# Comparing Coxeter functors made from exceptional and spherical objects

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Abstract Given two categories, one generated by an exceptional collection, the other generated by spherical objects and a reasonable functor between them, we establish a compatibility between two Coxeter-type autoequivalences on both categories. This has algebraic and geometric applications; as particular examples, we look at Kleinian and some Fuchsian singularities.

## 1 Introduction

A *Coxeter element* is a special isometry defined for some free abelian groups with a (not necessarily symmetric) bilinear pairing. For example, for a lattice with a basis of roots (i.e. vectors of square $-2$), the Coxeter element is the product of reflections along the basis roots. It depends on the ordering of the basis, but a permutation of the basis will lead to a conjugated Coxeter element. In various contexts, triangulated categories arise whose (numerical) K-groups, equipped with the Euler pairing, possess such a Coxeter element. By a *Coxeter functor*, we mean an autoequivalence of the category which lifts that Coxeter element.

In the study of derived categories, the special role of exceptional collections was noticed very early. As shown in [BK], the existence of a full exceptional collection means that the category is built from blocks of the most simple type — the category of graded vector spaces. It is also well-established that such collections can be mutated, giving rise to a Coxeter functor on the category.

More recently, collections of spherical objects have been found to be of importance. Spherical objects, in contrast to exceptional objects, are already interesting on their own, as they give rise to autoequivalences, the spherical twists. As originally noted by [ST], there are many cases where exceptional and spherical objects appear together, although in different categories.

In our setup, we begin with two categories, one of which is generated by an exceptional collection whereas the other one is generated by spherical objects, with the two categories connected by a functor with reasonable properties. This situation occurs rather often, both in algebraic and geometric situations. The generators mentioned above give rise to two Coxeter-type functors which we compare in a nice way.

Our applications include a purely algebraic statement in the language of Keller's derived preprojective algebras made from a dg category generated by a full exceptional collection [Ke]. Geometric applications are obtained using (equivariant) cones (also see [Bri1]) and include the weighted projective lines of Geigle and Lenzing [GL]. As specific

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examples, we can compare Coxeter functors obtained for Kleinian and certain Fuchsian singularities; [KST1, KST2, EP].

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We want to thank Sefi Ladkani for pointing out [L] to us. His work is related in spirit to this article, but different in approach and scope. It should be interesting to read [L] in conjunction with this paper.

## 2 Categories

### Exceptional objects, spherical objects and functors from evaluation maps

All our categories will be $\mathbf{k}$-linear enhanced triangulated categories, i.e. homotopy categories of pre-triangulated differential graded categories over a field $\mathbf{k}$. We also assume our categories to be Hom-finite, i.e. all Hom spaces have finite dimension over $\mathbf{k}$. A typical example is the bounded derived category $D^b(X)$ of coherent sheaves on a smooth, projective variety $X$ which is defined over $\mathbf{k}$. Another example is the bounded derived category of finite type modules over a $\mathbf{k}$-algebra of finite dimension.

Given such a category $\mathcal{T}$, we say that objects $E, F \in \mathcal{T}$ are *Serre dual* to each other if there is a functor isomorphism $\mathrm{Hom}(E, ?) \xrightarrow{\sim} \mathrm{Hom}(?, F)^*$. If a Serre dual object exists, it is unique up to unique isomorphism. If every object has a Serre dual, then one can form the *Serre functor* $\mathsf{S}\colon \mathcal{T} \to \mathcal{T}$. In other words, a Serre functor is an autoequivalence $\mathsf{S}\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ with bifunctorial isomorphisms $\mathrm{Hom}(X, Y) \xrightarrow{\sim} \mathrm{Hom}(Y, \mathsf{S}X)^*$. Again, Serre functors are unique up to unique isomorphism and in particular, they commute with all autoequivalences. Finally, $\mathcal{T}$ is said to be an *$n$-Calabi-Yau* category if the Serre functor is isomorphic to the $n$-fold shift: $\mathsf{S} \cong [n]$.

An object $E \in \mathcal{T}$ is called *exceptional* if the natural map $\mathbf{k} \to \mathrm{Hom}^\bullet_{\mathcal{T}}(E, E)$ is an isomorphism in $D(\mathbf{k})$. Here, $\mathrm{Hom}^\bullet_{\mathcal{T}}(A, B)$ is the complex of vector spaces with entries $\mathrm{Hom}^i_{\mathcal{T}}(A, B) := \mathrm{Hom}_{\mathcal{T}}(A, B[i])$ in degree $i$ and zero differentials. Assuming that $\mathcal{T}$ has a Serre functor, the object $E$ is called *$d$-spherical* if $\mathsf{S}(E) \cong E[d]$ for some $d \in \mathbb{Z}$ and $\mathrm{Hom}^\bullet(E, E) \cong \mathbf{k} \oplus \mathbf{k}[-d]$.

For any object $E \in \mathcal{T}$, we consider the endofunctor $\mathsf{T}_E\colon \mathcal{T} \to \mathcal{T}$ defined by triangles $\mathrm{Hom}^\bullet_{\mathcal{T}}(E, A) \otimes E \to A \to \mathsf{T}_E(A)$ for objects $A \in \mathcal{T}$. This definition uses the fact that $\mathcal{T}$ is given as the homotopy category of a pre-triangulated dg category, which ensures the existence of functorial cones.

There is a natural adjoint to $\mathsf{T}_E$, using the transpose of the evaluation map: $A \to \mathrm{Hom}^\bullet(A, E)^* \otimes E$ where dualising of the complex $\mathrm{Hom}^\bullet(A, E)$ includes inverting the degrees. We set $\mathsf{T}^!_E(A) := \mathrm{cone}(A \to \mathrm{Hom}^\bullet(A, E)^* \otimes E)$.

### Mutations and helices

Let $\mathcal{C}$ be a triangulated category as above. A sequence of exceptional objects $(E_0, \ldots, E_n)$ is called an *exceptional collection* if $\mathrm{Hom}^\bullet(E_i, E_j) = 0$ for $i > j$. The collection is called *full* if it generates the category $\mathcal{C}$; it is called *strong* if $\mathrm{Hom}^\bullet(E_i, E_j) = \mathrm{Hom}(E_i, E_j)$, i.e. all non-zero Homs are concentrated in degree 0.

From now on we shall assume that the given exceptional sequence $(E_0, \dots, E_n)$ is full and so generates $\mathcal{C}$. Thus by [BK2, Cor. 2.10, Cor. 3.5]), $\mathcal{C}$ has a Serre functor $\mathsf{S}$. We may then extend the exceptional sequence $(E_0, \dots, E_n)$ to a sequence $E_* = \{E_i\}_{i\in\mathbb{Z}}$ by requiring

$$E_i = \mathsf{S}(E_{i+n+1})[-n] \tag{1}$$

producing a *helix of period* $n+1$. Note that since the Serre functor $\mathsf{S}$ is an autoequivalence of $\mathcal{C}$, any subsequence $(E_j, \dots, E_{j+n})$ of the helix is again a full exceptional sequence in $\mathcal{C}$. Further, the whole helix can be recovered from any such subsequence by using (1). Such a subsequence is therefore referred to as a *foundation* of the helix that it generates.

The terms of the helix generated by a full exceptional collection can be constructed in a concrete way in terms of mutations. Let $E_* = \{E_i\}_{i\in\mathbb{Z}}$ be a helix of period $n+1$, which is generated by any of its exceptional subsequences $(E_j, \dots, E_{j+n})$. The *left mutation* of $E_i \in E_*$ is defined by $\mathsf{L}E_i := \mathsf{T}_{E_{i-1}}E_i[-1]$. Iterating this process, we put $\mathsf{L}^k E_i := \mathsf{T}_{E_{i-k}} \cdots \mathsf{T}_{E_{i-1}} E_i[-k]$. Dually, one can define *right mutations* $\mathsf{R}^k$ analogously using the adjoint twist functors. The $n$-fold mutation $\mathsf{L}^n$ is particularly important, since as observed by Bondal in [Bon, Prop. 4.2]

$$\mathsf{L}^n E_i \simeq \mathsf{S}(E_i)[-n] \tag{2}$$

In particular, we have the relation $\mathsf{L}^n E_i \simeq E_{i-n-1}$ for all $i \in \mathbb{Z}$. It will sometimes be convenient to abuse notation and extend $\mathsf{L}^n$ from an operation on objects of the helix $E_*$ to a functor defined on all of $\mathcal{C}$ by letting $\mathsf{L}^n = \mathsf{S}[-n]$.

