1 Some pointers to the literature

J. Humphreys: *Linear Algebraic Groups* Springer (1975, 1981)
A. Onishchik, E. Vinberg: *Lie Groups and Algebraic Groups* Springer (1990)

The first three books (Borel, Humphreys, Springer) are the classical textbooks on the subject. They share the following features: the ground field is algebraically closed and of arbitrary characteristic; rationality questions (i.e. non-algebraically closed fields of definition) are treated at the end; they cover the structure theory of linear algebraic groups including the classification of reductive/semisimple groups; they avoid schemes but use ringed spaces to define quotients. Of these books, I like Humphrey’s the best: it does a good job of explaining why things happen.

The other three books all work over the complex numbers, and have different aims. Tauvel/Yu is really concerned with Lie algebras. It covers basically all of the foundational results otherwise cited (including commutative algebra, sheaves, group theory, projective geometry, root systems) — at the price that Lie algebras are introduced as late as in §19 and linear algebraic groups in §21. Moreover, the book is extremely light on examples. It treats a lot of the finer structure theory, including reductive, Borel, parabolic, Cartan groups. In my opinion, this volume is more a reference than a textbook to learn the subject from. The sections relevant to the course are §10,18,21–23,25–28.

Goodman/Wallach starts with very explicit descriptions of the classical groups (SL\(_n\), SO\(_n\), Sp\(_n\)) and then develops the theories of Lie groups and of algebraic groups in parallel. Thus it is very example-driven: for example, maximal tori and roots are all developed first for the classical groups. Later on, Chapter 11 is devoted to a rapid development of linear algebraic groups, up to Borel subgroups and maximal tori (but without the classification). I took the construction of homogenous spaces \(G/H\) and the proof that morphisms of linear algebraic groups have closed image from Appendix A.

Onishchik/Vinberg is written in a unique style: the book grew out of a 1967 Moscow seminar; almost all of the theory is presented in a series of problems (generally with concise solutions). The book develops both (real and complex) Lie groups and complex linear algebraic groups, in a very quick and efficient fashion. They classify semisimple Lie algebras.

To a large extent, I have been following the lecture notes of Tamás Szamuely from a 2006 course at Budapast: [http://www.renyi.hu/~szamuely/lag.pdf](http://www.renyi.hu/~szamuely/lag.pdf). It has the advantage of starting out with low technical demands (no schemes or sheaves are used; only quasi-projective varieties). The proof of conjugacy of maximal tori using group cohomology is from these notes.
2 List of results proved in the course

Affine algebraic geometry

**Proposition 1:** Connected components of affine algebraic groups coincide with irreducible components. The identity component is a normal subgroup.

**Proposition 2:** \( \text{Mor}(X,Y) = \text{Hom}(A(Y), A(X)) \) for affine varieties \( X,Y \). For any finitely generated, reduced \( K \)-algebra there is a unique affine variety \( X \) with \( A(X) \cong A \).

**Proposition 3:** \( A(X \times Y) \cong A(X) \otimes_K A(Y) \) for affine varieties \( X,Y \).

Embedding theorem

**Theorem:** Any affine linear group \( G \) is a closed subgroup of some \( \text{GL}_n \).

**Chevalley-Lemma:** If \( H \subseteq G \) is a closed subgroup of a linear algebraic group, then there exists a finite-dimensional representation \( G \rightarrow \text{GL}(V) \) and a 1-dimensional linear subspace \( L \subseteq V \) such that \( H = \text{Stab}(L) \).

**Proposition 4:** For a normal subgroup \( H \subseteq G \) of a linear algebraic group \( G \), there is a finite-dimensional representation \( G \rightarrow \text{GL}(V) \) with kernel \( H \).

Jordan decomposition

**Theorem:** Jordan decomposition for a linear algebraic group \( G \).

**Theorem (Kolchin):** If \( G \subseteq \text{GL}_n \) is a unipotent group, then there exists a non-zero vector fixed by \( G \).

