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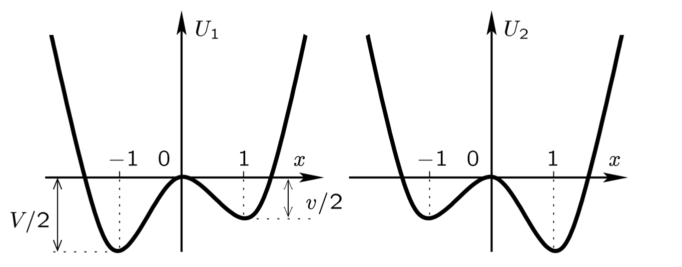
Stochastic Resonance

1. Stochastic Resonance in simplest framework*

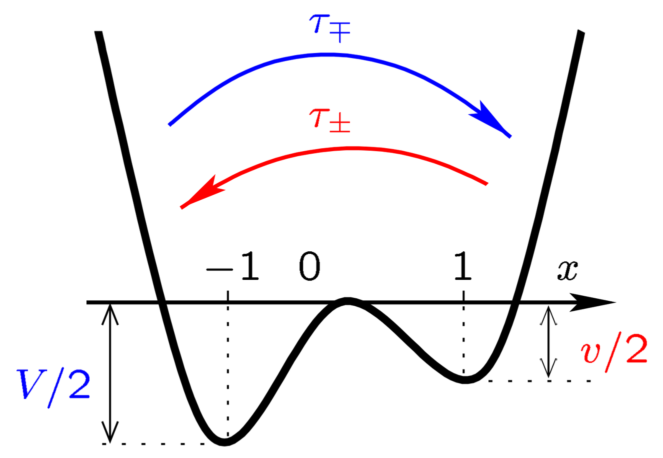
Brownian particle in a double-well potential described by SDE

$$dX_t^{\varepsilon,T} = -U'\left(X_t^{\varepsilon,T}, \frac{t}{T}\right) dt + \sqrt{\varepsilon}\, dW_t.$$



Flip model: Potential switches between two symmetric states U_1 and U_2 periodically in time with period T.
Due to the periodic flipping of the potential wells the sample paths acquire periodic properties. For which values of T and ε are they 'most' periodic?

2. Large deviations (Freidlin-Wentzell)*



Freidlin-Wentzell theory: the time-homogeneous system possesses two intrinsic time scales: exit times from the left resp. right well $\tau_\mp$, $\tau_\pm$.

In the small noise limit $\tau_\mp$ and $\tau_\pm$ are exponentially large in ε and have the order $e^{V/\varepsilon}$ and $e^{v/\varepsilon}$ (Kramers' law).

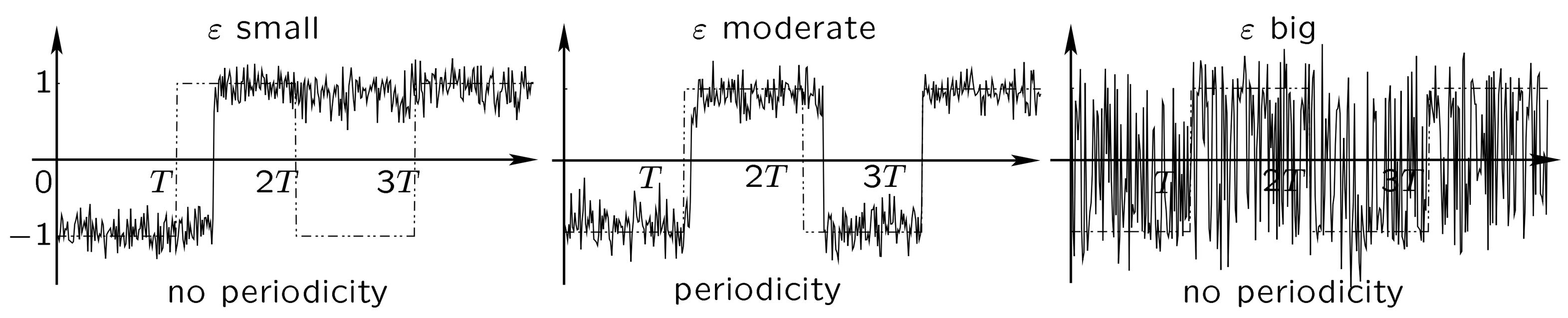
Lower bound for periodicity of $X^{\varepsilon,T}$ in the small noise limit:

