36. Compute the orbits of the actions of $GL_3$ on $\mathbb{P}^2$, of $GL_4$ on $(2,4)$, and of $GL_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ (diagonal action). Also compute isotropy groups for all orbits.

**Solution:** $GL_3$ acts transitively on $\mathbb{A}^3 \setminus \{0\}$, hence it also acts transitively on $\mathbb{P}^2$. The isotropy group of the point $(1 : 0 : 0)$ consists of the block matrices of type $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

For the action of $GL_4$ on $(2,4)$, we consider an arbitrary subspace $V \subset K^4$ with dim$(V) = 2$. If $v_1, v_2$ is a basis for $V$, then we can apply an automorphism of $K^4$, which maps $v_1$ to $e_1 = (1,0,0,0)$. Hence we can assume that the basis is $e_1, v$ for some $v \in K^4$. The subgroup $T_d \subset GL_4$ preserves $e_1$, up to scalars. Using an appropriate triangular matrix, we can fix $e_1$ and map $v$ to a vector of the form $(0, *, *, *)$. At this point we use the subgroup $GL_1 \times GL_2 \subset GL_4$, and with $GL_3$ acting transitively on $\mathbb{A}^3 \setminus \{0\}$, we can map $(0, *, *, *) \mapsto (0, 1, 0, 0) = e_2$. Hence $GL_4$ acts transitively on $(2,4)$. The isotropy group of the subspace $(*) = (0, 0, 1, 0, 0) \subset \mathbb{P}^4$ consists of matrices whose lower left $2 \times 2$-block is zero.

The diagonal action of $GL_2$ on $\mathbb{P}^1 \times \mathbb{P}^2$ has two orbits: one is the diagonal $\Delta := \{(p,p) \mid p \in \mathbb{P}^1\}$ (this is a closed orbit), the other is the open complement $U := \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$. It is obvious that $\Delta$ is $GL_2$-invariant. Moreover, $\Delta \cong \mathbb{P}^1$ and $GL_2$ acts transitively on $\mathbb{P}^1$; hence $\Delta$ is an orbit. For a point $(x_0 : x_1), (y_0 : y_1) \notin \Delta$, we have $x_0y_1 - x_1y_0 \neq 0$, and applying an appropriate matrix multiplication, we get

$$(\begin{pmatrix} y_1 \\ -x_1 \\ x_0 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix}) \mapsto ((x_0x_1 - y_0x_1 : 0), (y_0 : y_0x_0 - y_0x_1)) = (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) \in \mathbb{P}^1 \times \mathbb{P}^1.$$

The isotropy group of any point in $\mathbb{P}^1 \times \mathbb{P}^1$ is $D_2$.

37. Show that any non-degenerate conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. Deduce that homogeneous coordinate rings are not isomorphism invariants of projective varieties.

**Solution:** A conic in $\mathbb{P}^2$ is, by definition, the vanishing locus of some homogeneous polynomial of degree two: $a_{00}x_0^2 + a_{11}x_1^2 + a_{01}x_0x_1 + a_{20}x_0^2 + a_{12}x_0x_1 + a_{21}x_1^2 = 0$ with $a_{ij} \in K$. This zero set can be also described in matrix form $x^tMx = 0$, where the symmetric bilinear form $M$ can be diagonalised to

$$M = A = \begin{pmatrix} a_{00} & a_{01}/2 & a_{02}/2 \\ a_{01}/2 & a_{11} & a_{12}/2 \\ a_{02}/2 & a_{12}/2 & a_{22} \end{pmatrix} \sim \begin{pmatrix} c_0 & 0 & 0 \\ 0 & c_1 & 0 \\ 0 & 0 & c_2 \end{pmatrix}$$