We summarize the above discussion in the following proposition, which is from [BK2] and [Bon].

**Proposition 1. (Bondal, Kapranov)** *Let $\mathcal{C}$ be a triangulated category generated by an exceptional collection $(E_0, \dots, E_n)$. Then $\mathcal{C}$ has a Serre functor and the collection is the foundation of a helix $E_* = \{E_i\}_{i\in\mathbb{Z}}$. Also, $\mathsf{L}^n(E_i) \cong \mathsf{S}(E_i)[-n]$ for $E_i \in E_*$.*

### Helices and spherical collections

Let $\mathcal{C}$ be a triangulated category generated by a helix $E_* = \{E_i\}$. Assume we are given another triangulated category $\mathcal{D}$ with Serre functor together with a triangle functor $\iota\colon \mathcal{C} \to \mathcal{D}$. We will use the notation

$$\mathcal{E}_j := \iota E_j.$$

We are interested in two properties of the functor: $\iota$ should send the exceptional objects comprising the helix to spherical objects, and $\iota$ should be as faithful as possible given the first requirement. The latter condition is made precise as follows:

The functor $\iota$ is said to be *fully faithful on the foundation* $(E_j, \dots, E_{j+n})$ if the maps $\mathrm{Hom}^\bullet(E_j, E_k) \xrightarrow{\iota} \mathrm{Hom}^\bullet(\mathcal{E}_j, \mathcal{E}_k)$ are isomorphisms for all $j < k \le n+j$. By the next lemma, it does not matter on which foundation we test $\iota$.

**Lemma 2.** *Let $\mathcal{C}$ be a triangulated category generated by a helix $E_*$ and let $\iota\colon \mathcal{C} \to \mathcal{D}$ be a triangle functor which is fully faithful on a foundation $(E_0, \ldots, E_n)$ of the helix and such that $\iota(E_j) \in \mathcal{D}$ is $d$-spherical for all $0 \leq j \leq n$. Then $\iota$ is fully faithful on any foundation and $\iota(E_i) \in \mathcal{D}$ is $d$-spherical for all $i \in \mathbb{Z}$.*

*Proof.* We first prove the statement of the lemma for exceptional pairs. So let $(A, B)$ be an exceptional pair in $\mathcal{C}$. Let $L := \mathsf{T}_A B[-1]$ be its left mutation, defined by the triangle $L \to \mathrm{Hom}^\bullet(A, B) \otimes A \to B$ and denote $\mathcal{A} := \iota A$, $\mathcal{B} := \iota B$, $\mathcal{L} := \iota L$. Then $(L, A)$ is another exceptional pair. Assume that $\iota$ is fully faithful for $(A, B)$, i.e. that the map $\mathrm{Hom}^\bullet(A, B) \to \mathrm{Hom}^\bullet(\mathcal{A}, \mathcal{B})$ is an isomorphism. Further assume that $\mathcal{A}$ and $\mathcal{B}$ are $d$-spherical.

*Step 1.* Claim: $\iota$ is fully faithful for $(L, A)$.

Applying $\mathrm{Hom}^\bullet(?, A)$ to the triangle defining $L$ and applying $\iota$ to that yields a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^\bullet_{\mathcal{C}}(L, A) \arrow[d, "\iota"] & \mathrm{Hom}^\bullet_{\mathcal{C}}(A, B)^* \arrow[l] \arrow[d, "\iota"] & 0 \arrow[l] \arrow[d] \\
\mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{L}, \mathcal{A}) & \mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{A}, \mathcal{B})^* \oplus \mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{A}, \mathcal{B})^*[-d] \arrow[l] & \mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{B}, \mathcal{A}) \arrow[l]
\end{tikzcd}$$

where we have used $\mathrm{Hom}^\bullet_{\mathcal{C}}(A, A) = \mathbf{k}$, $\mathrm{Hom}^\bullet_{\mathcal{C}}(B, A) = 0$ and $\mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{A}, \mathcal{A}) = \mathbf{k} \oplus \mathbf{k}[-d]$. The lower sequence splits as a direct sum: the map $\mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{B}, \mathcal{A}) \to \mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{A}, \mathcal{B})^*[-d]$ is an isomorphism, and the map $\mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{B}, \mathcal{A}) \to \mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{A}, \mathcal{B})^*$ is zero. Therefore, we get the following square as a subdiagram

$$\begin{tikzcd}
\mathrm{Hom}^\bullet_{\mathcal{C}}(L, A) \arrow[d, "\iota"] & \mathrm{Hom}^\bullet_{\mathcal{C}}(A, B)^* \arrow[l, "\cong"'] \arrow[d, "\iota"] \\
\mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{L}, \mathcal{A}) & \mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{A}, \mathcal{B})^* \arrow[l, "\cong"']
\end{tikzcd}$$

The righthand map is an isomorphism, hence the lefthand is as well.

*Step 2.* Claim: $\mathcal{L}$ is $d$-spherical.

As $\iota$ is fully faithful for $(A, B)$, there is a triangle $\mathcal{L} \to \mathrm{Hom}^\bullet(\mathcal{A}, \mathcal{B}) \otimes \mathcal{A} \to \mathcal{B}$. With $\mathcal{A}$ and $\mathcal{B}$ being $d$-Calabi-Yau objects, the same is true for $\mathcal{L}$. Applying $\mathrm{Hom}^\bullet(\mathcal{B}, ?)$ and $\mathrm{Hom}^\bullet(?, \mathcal{L})$ produces two exact triangles in $D(\mathbf{k})$:

$$\begin{aligned}
&\mathrm{Hom}^\bullet(\mathcal{B}, \mathcal{L}) \to \mathrm{Hom}^\bullet(\mathcal{A}, \mathcal{B})^* \otimes \mathrm{Hom}^\bullet(\mathcal{A}, \mathcal{L}) \to \mathrm{Hom}^\bullet(\mathcal{L}, \mathcal{L}) \\
&\mathrm{Hom}^\bullet(\mathcal{B}, \mathcal{L}) \to \mathrm{Hom}^\bullet(\mathcal{A}, \mathcal{B}) \otimes \mathrm{Hom}^\bullet(\mathcal{B}, \mathcal{A}) \to \mathrm{Hom}^\bullet(\mathcal{B}, \mathcal{B}).
\end{aligned} \tag{3}$$

Furthermore, applying $\mathrm{Hom}^\bullet(\mathcal{A}, ?)$ to the triangle defining $\mathcal{L}$ and using that $\mathcal{A}$ is $d$-spherical shows $\mathrm{Hom}^\bullet(\mathcal{A}, \mathcal{L}) \cong \mathrm{Hom}^\bullet(\mathcal{A}, \mathcal{B})[-d]$. This in turn implies that the middle terms of the triangles (3) are isomorphic. Hence, $\mathrm{Hom}^\bullet(\mathcal{L}, \mathcal{L}) \cong \mathrm{Hom}^\bullet(\mathcal{B}, \mathcal{B})$ are isomorphic as complexes of vector spaces. Thus $\mathcal{L}$ is $d$-spherical, as $\mathcal{B}$ is.

*Step 3:* Propagation to arbitrary collections.

Now suppose there is a full exceptional collection $(M_1, \ldots, M_m, A, B, N_1, \ldots, N_n)$ which contains the pair $(A, B)$. Also suppose that $\iota$ is fully faithful on this foundation.

Then $(M_1, \ldots, M_m, L, A, N_1, \ldots, N_n)$ is again a full exceptional collection for which $\iota$ is fully faithful. To see this on pairs $(M_i, L)$, apply $\mathrm{Hom}^\bullet(M_i, ?)$ to the triangle defining $L$; to see this on pairs $(L, N_j)$, apply $\mathrm{Hom}^\bullet(?, N_j)$.

Finally, let $(E_0, \ldots, E_n)$ be the foundation of the statement of the lemma. We can repeat the above argument to find that $\iota$ is as well fully faithful on mutated foundations $(E_0, \ldots, E_{n-2}, \mathsf{L}^1 E_n, E_{n-1})$ up to $(\mathsf{L}^n E_n, E_0, \ldots, E_{n-1})$. Note that we are using both Steps 1 and 2 here. Hence, $\iota$ is fully faithful on $(E_{-1}, \ldots, E_{n-1})$. Iterating this process and applying the dual construction of right mutations shows that $\iota$ is fully faithful on any foundation.