**Burnside-Lemma:** If \( V \) is a finite-dimensional \( K \)-vector space and \( A \subseteq \text{End}(V) \) a subalgebra without \( A \)-stable subspaces except 0 and \( V \), then \( A = \text{End}(V) \).

**Theorem (structure of commutative linear algebraic groups \( G \)):** \( G_s,G_u \) are closed subgroups of \( G \), and \( G_s \times G_u \Rightarrow G \) is an isomorphism of affine algebraic groups.

Actions and representations

**Proposition 5:** \( G \) diagonalisable \( \iff \) \( G \) is isomorphic to a closed subgroup of \( D_n \) \( \iff \) \( G = G_s \) is commutative

**Lemma:** \( X^*(D_n) = \mathbb{Z}^n \) and \( X^*(D_n) \) is a basis of \( A(D_n) \).

**Theorem (structure of diagonalisable groups \( G \)):** \( G \) diagonalisable \( \iff \) \( X^*(G) \) is a finitely generated abelian group (without \( p \)-torsion if \( \text{char}(K) = p \)) and \( X^*(G) \) is a basis of \( A(G) \) \( \iff \) \( G \cong \mu_{d_1} \times \cdots \times \mu_{d_r} \times D_m \) with \( p \nmid d_i \).

**Proposition 6:** \( G \) diagonalisable \( \iff \) all finite-dimensional \( G \)-representations decompose into characters

**Theorem (Weyl groups are finite):** If \( G \) is a linear algebraic group and \( T \subseteq G \) a torus, then the quotient \( W(G,T) := N_G(T)/C_G(T) \) is finite.
Connected solvable groups

**Theorem (Lie-Kolchin):** If $G \subset \text{GL}_n$ is a connected solvable subgroup, then there exists a complete flag of $G$-invariant subspaces in $V$, i.e. $G$ can be conjugated into $T_n$.

**Theorem (structure of connected nilpotent groups):** If $G$ is a connected nilpotent linear algebraic group, then $G_u, G_s$ are closed, normal subgroups of $G$ and $G_s \times G_u \approx G$ is an isomorphism of affine algebraic groups, and $G_s$ is a torus.

Tangent spaces and Lie algebras

**Lemma:** If $X$ is an affine homogeneous space for a linear algebraic group $G$, then $X$ is smooth.

**Theorem:** If $G$ is a connected, 1-dimensional linear algebraic group, then either $G = G_a$ or $G = G_m$. [Only proved for char($K$) = 0.]

**Lemma:** If $G \subseteq \text{GL}_n$ is a linear algebraic group, then the action of $G$ on its Lie algebra $\mathfrak{g} = \{ A \in \text{gl}_n \mid X_A f \in I(G) \forall f \in I(G) \}$ is given by $g \cdot A = gAg^{-1}$.

**Proposition 7:** For a linear algebraic group $G$, tangent space at the unit and Lie algebra coincide: $L(G) \cong T_e G$.

Quotients by normal subgroups

**Theorem:** If $H \subseteq G$ is a closed normal subgroup of a linear algebraic group, then $G/H$ is a linear algebraic group.

**Proposition 8:** If $A \subset B$ are $K$-algebras, with $B$ finitely generated over $A$, then the extension property holds: $\forall b \in B \ b \neq 0 \ \exists \ a \in A \ a \neq 0 \ \exists \ \varphi : A \rightarrow K \ \exists \ \Phi : B \rightarrow K \ \Phi(b) \neq 0 : \Phi|_A = \varphi$.

**Proposition 9:** If $f : X \rightarrow Y$ is a morphism of affine varieties, then $f(X)$ contains an open subset of its closure $\overline{f(X)}$.

**Corollary:** If $f : G \rightarrow H$ is a morphism of affine algebraic groups, then the image $f(G)$ is a closed subgroup of $H$.

Quasi-projective varieties

**Proposition 10:** The Plücker map $p_d : \text{Gr}(d,V) \rightarrow \mathbb{P}(\Lambda^d V)$ is a closed embedding.

**Lemma:** $X$ proper, $Z \subseteq X$ closed $\Rightarrow$ $Z$ proper.

$X_1, X_2$ proper $\Rightarrow X_1 \times X_2$ proper.