$\varepsilon \ln T > v.$

Reduction to effective dynamics*

3. Diffusion $X^{\varepsilon,T}$

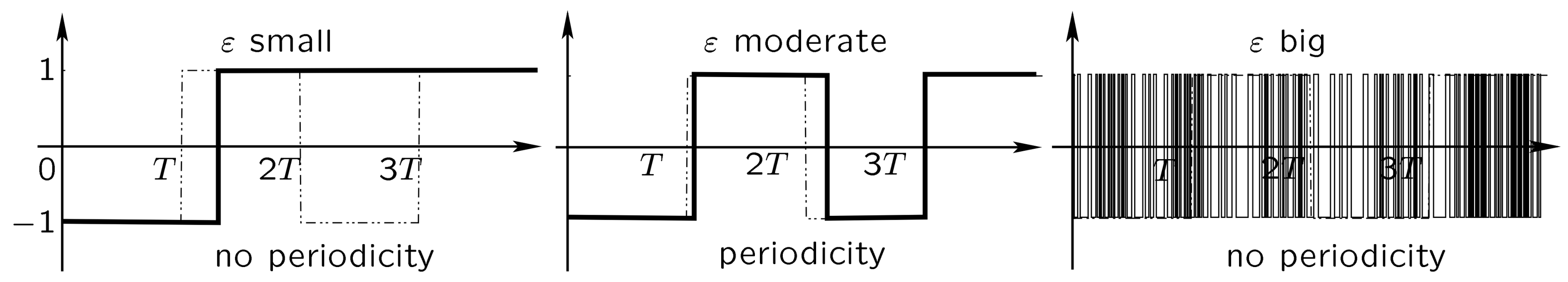
Potential minima ±1 are metastable states. Kramers' law suggests reduction to two-state process $Y^{\varepsilon,T}$ in the limit of small noise and large period. Is the optimal tuning the same for the diffusion and the Markov chain?



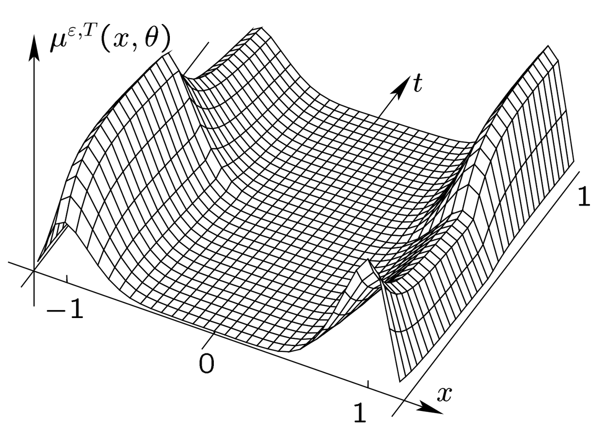
4. Markov chain $Y^{\varepsilon,T}$

Process on set of diffusion's metastable states ±1 with time-periodic transition rates

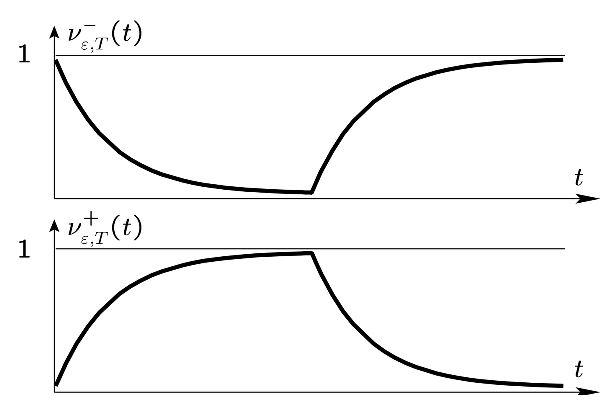
$$\varphi = \frac{\sqrt{U''(-1)|U''(0)|}}{2\pi}\, e^{-V/\varepsilon}, \quad \psi = \frac{\sqrt{U''(1)|U''(0)|}}{2\pi}\, e^{-v/\varepsilon}.$$



5. Invariant measures



Diffusion's invariant density μ is time-dependent and periodic. The mass is concentrated near the metastable states ±1. The invariant density is space-time antisymmetric. On each half-period it satisfies the forward Kolmogorov (Fokker-Planck) equation — parabolic PDE.



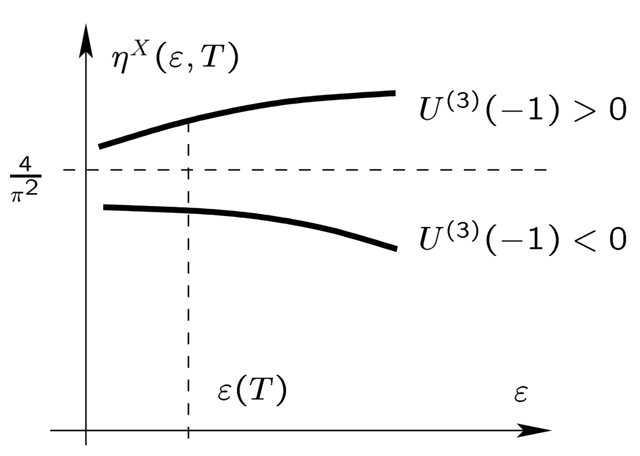
Markov chain's invariant measure is time-dependent, periodic and imitates the evolution of μ. The mass is concentrated in the metastable states ±1. The knowledge of the invariant measure allows us to study the evolution of the random processes in the stationary regime.