Still looking at $(E_0, \ldots, E_n)$, we see that $\mathcal{E}_0, \ldots, \mathcal{E}_n$ being $d$-spherical implies that $\iota(\mathsf{L}^1 E_n)$ is $d$-spherical by Step 2. We can iterate this process by Step 1, deducing that $\mathcal{E}_{-1} = \iota(\mathsf{L}^n E_n)$ is $d$-spherical. This can be extended to arbitrary elements of the helix, just as above. $\square$

**Theorem 3.** *Let $\mathcal{C}$ be a triangulated category generated by an exceptional collection $(E_0, \ldots, E_n)$. Let $\iota\colon \mathcal{C} \to \mathcal{D}$ be a triangle functor such that the objects $\mathcal{E}_i := \iota E_i$ are $d$-spherical for $i = 0, \ldots, n$. Assume that $\iota$ is fully faithful on the foundation $(E_0, \ldots, E_n)$ and has generating image. Then, $\mathsf{T}_{\mathcal{E}_0} \cdots \mathsf{T}_{\mathcal{E}_n} \iota \cong \iota \mathsf{S}[1-d]$.*

**Remark 4.** In our examples, $\mathcal{C}$ will be the bounded derived category of an abelian category for which we have a notion of dimension (geometric or homological) and so we shall speak of $\dim(\mathcal{C})$, as well as of the dimension $d$ of the spherical objects. Mostly we shall focus on the case that $d = 1 + \dim(\mathcal{C})$, but it is important to observe that there are examples of other dimensions and that there is no a priori relation between $\dim(\mathcal{C})$ and $d$. Indeed, take $\mathcal{C} = \langle E \rangle$, the category generated by one exceptional object, $\mathcal{D} = \langle \mathcal{E} \rangle$, the category generated by one $d$-spherical object, and $\iota$ the unique triangle functor such that $\iota E = \mathcal{E}$. Then the condition on the functor $\iota$ is vacuous here and the theorem obviously holds.

*Proof.* We proceed in a number of steps.

*Step 0.* Claim: There is at most one autoequivalence $\varphi \in \mathrm{Aut}(\mathcal{D})$ with $\varphi\iota \cong \iota\mathsf{S}$.

We use that $\mathcal{C}$ is generated by $E_0, \ldots, E_n$ and that $\mathcal{D}$ is generated by $\mathcal{E}_0, \ldots, \mathcal{E}_n$. The relation $\varphi\iota \cong \iota\mathsf{S}$ determines the value of $\varphi$ on the generators: $\varphi(\mathcal{E}_i) = \iota\mathsf{S}(E_i)$.

For $i < j$, we have

$$\begin{aligned}
\mathrm{Hom}^\bullet_{\mathcal{D}}(\varphi(\mathcal{E}_i), \varphi(\mathcal{E}_j)) &\simeq \mathrm{Hom}^\bullet_{\mathcal{D}}(\iota\mathsf{S}(E_i), \iota\mathsf{S}(E_j)) && \text{(by assumption)} \\
&\simeq \mathrm{Hom}^\bullet_{\mathcal{D}}(\iota\mathsf{L}^n E_i[n], \iota\mathsf{L}^n E_j[n]) && (\mathsf{S} = \mathsf{L}^n[n] \text{ on } E_*) \\
&\simeq \mathrm{Hom}^\bullet_{\mathcal{D}}(\iota E_{i-n-1}, \iota E_{j-n-1}) && \text{(helix property)} \\
&\simeq \mathrm{Hom}^\bullet_{\mathcal{C}}(E_{i-n-1}, E_{j-n-1}) && \text{(Lemma 2)} \\
&\simeq \mathrm{Hom}^\bullet_{\mathcal{C}}(\mathsf{S}(E_i), \mathsf{S}(E_j)) && (\mathsf{S} = \mathsf{L}^n[n] \text{ and helix}) \\
&\simeq \mathrm{Hom}^\bullet_{\mathcal{C}}(E_i, E_j) && (\mathsf{S} \text{ an autoequivalence})
\end{aligned}$$

where we use Lemma 2, which says $\iota$ is fully faithful on the foundation $(E_{-n-1}, \ldots, E_{-1})$, too. This settles the effect of $\varphi$ on maps $\mathcal{E}_i \to \mathcal{E}_j[k]$ with $i < j$.

If $i > j$, we use Serre duality in $\mathcal{D}$, which acts by a shift on spherical objects:

$$
\begin{aligned}
\mathrm{Hom}^\bullet_{\mathcal{D}}(\varphi(\mathcal{E}_i), \varphi(\mathcal{E}_j)) &\simeq \mathrm{Hom}^\bullet_{\mathcal{D}}(\varphi(\mathcal{E}_j), \varphi(\mathcal{E}_i)[d])^* && \text{(Serre duality)} \\
&\simeq \mathrm{Hom}^\bullet_{\mathcal{D}}(\iota\mathsf{S}(E_j), \iota\mathsf{S}(E_i)[d])^* && \text{(by assumption)} \\
&\simeq \mathrm{Hom}^\bullet_{\mathcal{C}}(\mathsf{S}(E_j), \mathsf{S}(E_i)[d])^* && \text{(fully faithful)} \\
&\simeq \mathrm{Hom}^\bullet_{\mathcal{C}}(E_j, E_i[d])^* && \text{($\mathsf{S}$ an autoequivalence)} \\
&\simeq \mathrm{Hom}^\bullet_{\mathcal{D}}(\iota E_j, \iota E_i[d])^* && \text{(fully faithful)} \\
&\simeq \mathrm{Hom}^\bullet_{\mathcal{D}}(\iota E_i[d], \iota E_j[d]) && \text{(Serre duality)} \\
&\simeq \mathrm{Hom}^\bullet_{\mathcal{D}}(\iota E_i, \iota E_j). && \text{(shift)}
\end{aligned}
$$

Thus we may define $\varphi$ on maps $\mathcal{E}_j \to \mathcal{E}_i[k]$.

For $i = j$, the action of $\varphi$ on maps $\mathcal{E}_i \to \mathcal{E}_i[k]$ is determined by functoriality $\varphi(\mathrm{id}_{\mathcal{E}_i}) = \mathrm{id}_{\varphi(\mathcal{E}_i)}$ and by Serre duality:

$$
\begin{aligned}
\mathrm{Hom}^\bullet_{\mathcal{D}}(\varphi(\mathcal{E}_i), \varphi(\mathcal{E}_i)) &= \mathrm{Hom}^0_{\mathcal{D}}(\varphi(\mathcal{E}_i), \varphi(\mathcal{E}_i)) \oplus \mathrm{Hom}^d_{\mathcal{D}}(\varphi(\mathcal{E}_i), \varphi(\mathcal{E}_i))[-d] \\
&= \mathrm{Hom}^0_{\mathcal{D}}(\varphi(\mathcal{E}_i), \varphi(\mathcal{E}_i)) \oplus \mathrm{Hom}^0_{\mathcal{D}}(\varphi(\mathcal{E}_i), \varphi(\mathcal{E}_i))^*[-d] \\
&= \mathbf{k} \cdot \mathrm{id}_{\varphi(\mathcal{E}_i)} \oplus \mathbf{k} \cdot \mathsf{S}(\mathrm{id}_{\varphi(\mathcal{E}_i)})
\end{aligned}
$$

Since the category $\mathcal{D}$ is generated by the objects $\mathcal{E}_0, \dots, \mathcal{E}_n$, the autoequivalence $\varphi$ is completely determined by its action on these objects and on maps between these objects, which we have see is determined by requiring $\varphi\iota \simeq \iota\mathsf{S}$. Thus if such a $\varphi \in \mathrm{Aut}(\mathcal{D})$ exists, it must be unique, as claimed.

Since any object of $\mathcal{C}$ is built functorially from the objects $E_0, \dots, E_n$, allows to extend the functor isomorphism to all of $\mathcal{C}$ if it is given on the exceptional collection. Furthermore, such an extension is unique.