$X$ proper, $\varphi : X \rightarrow Y$ morphism $\Rightarrow \varphi(X)$ closed, proper subvariety of $Y$.

$X$ proper and affine $\Rightarrow X$ is a finite set.

**Theorem:** Projective varieties are proper.

**Nakamaya-Lemma:** If $A$ is a commutative ring, $M \subset A$ a maximal ideal and $N$ a finitely generated $A$-module with $N = MN$, then there is an $f \in A \setminus M$ with $fN = 0$. 
Homogeneous spaces and quotients

**Theorem:** If $H \subset G$ is a closed subgroup of a linear algebraic group, then $G/H$ is a quasi-projective with a morphism $\pi: G \to G/H$ and if $\varphi: G \to X$ is any $G$-equivariant morphism of homogeneous $G$-spaces with $\varphi(H) = \varphi(1)$, then there exists a unique morphism $\psi: G/H \to X$ with $\varphi = \psi \pi$. Moreover, if $G$ acts on a quasi-projective variety $Y$ such that $H \subseteq G_y$ for some $y \in Y$, then the natural map $G/H \to G \cdot y$ is a morphism.

**Proposition 11:** Let $B$ be a finitely generated $K$-algebra without zero divisors and $A \subset B$ is a subalgebra. If there is $b \in B, b \neq 0$ such that all $\Phi: B \to K$ with $\Phi(b) \neq 0$ are uniquely determined by $\Phi|_A$, then $B \subseteq \text{Quot}(A)$.

**Proposition 12:** Let $f: M \to N$ and $h: M \to P$ be regular, dominant morphisms of affine varieties such that there exists a non-empty open set $U \subseteq M$ with $f(m_1) = f(m_2) \Rightarrow h(m_1) = h(m_2) \ \forall m_1, m_2 \in U$, then there is a rational map $g: N \dasharrow P$ with $h = gf$.

Borel and parabolic subgroups

**Orbit lemma:** If a linear algebraic group $G$ acts on a quasi-projective variety $X$, then (i) every orbit is open in its closure, (ii) orbits of minimal dimension are closed and, in particular, (iii) closed orbits exist.

**Borel Fixed Point Theorem:** Any action of a connected solvable group $G$ on a projective variety $X$ has a fixed point.

**Theorem:** Any two Borel subgroups of a linear algebraic group are conjugate.

**Proposition 13:** Borel subgroups are parabolic.

**Proposition 14:** If $\varphi: G \to G'$ is a surjective morphism of affine algebraic groups and $H \subseteq G$ is a parabolic (or Borel) subgroup, then so is $\varphi(H) \subseteq G'$.

Maximal tori

**Proposition 15:** For a connected solvable group $G$, there exists a torus $T \subset G$ such that $T \hookrightarrow G \to G/G_u$ is an isomorphism.

**Corollary:** A connected solvable group $G$ is a semi-direct product $G = G_u \rtimes T$, where $T \cong G/G_u$ is a maximal torus.

**Theorem:** Any maximal tori in a connected linear algebraic group are conjugate.

Structure results

**Theorem:** Let $G$ be a connected linear algebraic group, $B \subseteq G$ a Borel subgroup and $T \subseteq G$ a maximal torus. Then $G$ is covered by all Borel subgroups, and $G_s$ is covered by all maximal tori: $G = \bigcup_{g \in G} gBg^{-1}$ and $G_s = \bigcup_{g \in G} gTg^{-1}$.

**Proposition 16:** Let $G$ be a connected linear algebraic group and $T \subseteq G$ a torus. Then $C_G(T)$ is connected.
**Theorem:** Given $B \subseteq G$, a Borel subgroup of a connected linear algebraic group, then $N_G(B) = B$.

**Proposition 17:** There is a bijection between the generalised flag variety $G/B$ and the set $\mathfrak{B}$ of all Borel subgroups of $G$. 