NOISE}$$

U — double-well potential, wells correspond to the climate states.
P. Ditlevsen (*Geophys. Res. Lett. 1999*): spectral analysis of the data.
NOISE has an α-stable component with $\alpha \approx 1.75$.

4. Object of Study



SDE driven by symmetric Lévy process of small intensity:

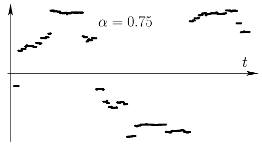
$$X_t^\varepsilon = x - \int_0^t U'(X_{s-}^\varepsilon)\,ds + \varepsilon L_t, \quad \varepsilon \downarrow 0$$

Unique stable point 0, $U''(0) > 0$.

$$\sigma(\varepsilon) = \inf\{t \geq 0 \,:\, X_t^\varepsilon \notin [-b, a]\}, \quad a, b < \infty \ (b = \infty)$$

5. The Driving Process L

$$\mathbf{E} e^{i\lambda L_1} = \exp\left\{ -\frac{d}{2}\lambda^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\lambda y} - 1 - i\lambda y\, \mathbb{I}_{\{|y|\leq 1\}}(y) \right) \frac{dy}{|y|^{1+\alpha}} \right\}$$



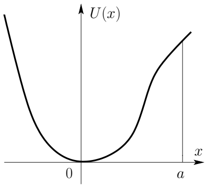
6. Kramers' Law for Gaussian Diffusion

H.A. Kramers (*Physica 7, 1940*): Chemical reaction rate theory
Diffusion model of chemical reactions

$$\hat X_t^\varepsilon = x - \int_0^t U'(\hat X_{s-}^\varepsilon)\,ds + \varepsilon W_t$$

$$\hat\sigma(\varepsilon) = \inf\{t \geq 0 \,:\, \hat X_t^\varepsilon = a\}, \quad \varepsilon \downarrow 0$$

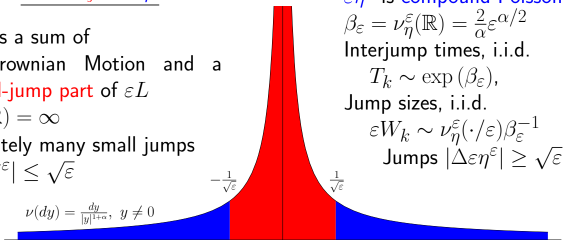
$$\mathbf{E}_x \hat\sigma \approx \frac{\varepsilon\sqrt{\pi}}{|U'(a)|\sqrt{U''(0)}} \exp\left(\frac{2U(a)}{\varepsilon^2}\right)$$



7. Process L: ε-dependent representation

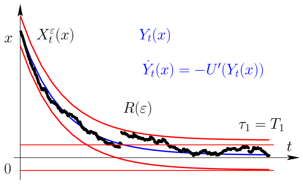
$$\underline{\varepsilon L = \varepsilon\xi^\varepsilon + \varepsilon\eta^\varepsilon}$$

$\varepsilon\xi^\varepsilon$ is a sum of a Brownian Motion and a small-jump part of εL
$\nu_\xi^\varepsilon(\mathbb{R}) = \infty$
Infinitely many small jumps
$|\Delta\varepsilon\xi^\varepsilon| \leq \sqrt{\varepsilon}$



$\varepsilon\eta^\varepsilon$ is compound Poisson
$\beta_\varepsilon = \nu_\eta^\varepsilon(\mathbb{R}) = \frac{2}{\alpha}\varepsilon^{\alpha/2}$
Interjump times, i.i.d.
$T_k \sim \exp(\beta_\varepsilon)$,
Jump sizes, i.i.d.
$\varepsilon W_k \sim \nu_\eta^\varepsilon(\cdot/\varepsilon)\beta_\varepsilon^{-1}$
Jumps $|\Delta\varepsilon\eta^\varepsilon| \geq \sqrt{\varepsilon}$

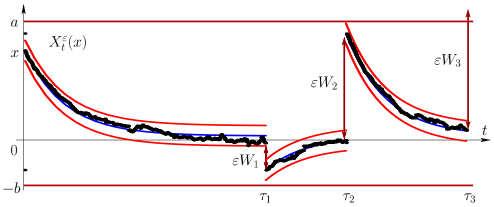
8. Small-jumps Dynamics



The deviation probability
$\mathbf{P}\left(\sup_{[0,T_1)} |X_t^\varepsilon - Y_t| \geq \frac{\varepsilon^\gamma}{2}\right)$
is exponentially small, $\gamma > 0$

Relaxation time for $Y_t(x)$
to reach ε^γ-neighbourhood of 0
$R(\varepsilon) = \mathcal{O}(|\ln\varepsilon|)$

9. Predominant Behaviour



Between the **"large"** jumps of η^ε, X^ε is driven by **"small jumps"** $\varepsilon\xi^\varepsilon$.
Interjump times $\mathbf{E}T_k = \frac{1}{\beta_\varepsilon} = \frac{\alpha}{2}\varepsilon^{-\alpha/2} \to \infty$ **polynomially fast**.
Relaxation time $R(\varepsilon) \ll \mathbf{E}T_k$
$\Rightarrow$ at $t = \tau_k$, X^ε **jumps from a neighbourhood of** 0 at εW_k.

10. Kramers' Type Law: Heuristic Proof

With **high** probability, X^ε exits $I = [-b, a]$ at arrival times of η^ε, $t = \tau_k$.
$\tau_k = T_1 + T_2 + \cdots + T_k$, $\mathbf{E}T_1 = 1/\beta_\varepsilon$
Jump-out probability

$$\mathbf{P}(\varepsilon W_1 \notin [-b, a]) = \frac{1}{\beta_\varepsilon}\Big(\int_{\frac{a}{\varepsilon}}^\infty + \int_{\frac{b}{\varepsilon}}^\infty\Big)\frac{dy}{y^{1+\alpha}} = \frac{1}{\beta_\varepsilon}\cdot\frac{\varepsilon^\alpha}{\alpha}\cdot\left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\right]$$

$$\mathbf{E}_x\sigma(\varepsilon) \approx \sum_{k=1}^\infty \mathbf{E}\tau_k \cdot \mathbf{P}(\sigma(\varepsilon) = \tau_k)$$

$$= \sum_{k=1}^\infty k \cdot \mathbf{E}T_1 \cdot \mathbf{P}(\varepsilon W_1 \in I, \ldots, \varepsilon W_{k-1} \in I, \varepsilon W_k \notin I)$$

$$= \frac{\mathbf{P}(\varepsilon W_1 \notin I)}{\beta_\varepsilon}\frac{1}{\mathbf{P}(\varepsilon W_1 \notin I)^2} = \frac{\alpha}{\varepsilon^\alpha}\cdot\left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\right]^{-1}$$

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[⋆] P. Imkeller and I. Pavlyukevich, First exit times of solutions of non-linear stochastic differential equations driven by symmetric Lévy processes with α-stable component, `arXiv:math.PR/0409246`, 2004