*Step 1.* We introduce some temporary notation:

$$\tau_j^{(l)} := \mathsf{T}_{\mathcal{E}_{j-l+1}} \cdots \mathsf{T}_{\mathcal{E}_{j-1}} \mathsf{T}_{\mathcal{E}_j} \in \mathrm{Aut}(\mathcal{D}).$$

Fix $i \in \mathbb{Z}$. We first show by an induction on $l = 1, \dots, n$

(i) There are isomorphisms $\alpha_i \colon \iota\mathsf{L}^l E_i[l] \xrightarrow{\sim} \tau_{i-1}^{(l)}\mathcal{E}_i$,

(ii) $\iota \colon \mathrm{Hom}^\bullet_{\mathcal{C}}(E_t, \mathsf{L}^{l-1}E_i) \to \mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{E}_t, \iota\mathsf{L}^{l-1}E_i)$ is an isomorphism for $0 \le t \le i - l$.

For $l = 1$, (ii) is true by assumption, the functor $\iota$ being fully faithful on the foundation. Using (ii), we get $\mathrm{Hom}^\bullet(\mathcal{E}_{i-1}, \mathcal{E}_i) \otimes \mathcal{E}_{i-1} \to \mathcal{E}_i \to \iota\mathsf{L}E_i[1]$ by applying $\iota$ to the triangle defining $\mathsf{L}E_i[1]$. But this triangle defines $\mathsf{T}_{\mathcal{E}_{i-1}}\mathcal{E}_i$.

For $l > 1$, use $\mathrm{Hom}(E_t, ?)$ on the triangle defining $\mathsf{L}^{l-1}E_i$ and apply $\iota$ to that, resulting in a commutative diagram

$$
\begin{tikzcd}
\mathrm{Hom}^\bullet_{\mathcal{C}}(E_t, \mathsf{L}^{l-1}E_i) \arrow[r] \arrow[d, "\iota"] & \mathrm{Hom}^\bullet_{\mathcal{C}}(E_t, E_{i-l+1}) \otimes \mathrm{Hom}^\bullet_{\mathcal{C}}(E_{i-l+1}, \mathsf{L}^{l-2}E_i) \arrow[r] \arrow[d, "\iota"] & \mathrm{Hom}^\bullet_{\mathcal{C}}(E_t, \mathsf{L}^{l-2}E_i) \arrow[d, "\iota"] \\
\mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{E}_t, \iota\mathsf{L}^{l-1}E_i) \arrow[r] & \mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{E}_t, \mathcal{E}_{i-l+1}) \otimes \mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{E}_{i-l+1}, \iota\mathsf{L}^{l-2}E_i) \arrow[r] & \mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{E}_t, \iota\mathsf{L}^{l-2}E_i)
\end{tikzcd}
$$

The last two vertical arrows are isomorphisms by induction and assumption on $\iota$, hence the first one is as well, thereby proving (ii). Writing down the triangle defining $\mathsf{L}^l E_i$ and applying $\iota$ gives a diagram

$$
\begin{tikzcd}
\mathrm{Hom}^\bullet_{\mathcal{C}}(E_{i-l}, \mathsf{L}^{l-1}E_i) \otimes \iota E_{i-j} \arrow[r] \arrow[d, "\iota\otimes\mathrm{id}"] & \iota\mathsf{L}^{l-1}E_i \arrow[r] \arrow[d, equal] & \iota\mathsf{L}^l E_i[1] \arrow[d, dotted] \\
\mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{E}_{i-l}, \iota\mathsf{L}^{l-1}E_i) \otimes \mathcal{E}_{i-l} \arrow[r] & \iota\mathsf{L}^{l-1}E_i \arrow[r] & \mathsf{T}_{\mathcal{E}_{i-l}}(\iota\mathsf{L}^{l-1}E_i)
\end{tikzcd}
$$

where the dotted arrow is an isomorphism. By induction, we get actual isomorphisms $\iota \mathsf{L}^l E_i[l] \xrightarrow{\sim} \tau_{i-1}^{(l)} \mathcal{E}_i$.

*Step 2.* Next, we show that the functor $\tau_n^{(n+1)}$ is invariant under cycling the spherical collection, i.e. there are functor isomorphisms $\beta_i \colon \tau_n^{(n+1)} \xrightarrow{\sim} \tau_i^{(n+1)}$ for any $i \in \mathbb{Z}$.

As a preparatory remark, note that an isomorphism $f \colon \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ yields a functor isomorphism $\mathsf{T}_f \colon \mathsf{T}_{\mathcal{E}} \xrightarrow{\sim} \mathsf{T}_{\mathcal{E}'}$ in view of the diagram

$$\begin{tikzcd}
\mathrm{Hom}^\bullet(\mathcal{E}, X) \otimes \mathcal{E} \arrow[r] \arrow[d, "\mathrm{Hom}(f^{-1}, X) \otimes f"'] & X \arrow[r] \arrow[d, "\mathrm{id}"] & \mathsf{T}_{\mathcal{E}} X \arrow[d, dotted, "\mathsf{T}_f"] \\
\mathrm{Hom}^\bullet(\mathcal{E}', X) \otimes \mathcal{E}' \arrow[r] & X \arrow[r] & \mathsf{T}_{\mathcal{E}'} X
\end{tikzcd}$$

We have $\mathcal{E}_0[n] \xrightarrow{\sim} \tau_n^{(n)}(\mathcal{E}_{n+1})$ by step 1, hence

$$\mathsf{T}_{\mathcal{E}_0} \xrightarrow{\sim} \mathsf{T}_{\tau_n^{(n)}(\mathcal{E}_{n+1})[-n]} \xrightarrow{\sim} \mathsf{T}_{\tau_n^{(n)}(\mathcal{E}_{n+1})} \xrightarrow{\sim} \tau_n^{(n)} \mathsf{T}_{\mathcal{E}_{n+1}} (\tau_n^{(n)})^{-1},$$

thus $\tau_n^{(n+1)} \xrightarrow{\sim} \tau_{n+1}^{(n+1)}$ and the general case proceeds by induction.

*Step 3.* Therefore,

$$\begin{aligned}
\tau_n^{(n+1)}(\mathcal{E}_i) &\xrightarrow{\sim} \tau_i^{(n+1)}(\mathcal{E}_i) && \text{by step 2} \\
&\xrightarrow{\sim} \tau_{i-1}^{(n)}(\mathcal{E}_i)[1-d] && \mathsf{T}_{\mathcal{E}}(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}[1-d] \\
&\xrightarrow{\sim} \mathcal{E}_{i-n-1}[n+1-d] && \text{by step 1 and } \mathsf{L}_{\mathcal{E}} = \mathsf{T}_{\mathcal{E}}[1].
\end{aligned}$$

By the helix condition, using that the period is $p = n+1$, we find

$$\begin{aligned}
\tau_n^{(n+1)} \iota E_i = \tau_n^{(n+1)}(\mathcal{E}_i) &\cong \mathcal{E}_{i-n-1}[n+1-d] = \iota E_{i-n-1}[n+1-d] \\
&\xrightarrow{\sim} \iota \mathsf{S} E_{i-n-1+p}[n+1-d+1-p] = \iota \mathsf{S} E_i[1-d].
\end{aligned}$$

*Step 4.* We want an isomorphism between functors $\tau_n^{(n+1)} \iota \xrightarrow{\sim} \iota \mathsf{S}[1-d]$, or, what amounts to the same, $\tau_n^{(n+1)}[d-1] \xrightarrow{\sim} \iota \mathsf{S}$. For this, note that the above steps provide isomorphisms $\alpha(E_i) \colon \tau_n^{(n+1)} \iota E_i[d-1] \xrightarrow{\sim} \iota \mathsf{S} E_i$ for all $i \in \mathbb{Z}$. Thus, $\tau_n^{(n+1)}[d-1]$ is an autoequivalence of $\mathcal{D}$ with the correct values on the generators. By step 0 we can thus conclude the proof. □

## 3 Applications

All of the upcoming examples generalise the following paradigm: Let $\mathcal{C} = D(\mathbb{P}^1)$ be the bounded derived category of the projective line; let $V = T^*_{\mathbb{P}^1}$ be the cotangent bundle, which is a non-proper surface containing $\mathbb{P}^1$ as the zero section, and let $\mathcal{D} = D_o(V)$ be the full subcategory of $D(V)$ of objects supported on the zero section. The inclusion $i \colon \mathbb{P}^1 \to V$ yields a functor $\iota := i_* \colon \mathcal{C} \to \mathcal{D}$. There is a full exceptional collection on $\mathcal{C}$, given by $(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1))$, and the main theorem applies in this setting.