with $c_i$ either 0 or 1 ($K$ is algebraically closed). This is given by a linear coordinate change, i.e. an automorphism of $\mathbb{P}^2$. However, if one of the $c_i = 0$, then the quadric is degenerate. Hence every non-degenerate quadric in $\mathbb{P}^2$ is isomorphic to $V(x_0^2 + a_1x_1^2 + a_2x_2^2)$. Moreover, $\mathbb{P}^1 \rightarrow \mathbb{P}^2, (y_0 : y_1) \mapsto (y_0^2 : y_0y_1 : y_1^2)$ is a morphism which is injective and whose image is cut out by $V(x_0^2 + a_1x_1^2 + a_2x_2^2)$. Hence $\mathbb{P}^1 \cong V(x_0^2 + a_1x_1^2 + a_2x_2^2)$ using the coordinate change $x_1 \mapsto x_1$, and all non-degenerate conics are isomorphic to $\mathbb{P}^2$. For the second claim, note that the homogeneous coordinate ring of $\mathbb{P}^1$ is $K[x,y]$, the polynomial ring in two variables (with its standard grading). However, the homogeneous coordinate ring of a quadric is e.g. $S = K[x,y]/(x^2 + y^2 + z^2)$. These rings are not isomorphic, for example because the associated affine varieties aren’t $V(x^2 + y^2 + z^2) \subset \mathbb{A}^3$ is singular.

38. Show that $\mathbb{A}^1$ and $\mathbb{P}^1$ are homeomorphic, but $\mathbb{A}^2$ and $\mathbb{P}^2$ are not.

**Solution:** As topological spaces, both $\mathbb{A}^1$ and $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ are sets of the same cardinality as $K$, with the cofinite topology. Hence they are homeomorphic. When comparing $\mathbb{A}^2$ and $\mathbb{P}^2$, let a curve be a closed, irreducible subset which is neither a point nor the whole space. We claim that for any curve $C \subset \mathbb{A}^2$, there exists another curve $C' \subset \mathbb{P}^2$ with $C \cap C' = \emptyset$. (If $C = V(f)$, then $C' = V(f + 1)$ will work.) However, in $\mathbb{P}^2$ there are curves which intersect all other curves. (For example, a line $V(x_0)$ will work: if $V(y)$ is a curve given by a homogeneous polynomial $g \in K[x_0, x_1, x_2]$, then either $g \in (x_0)$ and $V(y) \cap V(x_0)$ is even infinite, or $g(0, x_1, x_2) \neq 0$ and defines a finite set of points in $\mathbb{P}^2$ — this set is non-empty because $K$ is algebraically closed.) These are topological properties, so $\mathbb{A}^2$ and $\mathbb{P}^2$ cannot be homeomorphic.

39. For an affine, irreducible variety $X$ and a point $p \in X$, show that the local ring of $X$ at $p$ is given by the localisation of $A(X)$ at the maximal ideal $M_p$.

40. Show that $\text{Aut}(\mathbb{P}^1) = \text{PGL}_2$.

**Solution:** We already know that $GL_2$ acts on $\mathbb{P}^1$ with kernel $Z(GL_2) \cong \mathbb{G}_m$. Hence $PGL_2$ acts faithfully on $\mathbb{P}^1$, and we have to show that all automorphisms of the projective line come from linear maps.

Let $\varphi \in \text{Aut}(\mathbb{P}^1)$ be an arbitrary automorphism. Put $[x_0 : x_1] := \varphi([1 : 0])$. If $[x_0 : x_1] \neq [1 : 0]$, then $x_1 \neq 0$, and then matrix $\varphi := \begin{pmatrix} 0 & 1 \\ 1 & \varphi^{-1}(1) \end{pmatrix}$ satisfies $\varphi([1 : 0]) = [1 : 0]$. Hence $\varphi$ induces an automorphism of $\mathbb{P}^1 \setminus \{[1 : 0]\} = \mathbb{A}^1$.

As $\mathbb{A}^1$ is an affine variety, we have $\text{End}(\mathbb{A}^1) = \text{End}(\mathbb{A}(\mathbb{A}^1)) = \text{End}(K[x])$, so the automorphism must be of the form $\varphi(x) = ax + b$, i.e. a linear polynomial. Then $\varphi := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ maps $x + 1 \mapsto (ax + b + 1) = \psi(x + 1)$ and $(1 : 0) \mapsto (a : 0) = (1 : 0)$. Hence $\theta = \psi^{-1}$ and $\varphi = \psi^{-1} \theta \in \text{PGL}_2$.

[It is also true that $\text{Aut}(\mathbb{P}^n) = \text{PGL}_{n+1}$ but the proof requires considerably harder methods.]