51. Find Borel subgroups in \( \text{SO}_4 \), \( \text{Sp}_4 \), \( T_4 \) and \( U_4 \).

**Solution:** Recall that \( \text{GL}_4 / T_4 \cong \text{Fl}(K^4) \), the variety of full flags in \( K^4 \). Given the subgroup \( \text{SO}_4 \subset \text{GL}_4 \), we consider the subset of isotropic flags \( \text{Fl}_0(K^n, \beta) = \{ V^0 \subset V^1 \subset V^2 \mid \dim(V^i) = i, \beta|_{V^i \times V^i} = 0 \} \) where \( \beta : K^4 \times K^4 \to K \) is the bilinear form. This is a closed subset, so that \( \text{Fl}_0(K^n, \beta) \) is a projective variety. \( \text{SO}_4 \) acts transitively on isotropic flags (similar proof as with \( \text{GL}_n \) and full flags). Using the bilinear form \( \beta \) from Exercise 31, a particular isotropic flag is given by \( V^0 = 0 \subset V^1 = K e^1 \subset V^2 = K e^1 + K e^3 \). From \( g \in \text{GL}_4 \) with \( g e^1 = e^1 \), \( g e^3 = e^3 \) and \( g^t \beta g = \beta \), we get

\[
\beta = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad g = \begin{pmatrix}
1 & * & 1 & * \\
0 & 1 & 0 & * \\
0 & 0 & 1 & * \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

and in particular the stabiliser group in \( \text{SO}_4 \) is solvable.

\( T_4 \) and \( U_4 \) are solvable and connected, hence their Borel groups are the full group in each case.

52. Find all parabolic subgroups \( P \) with \( T_4 \subset P \subset \text{GL}_4 \).

**Solution:** There are 8 such parabolic subgroups, there are given by block matrices of the formats

\[
\text{GL}_4 = \begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
0 & * & * & * \\
0 & 0 & * & *
\end{pmatrix}, \quad \begin{pmatrix}
* & * & * & * \\
0 & * & * & * \\
0 & 0 & * & * \\
0 & 0 & * & *
\end{pmatrix}, \quad \begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
0 & * & * & * \\
0 & 0 & * & *
\end{pmatrix}, \quad \begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
0 & * & * & * \\
0 & 0 & * & *
\end{pmatrix}, \quad \begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
0 & * & * & * \\
0 & 0 & * & *
\end{pmatrix}, \quad \begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{pmatrix}, \quad \begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{pmatrix}, \quad \begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{pmatrix} = T_4.
\]

53. Let the connected linear algebraic group \( G \) act on a quasi-projective variety \( X \) with finitely many orbits. Show that every irreducible \( G \)-invariant subset in \( X \) is the closure of a \( G \)-orbit. Find a counterexample for an action with infinitely many orbits.

**Solution:** We know that orbits in general are quasi-projective and open in their closure. An irreducible \( G \)-invariant subset \( Z \subset X \) is a union of orbits. Since there are only finitely many orbits, \( Z \) contains an orbit \( O \) as an open subset; the closure \( \overline{O} \) must then be \( Z \).

Counterexamples abound. (For a trivial one, take \( G = 1, Z = X = \mathbb{A}^1 \).)

54. Find a connected linear algebraic group \( G \) and a maximal solvable subgroup \( U \subset G \) such that \( U \) is disconnected.

**Solution:** This is problem 17 in disguise: we take \( G = \text{GL}_2 \) and \( U := N(D_2) = (0^* \ 0) \cup (\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}) \). Then \( U \) is obviously disconnected, it is solvable by problem 17. It is a matrix calculation to show that the only subgroup of \( G \) strictly containing \( U \) is \( G \).

55. Classify all root systems in the Euclidean plane \( E := \mathbb{R}^2 \).