We will provide several geometric and algebraic settings in which our theorem can be applied. Several of these involve global quotient stacks $[X/G]$, where $X$ is a $\mathbf{k}$-variety carrying the action of an algebraic group $G$. By the very definition of sheaves on stacks, there is a tautological equivalence of abelian categories

$$\mathrm{Coh}([X/G]) \cong \mathrm{Coh}^G(X)$$

between coherent sheaves on $[X/G]$ and $G$-linearised coherent sheaves on $X$. We are therefore free to switch between these notations.

### 3.1 Derived preprojective algebras

The most general and algebraic application uses Keller's Calabi-Yau completions or preprojective algebras [Ke]. To any differential graded algebra $A$, there is associated an $n$-Calabi-Yau differential graded algebra $\Pi_n(A)$, called the *derived $n$-preprojective algebra of $A$* or *$n$-Calabi-Yau completion*.

For the purposes of this paper, we are interested in a small, idempotent complete triangulated category $\mathcal{C}$ generated by an exceptional collection. Then $\mathcal{C}$ is equivalent to the subcategory of $D(A)$ consisting of perfect dg modules over $A$, the dg endomorphism algebra of the collection.

There is a natural projection map $f\colon \Pi_n(A) \to A$ so that the restriction functor $i\colon D(A) \to D(\Pi_n(A))$ of scalars functor sends the exceptional generators to spherical objects. More generally, there is the following relation (see [Ke, Lemma 4.4b]) ????

$$\begin{aligned}\mathrm{Hom}^\bullet_{\Pi_n(A)}(i(E), i(E')) &= \mathrm{Hom}^\bullet_A(E, E') \oplus \mathrm{Hom}^\bullet_A(E \otimes \Theta, E')[-n] \\ &= \mathrm{Hom}^\bullet_A(E, E') \oplus \mathrm{Hom}^\bullet_A(E', E)^*[-n]\end{aligned}$$

for $E, E' \in D_{fd}(A)$ (i.e. of finite dimension over the field). Here, $\Theta$ is the inverse dualising complex (see [Ke, §2]) and $V^*$ denotes the dual vector space. We see that the functor induced by $f$ is fully faithful on the foundation, allowing us to employ Theorem 3.

Most of the examples below are known to be derived equivalent to the situation above (and we expect the same for the other examples). However, we believe that is generally worthwhile to find geometric descriptions for the categories and functors when the original data is geometric.

### 3.2 Cones

#### Cones and $\mathbb{G}_m$-equivariant sheaves

Let $A = \bigoplus_{i\geq 0} A_i$ be a non-negatively graded, commutative unital, $\mathbf{k}$-algebra. Assume furthermore that each graded piece $A_i$ is finite dimensional, that $A_0 = \mathbf{k}$, and that $A$ is finitely generated over $\mathbf{k}$ (though not necessarily in degree one).

The grading on the algebra $A$ corresponds to a $\mathbb{G}_m$-action on the affine scheme $X = \mathrm{Spec}(A)$. We shall refer to $\mathrm{Spec}(A)$ endowed with this $\mathbb{G}_m$-action as a (quasi-homogeneous) cone. The point 0 corresponding to the maximal ideal $\mathfrak{m} = \bigoplus_{i\geq 1} A_i$ is fixed by the $\mathbb{G}_m$-action and will be referred to as the vertex of the cone. The complement $U = X \setminus \{0\}$ is $\mathbb{G}_m$-stable. We shall be interested in the $\mathbb{G}_m$-equivariant geometry of the complement $U$ as giving rise to a situation in which our main theorem can be applied.

When the algebra $A$ is generated in degree 1, the $\mathbb{G}_m$-action is free and the quotient $[U/\mathbb{G}_m]$, which a priori is an algebraic stack, is represented by the scheme $\mathrm{Proj}(A)$, and we have an equivalence of abelian categories $\mathrm{Coh}([U/\mathbb{G}_m]) \cong \mathrm{Coh}(\mathrm{Proj}(A))$. When the algebra $A$ is not generated in degree 1, the $\mathbb{G}_m$-action will have stabilisers and the quotient stack $[U/\mathbb{G}_m]$ will not be represented by a scheme.

Since we are interested in good categories of sheaves for applications of our main theorem, we shall therefore work with $\mathbb{G}_m$-equivariant coherent sheaves on the punctured cone $U$ or, equivalently, with coherent sheaves on the quotient stack $[U/\mathbb{G}_m]$.

Using the quotient stack $[U/\mathbb{G}_m]$ rather than $\mathrm{Proj}(A)$ has the advantage that Serre's theorem on the equivalence of coherent sheaves on $\mathrm{Proj}(A)$ with finitely generated $A$-modules up to torsion now holds without the assumption that $A$ is generated in degree 1. More precisely, there is a short exact sequence of abelian categories $\mathrm{Coh}^{\mathbb{G}_m}(X)_{\mathrm{tors}} \to \mathrm{Coh}^{\mathbb{G}_m}(X) \to \mathrm{Coh}^{\mathbb{G}_m}(U)$ where the right hand arrow is just restriction to the open subscheme $U$.

The following theorem of Orlov [O] relates the derived categories of stacks $[U/\mathbb{G}_m]$ to triangulated categories of singularities for the original singularity $X = \mathrm{Spec}(A)$, namely $D^{\mathbb{G}_m}_{\mathrm{sing}}(X) = D^{gr}_{\mathrm{sing}}(A) := D^{\mathbb{G}_m}(X)/\operatorname{Perf}^{\mathbb{G}_m}(X)$. If $A$ is a graded algebra as above and in addition Gorenstein, then the *Gorenstein parameter* of a $A$ is defined as the integer number $a$ with $\mathrm{Ext}^{\dim A}(\mathbf{k},\mathbf{k}) = \mathbf{k}(a)$.

**Theorem 5. (Orlov)** *Let $A$ be an algebra as above which is Gorenstein with Gorenstein parameter $a$. Then,*

$$\begin{aligned}
a > 0 &\implies D^b([U/\mathbb{G}_m]) = \langle \mathcal{O}(-i-a+1), \ldots, \mathcal{O}(-i), D^{\mathrm{gr}}_{\mathrm{sing}}(A)\rangle,\\
a < 0 &\implies D^{\mathrm{gr}}_{\mathrm{sing}}(A) = \langle \mathbf{k}(-i), \ldots, \mathbf{k}(-i+a+1), D^b([U/\mathbb{G}_m])\rangle,\\
a = 0 &\implies D^b([U/\mathbb{G}_m]) \cong D^{\mathrm{gr}}_{\mathrm{sing}}(A)
\end{aligned}$$

*where for $a \neq 0$, the right hand sides are semi-orthogonal decompositions and $i \in \mathbb{Z}$ is arbitrary.*

## Canonical bundles and Calabi-Yau categories

We assume now that the punctured cone $U$ is smooth of dimension $n$ and we consider the total space $V := \mathbb{V}(\omega_U)$ of its canonical bundle endowed with the induced $\mathbb{G}_m$-action. We will study $D^b([V/\mathbb{G}_m]) = D^{\mathbb{G}_m}(V)$, where in general we shall use the latter notation. Let $\pi$ denote the projection and $i$ the zero-section, both of which are equivariant. We therefore have (derived) functors $\pi^*\colon D^{\mathbb{G}_m}(U) \to D^{\mathbb{G}_m}(V)$, $i_*\colon D^{\mathbb{G}_m}(U) \to D^{\mathbb{G}_m}(V)$, $i^*\colon D^{\mathbb{G}_m}(V) \to D^{\mathbb{G}_m}(U)$ and the relation $i^* \circ \pi^* \cong \mathrm{id}$. Further, the pushforward $i_*$ factors through the inclusion of the category $\mathcal{D}$ consisting of complexes of equivariant sheaves (set-theoretically) supported on the zero section.

**Lemma 6.** *The category $\mathcal{D}$ is $n$-Calabi-Yau.*

*Proof.* The category $D^{\mathbb{G}_m}(V)$ has a duality theory using the canonical bundle $\omega_V$ which is naturally $\mathbb{G}_m$-linearised. This does not give rise to a Serre functor on $D^{\mathbb{G}_m}(V)$ as this category is not Hom-finite. However, this does produce a Serre functor on the Hom-finite subcategory $\mathcal{D}$.

