6. Prove that the group $U_n$ of unipotent upper triangular matrices is nilpotent.

**Solution:** This is a straightforward computation with commutators of matrices. For example, $C^2(U_n) = [U_n, U_n]$ consists of matrices with zeros on the secondary diagonal. In the central series, $C^n(U_n)$ is trivial.

7. Prove that a group $G$ is solvable if and only if it has a composition series with abelian factors, i.e. there is a chain of subgroups $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_m = \{1\}$ such that each $G_{i+1}$ is normal in $G_i$ and all $G_i/G_{i+1}$ are abelian.

**Solution:** We use the fact that for a subgroup $H \leq G$ holds: $H \supseteq D(G) = [G, G]$ if and only if $H$ is normal in $G$ and $G/H$ is abelian.

If $G$ is solvable, then by definition the derived series $D^n(G)$ trivialises. Put $G_i := D^i(G)$. Then by the fact, each $G_{i+1} = D(G_i)$ is normal in $G_i$ with abelian quotient. The series terminates after finitely many steps because $G$ is solvable.

If we are given the chain of subgroups, then invoking the fact in the reverse direction, we get $D(G_i) \subseteq G_{i+1}$. Thus inductively, $D^i(G) \subseteq G_i$ and therefore the derived series trivialises.

8. What is the Jordan decomposition for a finite group? For $G_a$?

**Solution:** If char($K$) = 0, then all elements of a finite group $G$ (considered embedded in some $GL_n$) are semisimple. To see this, note that a unipotent matrix $U$ has all eigenvalues 1. If $U \neq I_n$, then $U$ has non-trivial Jordan blocks and $U^k \neq I_n$ for $k \geq 1$. Therefore $U$ can never have finite order, i.e. belong to $G$. Note that this argument fails in finite characteristic: for example, the matrix $U = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$ over a field $K$ with char($K$) = 2 is unipotent with $U \neq I_2$ but $U^2 = I_2$.

$G_a \cong \mathbb{Z}_2$, so all elements of $G_a$ are unipotent.

9. Find a closed subgroup $G$ of $GL_2$ such that $G$ is not a closed subset.

**Solution:** The subgroup $T_2$ of upper triangular matrices $\left( \begin{array}{cc} a & b \\ 0 & 1 \end{array} \right)$. The subset of semisimple elements is

$(T_2)_s = \left\{ \left( \begin{array}{cc} a & b \\ 0 & 1 \end{array} \right) \mid a, d \in K^*, b \in K \text{ with } a \neq d \text{ or } a = d, b = 0 \right\}$.

It is the complement of matrices $\left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right)$ with $a = d, b \neq 0$, and this subset is not Zariski-open: for example, its closure are the matrices $\left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right)$ with $a = d$ (arbitrary $b$), which is not all of $T_2$.

10. Compute the centre $C$ of $SL_2(K)$, assuming char($K$) $\neq 2$. Show that the quotient group $PSL_2(K) := SL_2(K)/C$ is an affine algebraic group.

(Hint: embed $SL_2 \subset \mathbb{A}^4$ as a Zariski-closed subset, then check that the action of $C$ on $SL_2$ extends to an action of $C$ on $\mathbb{A}^4$. Now map $\mathbb{A}^4/C$ to some affine space as a Zariski-closed subset.)

**Solution:** By direct computation, $C = \left\{ \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right) \right\}$ is the group with two elements. With $SL_2 = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{A}^4 \mid x_1 x_4 - x_2 x_3 = 1 \right\}$, the group is a closed subset of $\mathbb{A}^4$. The group $C$ acts by $\pm \text{id}$, which obviously extends to $\mathbb{A}^4$. An element of $\mathbb{A}^4/C$ is a quadruple up to sign, and we can map

$\mathbb{A}^4/C \to \mathbb{A}^5$, $\pm(x_1, x_2, x_3, x_4) \mapsto (x_1 x_2, x_1 x_3, x_2 x_3, x_2 x_4, x_3 x_4)$.

It is straightforward (and necessary) to check that this map is (a) well-defined and (b) cut out by the three polynomial equations $y_1 y_6 = y_2 y_5 = y_3 y_4$, with coordinates $A(\mathbb{A}^4) = K[x_1, \ldots, x_4]$ and $A(\mathbb{A}^5) = K[y_1, \ldots, y_6]$. The map is not injective, but it is injective on the subset $PSL_2$.

This exhibits $PSL_2 = SL_2/C \subset \mathbb{A}^5$ as a Zariski-closed subset. (The equation det = 1, i.e. $x_1 x_4 - x_2 x_3 = 1$, for $SL_2$ is invariant under $C$.) It remains to observe that multiplication $PSL_2 \times PSL_2 \to PSL_2$ and inversion $PSL_2 \to PSL_2$ are given by polynomial maps.

Alternative: the map $\mathbb{A}^4/C \to \mathbb{A}^{10}$, $\pm(x_1, x_2, x_3, x_4) \mapsto (x_1^2, x_2^2, x_3^2, x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, x_3 x_4)$ is injective and can therefore also be used to embed $PSL_2$ in an affine space.

Note: This exercise shows that some linear algebraic groups do not come with a natural embedding into a linear group, for such groups, it is easier to check that they are affine algebraic groups.