6. Spectral Power Amplification coefficient (SPA)

SPA coefficient is one of the physicists' favourite measure of SR based on the study of the process's invariant law. It gives the spectral energy carried by the averaged trajectories of the random process on the period T.

For example, for the diffusion X the SPA coefficient is defined by

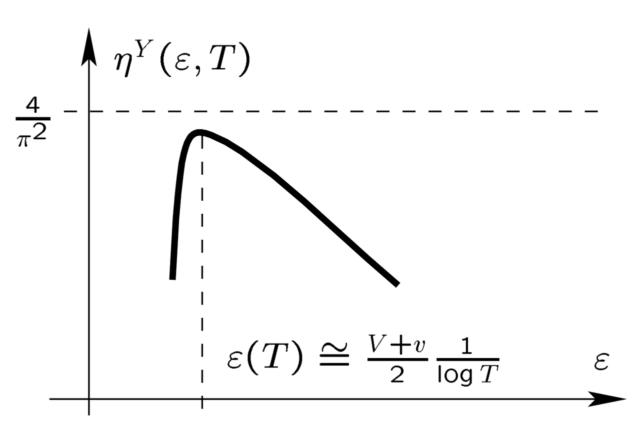
$$\eta^X(\varepsilon, T) = \left| \int_0^1 \mathbf{E}_\mu X_{sT}^{\varepsilon,T} e^{2\pi i s}\, ds \right|^2.$$



SPA coefficient approximately equals

$$\frac{4}{\pi^2}\left(\frac{\int y e^{-2U/\varepsilon}\, dy}{\int e^{-2U/\varepsilon}\, dy}\right)^2 \frac{\lambda_1^2 T^2}{4\pi^2 + \lambda_1^2 T^2}.$$

It has no local maximum. SPA coefficient depends on the fine geometry of the potential.

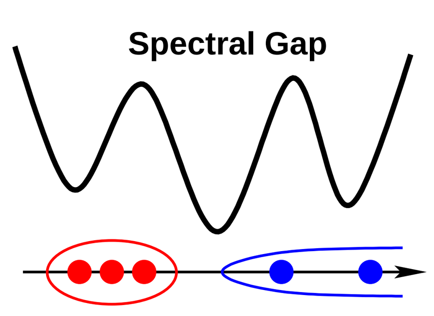


SPA coefficient is found explicitely:

$$\frac{4}{\pi^2}\frac{(\varphi-\psi)^2 T^2}{4\pi^2 + (\varphi+\psi)^2 T^2}.$$

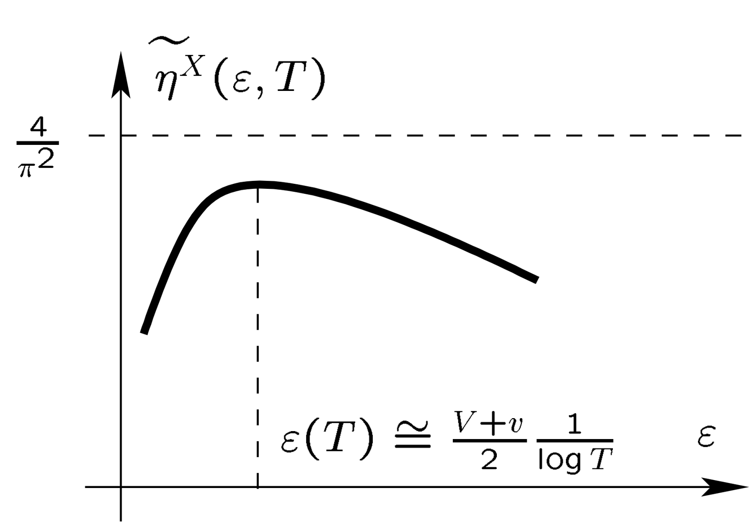
The uniquely determined optimal tuning is given by the coordinate of the maximum.

7. Spectral Analysis



The spectral analysis of the diffusion's infinitesimal generator mainly contributes to a good approximation of the invariant density μ. Spectral gap between the low lying eigenvalues plays a crucial role and justifies a Floquet type expansion of μ. Only two first terms of this expansion are important. They remind of the equilibrium measure of Y. However, the potential geometry must not be underestimated! Fluctuations near the metastable states completely change the optimal tuning properties of the processes.

8. Cut off fluctuations



SPA η^X can be modified to 'catch' only interwell dynamics of the diffusion. 'Cutoff' for fluctuations in the potential's bottoms — identify neighbourhoods of ±1 with metastable states.
Local maximum exists and optimal tuning corresponds to that of the adapted Markov chain.

* P. Imkeller and I. Pavlyukevich, Stochastics and Dynamics 2 (2002),   * M. Freidlin, Physica D 137 (2000)