Reason or reference!

Using the short exact sequence $0 \to \pi^*\omega_U^{-1} \to \mathcal{O}_V \to i_*\mathcal{O}_U \to 0$, which is also the Koszul resolution of $i_*\mathcal{O}_U$, we immediately get $\pi^*\omega_U^{-1} = \mathcal{O}_V(-U)$, hence $\omega_U^{-1} = \mathcal{O}_V(-U)|_U$. For $E \in D^{\mathbb{G}_m}(U)$, we use projection formula and adjunction to find $i_*E \otimes \omega_V = i_*(E \otimes i^*\omega_V) = i_*(E \otimes \omega_U \otimes \mathcal{O}_V(-U)|_U) = i_*E$, so that $\mathsf{S}(i_*E) = i_*E \otimes \omega_V[n] = i_*E[n]$. This shows that the Serre functor on $\mathcal{D}$ is the shift $[n]$, as claimed. □

This provides an abundant and concrete source of Calabi-Yau categories. Such examples were considered by Bridgeland when $[U/\mathbb{G}_m]$ is a smooth projective variety, [Bri1].

**Lemma 7.** *If $E, F \in D^{\mathbb{G}_m}(U)$, then* $\mathrm{Hom}^j(i_*E, i_*F) \cong \mathrm{Hom}^j(E,F) \oplus \mathrm{Hom}^{n-j}(F,E)^*$. *In particular, if $E \in D^{\mathbb{G}_m}(U)$ is an exceptional object, then $i_*E \in \mathcal{D}$ is $n$-spherical.*

*Proof.* For $E, F \in D^{\mathbb{G}_m}(U)$, adjunction yields $\mathrm{Hom}^j(i_*E, i_*F) = \mathrm{Hom}^j(i^*i_*E, F)$. We thus wish to understand $i^*i_*E$. By the projection formula, $i_*E \cong \pi^*E \otimes i_*\mathcal{O}_U$. Now consider again the short exact sequence $0 \to \pi^*\omega_U^{-1} \to \mathcal{O}_V \to i_*\mathcal{O}_U \to 0$. Tensoring this with $\pi^*E$ and and then pulling-back along $i$ therefore produces an exact triangle $E \otimes \omega_U^{-1} \to E \to i^*i_*E$. Note that the first arrow in this triangle vanishes since the corresponding arrow in the Koszul resolution vanishes along the zero section. Thus the triangle splits and we have an isomorphism $i^*i_*E \cong E \oplus E \otimes \omega_U^{-1}[1]$. Altogether then we have isomorphisms $\mathrm{Hom}^j(i_*E, i_*F) \cong \mathrm{Hom}^j(i^*i_*E, F) \cong \mathrm{Hom}^j(E,F) \oplus \mathrm{Hom}^j(E \otimes \omega_U^{-1}[1], F) \cong \mathrm{Hom}^j(E,F) \oplus \mathrm{Hom}^{n-j}(F,E)^*$, where the last isomorphism uses Serre duality on $[U/\mathbb{G}_m]$.

For the second claim, suppose $E \in D^{\mathbb{G}_m}(U)$ is exceptional, so that $\mathrm{Hom}^j(E,E) \cong \mathbf{k}$ if $j = 0$, and is otherwise zero. Then by the first claim, we have an isomorphism $\mathrm{Hom}^j(i_*E, i_*E) \cong \mathrm{Hom}^j(E,E) \oplus \mathrm{Hom}^{n-j}(E,E)^*$, which is non-zero if and only if $j = 0, n$, in which cases it is one-dimensional. Thus $i_*E$ is spherical, as claimed. □

**Lemma 8.** *Suppose that $D^{\mathbb{G}_m}(U)$ is generated by an object $E$. Then $\mathcal{D}$ is generated by the pushforward $i_*E$.*

*Proof.* We must argue that every object in $\mathcal{D}$ is in the thick triangulated category generated by $i_*E$.

Every object in $\mathcal{D}$ can be written as a successive extension of equivariant coherent sheaves on $V = \omega_U$ that are set-theoretically supported on the zero-section. Each such coherent sheaf has a canonical filtration whose graded pieces are scheme-theoretically supported on the zero section and therefore pushed-forward from equivariant coherent sheaves on $U$. Finally, by assumption, each equivariant coherent sheaf on $U$ is in the thick triangulated subcategory generated by $E$, and thus the pushforward of such a coherent sheaf is in the thick subcategory generated by the pushforward of $E$. □

### 3.3 Weighted projective lines

Let $Z$ be a smooth projective curve and $G \subset \mathrm{Aut}(Z)$ be a finite subgroup such that the quotient (as variety) is the projective line: $Z/G \cong \mathbb{P}^1$. Then the quotient stack (orbifold)

$[Z/G]$ is the geometric version of a *weighted projective line* of Geigle and Lenzing, in the sense that $\mathrm{Coh}([Z/G])$ coincides with the category of coherent on some weighted projective line $\mathbb{X}$.

By their fundamental result [GL], there exists a full exceptional collection for the derived category $D([Z/G])$. Consider the functor $i_*\colon D([Z/G]) \to D([T_Z^*/G])$ where $i\colon Z \to T_Z^*$ is the $G$-equivariant embedding of the zero section into the cotangent bundle. Of course, the functor can also be seen as $i_*\colon D^G(Z) \to D^G(T_Z^*)$.

**Lemma 9.** *The functor $i_*$ sends expectional objects to spherical objects and is fully faithful on exceptional collections.*

Therefore, we can apply Theorem 3 in this setup. In the following, we will explain how this natural functor can be used to compare Coxeter functors of certain categories associated to singularities.

# 4 Examples

The lemmas from the last section suggest that we can apply our main theorem to the categories $\mathcal{C} = D^b([U/\mathbb{G}_m])$ and $\mathcal{D}$ when the former admits a full exceptional collection. In the next subsection, we consider some classic examples coming from singularity theory.

## 4.1 Kleinian singularities

(Should really restrict to $n$ even for type $A$? Damn.)

References: [KST2], [Brav].

I replaced all $\mathbb{C}$ with $\mathbf{k}$ and all $\mathbb{C}^*$ with $\mathbb{G}_m$. I hope that's okay — we only need algebraically closed and characteristic 0, right?

Given a finite group $G \subset SL_2(\mathbf{k})$, consider the quotient $X := \mathbb{A}^2/G$, which is well-known since the work of Klein to be a hypersurface with a single, isolated singularity at 0, of type $A$,$D$, or $E$. Since the $G$-action on $\mathbb{A}^2$ is linear, the quotient $\mathbb{A}^2/G$ inherits an action of $\mathbb{G}_m$. We consider $U = X \setminus \{0\}$, which is a smooth punctured cone of dimension 2, as in the previous section.

The category $D^b([U/\mathbb{G}_m])$ admits a full, strong exceptional collection. Let $V_j$ be the irreducible representations of the group $G$, which via the McKay correspondence label the nodes of an extended affine Dynkin diagram $\widehat{\Gamma}$ of the appropriate type. Now consider the associated $G\times\mathbb{G}_m$-equivariant bundles $V_j\otimes\mathcal{O}_{\mathbb{A}^2}$ on $\mathbb{A}^2$. Taking $G$-invariants produces $\mathbb{G}_m$-equivariant coherent sheaves on the quotient $\mathbb{A}^2/G$, and restricting to $U$ gives the desired full, strong exceptional collection $E_j$, $j \in \widehat{\Gamma}$. The endomorphism algebra of this collection isomorphic to the path algebra for an appropriate orientation of the diagram $\widehat{\Gamma}$.

If $A = k[X]$ is the coordinate algebra of the singularity, then it is graded (the singularity being quasi-homogenous) and Gorenstein with Gorenstein parameter $-1$. Applying Orlov's theorem, there is a semi-orthogonal decomposition $D^{\mathrm{gr}}_{\mathrm{sg}}(A) = \langle \mathbf{k}, D^{\mathbb{G}_m}(U)\rangle$, thus giving an equivalence of the triangulated category of singularities with modules for the (non-extended) Dynkin quiver. (See [KST2] and especially the appendix by Ueda.)

Let $V = \mathbb{V}(\omega_U)$ be the geometric canonical bundle, as before, and let $\mathcal{D}$ be the subcategory of $D^{\mathbb{G}_m}(V)$ consisting of objects supported along the zero section. We see by Lemma 7 and Lemma 8 that the objects $i_*E_j \in \mathcal{D}$ are spherical and generate $\mathcal{D}$. Further, a simple computation shows that they form a $\widehat{\Gamma}$-configuration of spherical objects (see [ST] or [Brav]), and in particular that the pushforward $i_*$ is fully faithful on the base of the helix. We may therefore conclude by Theorem 3 that the natural Coxeter functor on $D^{\mathbb{G}_m}(U)$ is compatible under the pushforward with any Coxeter functor on $\mathcal{D}$ given as the composition of all spherical twists $T_{i_*E_j}$ in some order.

Thus from the point of view of categorifying the Coxeter transformation on the affine root lattice, the categories $D([U/\mathbb{C}^*])$ and $\mathcal{D}$ are interchangeable. The latter category, however, possesses a richer structure in that the spherical twists along the objects $i_*E_j$ generate an action of the affine braid group, which factors through the action of the affine Weyl group on the Grothendieck group, the latter being identified with the affine root lattice. This braid group action is known to be faithful by recent work of Brav-Thomas [BT].

This example can also be seen as an instance of a weighted projective line and its cotangent bundle: we are dealing with an action of a finite group $G$ on $\mathbb{P}^1$ and hence also on $T^*_{\mathbb{P}^1}$. Theorem 3 then applies to the inclusion of orbifolds, as in $i_* \colon D^G(\mathbb{P}^1) \to D^G_0(T^*_{\mathbb{P}^1})$.

## 4.2 Fuchsian singularities

Let $(X, 0)$ be a Fuchsian singularity of genus 0. This means that $X = \mathrm{Spec}(A)$ is affine and has a good $\mathbb{G}_m$-action. In particular, $A$ is non-negatively graded with $A_0 = \mathbf{k}$. The genus of the singularity is by definition the genus of the curve $\mathrm{Proj}(A)$.

By results of Lenzing and de la Peña ([LP]), the graded category of singularities $D^{\mathbb{G}_m}_{\mathrm{sing}}(X)$ of such a singularity has a strong, full exceptional collection. This allows one to identify $K_0(D^{\mathbb{C}^*}_{sg}(X))$, endowed with the negative of the symmetrised Euler form, with the Milnor lattice of $X$. Further, the Coxeter element of the Milnor lattice is induced by a canonical Coxeter functor (shift of the Serre functor) acting on $D^{\mathbb{G}_m}_{\mathrm{sing}}(X)$.

On the other hand, [EP] use partial smoothings to obtain a 2-Calabi-Yau category $\mathcal{D}(X)$ generated by spherical objects. Equipped with the negative Euler form, $K_0(\mathcal{D}(X))$ is again the Milnor lattice of $X$. In this setup, the Coxeter element is induced by a Coxeter functor given as the composition of spherical twists, one for each object in the generating collection. To form such a composition, we must choose an ordering of the collection, so in fact we have many Coxeter functors in this context.

Ideally, we would like to compare these two liftings of the Coxeter element to a Coxeter functor using our comparison theorem. To this end, we can use the derived Calabi-Yau completions of Section 3.1, which provide yet another categorification of the Milnor lattice. In the case of Fuchsian singularities, however, it is not clear at this point whether the derived endomorphisms algebras of the spherical collections from the partial smoothings of [EP] and from Keller's Calabi-Yau completion are quasi-isomorphic. The latter differential graded algebra is formal, but it remains an interesting question whether the endomorphism algebra of the geometric spherical collection is formal as well. If so, the comparison of Coxeter functors applies precisely. If not, the two spherical collections, from partial smoothings and from Calabi-Yau completions, would provide distinct 2-Calabi-Yau categorifications, which would be quite surprising.

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Still to mention:

The derived category has a full, strong exceptional collection (the squid). Related to extended canonical algebra via Orlov's theorem. Again, apply the main theorem. Compatibility of Coxeters. Suspect derived equivalent to Ebeling-Ploog, minus the pesky extending vertex. Would have to prove formality in Ebeling-Ploog example. (Formality in our case holds, according to Bernhard.)

In the future, interesting to study the group of autoequivalences generated by spherical twists for this Fuchsian example. Expect to induce monodromy on the appropriate summand of the Milnor lattice? Would Wolfgang give a vote of confidence on that?

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## 4.3 Singularities

An affine, normal surface singularity $X$ with good $\mathbb{G}_m$-action corresponds to a non-negatively graded coordinate algebra $A = \bigoplus_{k\geq 0} A_k$ with $A_0 = K$. Let $X^* := X_{reg}$ be the pointed quasi-cone. The quotient stack $[X^*/\mathbb{G}_m]$ has a representation $[Z/G]$, but the pair $(Z, G)$ is not unique.[2]

We want to apply the theorem to the category of graded singularities $D^{\mathrm{gr}}_{\mathrm{sg}}(A)$ on the one hand and a category obtained from $D(\mathbb{P}\mathrm{roj}(A)) = D([X^*/\mathbb{G}_m])$. For this, we use Orlov's theorem, which relates the two by means of the *Gorenstein parameter* $a \in \mathbb{Z}$ of $A$, i.e. $\omega_A = A(a)$ as graded module.[3]

**Lemma 10.** *(i)* $\mathrm{Coh}^G(Z) = \mathrm{Coh}([Z/G]) \cong \mathrm{Coh}(\mathbb{P}roj(A))$
*(ii)* $CM^{gr}(A) = VB^G(X^*)$
*(ii)* $D^G(T^*_Z) \cong D(S)$

(i) is okay.
(i) is [BD, Cor. 3.12] and should give a relation of $D^{\mathrm{gr}}_{\mathrm{sg}}(A)$ to categories built of $(Z, G)$.
(iii) BKR.

If $a = -1$, then $X$ is a Kleinian singularity and Orlov's theorem gives $D(\mathbb{P}\mathrm{roj}(A)) \cong \langle \mathcal{O}, D^{\mathrm{gr}}_{\mathrm{sg}}(A)\rangle$. In this case, [KST2] shows that there is an exceptional collection for $D^{\mathrm{gr}}_{\mathrm{sg}}(A)$ made up of vector bundles, relying on Auslander's classification of MCM modules on simple singularities. [4] Here, $\mathbb{P}\mathrm{roj}(A) = [\mathbb{P}^1/G]$ is a global quotient of $\mathbb{P}^1$ by a finite group $G$. Let $\tilde{X} \to X$ be the minimal resolution. Then, by derived McKay correspondence, $D(\tilde{X}) \cong D^G(\mathbb{A}^2)$. If $\mathcal{E} \subset \tilde{X}$ denotes the exceptional divisor, a configuration of smooth, rational $-2$-curves, then $D_{\mathcal{E}}(\tilde{X}) \cong D^G_0(\mathbb{A}^2)$ for the subcategories with support at $\mathcal{E}$ and the origin, respectively. It is well-known that in fact $\tilde{X} \cong T^*_{\mathbb{P}^1}$.

unfinished ...
Plan:
(1) Comparison theorem applies to $[Z/G]$ (done).
(2) Some singularities fit into framework of (1):
(2.1) Can express $D^{\mathrm{gr}}_{\mathrm{sg}}(A)$ (or related category) by $[Z/G]$;
(2.2) Can express $D(\tilde{X})$ (or related surface category) by $[T^*_Z/G]$.

Not sure if the stacks from cones ($[U/\mathbb{G}_m]$ etc.) deserve an extra section.

[2] reference needed, ideally in this framework; in the classical language, this should be Dolgachev
Going from $[Z/G]$ back to $A$: $C_Z := \mathrm{Spec}(\bigoplus_{n\geq 0} H^0(Z, \omega_Z^{an}))$ is the cone of $Z$ and $A = (C_Z)^G$.
[3] sign of $a$ might be off
[4] A simpler proof by Ueda is given in an appendix of [KST2], providing only an exceptional collection of sheaves. This proof is much closer to the spirit of this article, using weighted projective lines and Orlov's theorem.

### 4.4 Cones

Let $U$ be any smooth, affine variety. Then its geometrical canonical bundle $V := \mathbb{V}(\omega_U)$ comes with natural maps $\pi\colon V \to U$ and $i\colon U \to V$. The short exact sequence $0 \to \pi^*\Omega_U \to \Omega_V \to \pi^*\omega_U \to 0$ shows that the canonical bundle of $V$ is trivial.

If $U$ has a $\mathbb{G}_m$-action, then so does $V$. We claim that $i_*$ sends exceptional objects on $U$ to spherical objects on $V$. Also, $i_*$ is fully faithful on exceptional collections.

Remark: Brian points out that the (graded) class group of $A$ can be trivial (free of rank 1), in which case $[V/\mathbb{G}_m] = U$.

### 4.5 Kleinian singularities

For both types of categorical invariants of Kleinian singularities, there are various presentations of the categories, both geometric and algebraic:

$$\mathcal{C} := D^{\mathrm{gr}}_{\mathrm{sg}}(R) \cong HMF^{\mathrm{gr}}(f) \cong D^{k^*}_{\mathrm{sg}}(\mathbb{A}^2/\widetilde{G})$$
$$\mathcal{D} := D_{\mathcal{E}}(\tilde{X}) \cong D_0(\widetilde{G} - \mathrm{Hilb}(\mathbb{A}^2)) \cong D^{\widetilde{G}}_0(\mathbb{A}^2)$$

Here, $G \subset PSL_2(\mathbb{C})$ is a finite subgroup, $\widetilde{G} \subset SL_2(\mathbb{C})$ its double (binary) cover. (We may lose the odd $A_n$'s here.) Further, $R = \mathbb{C}[x, y, z]/f$ and $\mathbb{A}^2/\widetilde{G} = \mathrm{Spec}(R)$. Let $\tilde{X} \to \mathbb{A}^2/\widetilde{G}$ be a resolution (for example given by the equivariant Hilbert scheme) and $\mathcal{E} \subset \tilde{X}$ the exceptional locus.

Kajiura, Saito and Takahashi prove in [KST2] that $D^{\mathrm{gr}}_{\mathrm{sg}}(R)$ possesses a full, strong exceptional collection. They use Auslander's classification of maximal Cohen-Macaulay modules on simples singularities. Another proof by Ueda uses Orlov's theorem. The K-groups of $\mathcal{C}$ and $\mathcal{D}$ correspond to the root lattices for the affine Dynkin diagrams.

$\mathcal{C}$ and $\mathcal{D}$ are subcategories of

$$\widetilde{\mathcal{C}} := D^G(\mathbb{P}^1),$$
$$\widetilde{\mathcal{D}} := D^G_0(T^*) = \{A \in D^G(T^*) \text{ supported on the zero section}\},$$

respectively, where $T^* := T^*\mathbb{P}^1 = \mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(-2))$ is the cotangent bundle of the projective line ($T^*$ is also just the quadric in $\mathbb{P}^3$).

There are two natural maps, $i : \mathbb{P}^1 \hookrightarrow T^*$ (zero section) and $\pi : T^* \to \mathbb{P}^1$ (projection). We use $\iota := i_* : \tilde{\mathcal{C}} \to \tilde{\mathcal{D}}$. The theorem of the last section gives the desired compatibility of the Coxeter functors: $\mathsf{S} = \mathsf{M}_\omega[1]$ and $d = 2$, so the shifts cancel. Also, $\iota\mathsf{M}_\omega \cong \mathsf{M}_{\pi^*\omega}\iota$, so we have

$$C_{\mathrm{sph}}\iota = \mathsf{T}_{\mathcal{E}_0} \cdots \mathsf{T}_{\mathcal{E}_n}\iota \cong \iota C_{\mathrm{AR}} = \iota\mathsf{S}[1-2] = \iota\mathsf{M}_\omega = \mathsf{M}_{\pi^*\omega}\iota.$$

We have $\mathcal{C} = c^\perp$ and $\mathcal{D} = d^\perp$ for $c = \mathcal{O}_{\mathbb{P}^1} \in \tilde{\mathcal{C}}$, $d = \mathcal{O}_{s(\mathbb{P}^1)} \in \tilde{\mathcal{D}}$ (structure sheaf of $\mathbb{P}^1$ and the zero section, respectively; both with the trival action). Since $\iota(c) = d$, we get a functor $\iota\colon \mathcal{C} \to \mathcal{D}$ with the compatibilities of Coxeters as well (the introduction actually talks about this case). Alternatively, one may try to consider $[\mathbb{P}^1/G]$ as a stacky curve.

### 4.6 Fuchsian singularities of genus 0

Theorem 3 should also apply to the hypersurface singularities from weight systems with $R = -\varepsilon = 1$ (these are hypersurface Fuchsian singularities with genus 0). If the nice geometric picture for the Euclidean Dynkin diagrams in the ADE case continues to hold, the categories in question should be related to $D^\Gamma(\mathbb{H})$ and $D^\Gamma(T_\mathbb{H})$ (or the derived categories of the quotient stacks $[\mathbb{H}/\Gamma]$ and $[T_\mathbb{H}/\Gamma]$) where $\Gamma$ is the Fuchsian group defining the singularity. (The hypersurface assumption guarantees by another result of Kajiura, Saito and Takahashi that the category of graded matrix factorisations has a full and strong exceptional collection.)

However, that result one addresses the 22 Fuchsian hypersurface singularities with $g = 0$ and $r \leq 5$ ($r$ is the number of legs of the resolution graph over the vertex). Yet the full exceptional collections should exist in generality. Here is the plan: a Fuchsian singularity is by definition an affine cone, $X = \mathrm{Spec}(A)$ where $A$ is graded but not generated in degree 1. $\mathrm{Proj}(A)$ is a smooth curve of genus $g$. If we (in the $g = 0$ case) instead look at the quotient stack $[X^\times/k^*]$, we get a stacky $\mathbb{P}^1$ — this is exactly described in the old language by assigning finitely many points on a normal projective line and attack finite (cyclic) groups to them. It should be possible to use Orlov's theorem for proving the existence of a full exceptional collecting in $D^{gr}_{sg}(X)$ from this.

### 4.7 Quasi-cone surface singularities

One could hope that the theory applies to the full picture of Dolgachev: to any normal, affine, homogenous (i.e. good $\mathbb{C}^*$-action) surface with an isolated singularity, $(X, x)$, he associates a triple $(C, L, G)$ where $C$ is a Riemann surface, $G$ a group acting on $C$ (not necessarily finite) such that $C/G$ is compact and $L$ a $G$-linearised bundle on $C$. This data gives back $(X, x)$: the graded ring of $G$-invariant sections of powers of $L$ is the coordinate ring of $X$.

One could hope that $D^G(C)$ and $D^G(\mathbb{V}(L))$ (which generalise $D^G(\mathbb{P}^1)$ and $D^G(\omega_{\mathbb{P}^1})$ for Kleinian singularities) allow a description as above. However, it is not clear whether the curve category has a full exceptional collection (that they exist in the Kleinian and Fuchsian hypersurface $g = 0$ cases are theorems cited above) and whether the surface category is generated by spherical objects (for the Fuchsian singularities, Pinkham's generic deformations are used).

There should be a more conceptual picture than the one given above.

Let $S$ be a quasi-homogenous surface, i.e. an affine, normal surface with a good $\mathbb{C}^*$-action and a single singularity $0 \in S$. Let $S^* := S \setminus \{0\}$ be the punctured surface. Then there is a Orlov-type relation between $D^{\mathbb{C}^*}_{sing}(S)$ and $D([S^*/\mathbb{C}^*]) = D^{\mathbb{C}^*}(S^*)$. (In the Fuchsian case, $D^{\mathbb{C}^*}_{sing}(S) = HMF^{gr}(f)$ and $D([S^*/\mathbb{C}^*]) = D^{\tilde{G}}(\mathbb{P}^1)$.) There should be a line bundle $L$ on $S^*$ giving a derived resolution, i.e. $D([L/\mathbb{C}^*])$ is 2-CY etc. There is an obvious functor $i_*\colon D([S^*/\mathbb{C}^*]) \to D([L/\mathbb{C}^*])$ pushing forward on the zero section (this functor will map exceptionals to sphericals).

In good cases, $S \subset \mathbb{A}^3$ will be a hypersurface (with $\mathbb{C}^*$ acting on both). Then, the line bundle $L$ should actually come from a $\mathbb{C}^*$-linearised bundle on $\mathbb{A}^3 \setminus 0$.