Linear Algebraic Groups

These are the exercises (with solutions) and the exam for the course Algebra II, winter term 2014/2015 at Bonn.
Students were not assumed to have heard algebraic geometry before, just algebra. In particular, basic affine and projective geometry were introduced from scratch. The course covered the standard material up to the structure results about maximal tori and Borel subgroups. Root systems and data were covered, as well as how to associate a root system to a semisimple group.

Contents

1 Some pointers to the literature

A. Borel:	*Linear Algebraic Groups*	Springer (1969, 1997)
J. Humphreys:	*Linear Algebraic Groups*	Springer (1975, 1981)
T. Springer:	*Linear Algebraic Groups*	Birkhäuser (1981, 1998)
P. Tauvel, R. Yu:	*Lie Algebras and Algebraic Groups*	Springer (2005)
A. Onishchik, E. Vinberg:	*Lie Groups and Algebraic Groups*	Springer (1990)
R. Goodman, N.Wallach:	*Symmetry, Representations and Invariants*	Springer (2009)

The first three books (Borel, Humphreys, Springer) are the classical textbooks on the subject. They share the following features: the ground field is algebraically closed and of arbitrary characteristic; rationality questions (i.e. non-algebraically closed fields of definition) are treated at the end; they cover the structure theory of linear algebraic groups including the classification of reductive/semisimple groups; they avoid schemes but use ringed spaces to define quotients. Of these books, I like Humphrey's the best: it does a good job of explaining why things happen.

The other three books all work over the complex numbers, and have different aims. Tauvel/Yu is really concerned with Lie algebras. It covers basically all of the foundational results otherwise cited (including commutative algebra, sheaves, group theory, projective geometry, root systems) — at the price that Lie algebras are introduced as late as in §19 and linear algebraic groups in §21. Moreover, the book is extremely light on examples. It treats a lot of the finer structure theory, including reductive, Borel, parabolic, Cartan groups. In my opinion, this volume is more a reference than a textbook to learn the subject from. The sections relevant to the course are §10,18,21–23,25–28.

Goodman/Wallach starts with very explicit descriptions of the classical groups $(\mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_n)$ and then develops the theories of Lie groups and of algebraic groups in parallel. Thus it is very example-driven: for example, maximal tori and roots are all developed first for the classical groups. Later on, Chapter 11 is devoted to a rapid development of linear algebraic groups, up to Borel subgroups and maximal tori (but without the classification). I took the construction of homogenous spaces G/H and the proof that morphisms of linear algebraic groups have closed image from Appendix A.

Onishchik/Vinberg is written in a unique style: the book grew out of a 1967 Moscow seminar; almost all of the theory is presented in a series of problems (generally with concise solutions). The book develops both (real and complex) Lie groups and complex linear algebraic groups, in a very quick and efficient fashion. They classify semisimple Lie algebras.

To a large extent, I have been following the lecture notes of Tamás Szamuely from a 2006 course at Budapest: http://www.renyi.hu/~szamuely/lag.pdf. It has the advantage of starting out with low technical demands (no schemes or sheaves are used; only quasi-projective varieties). The proof of conjugacy of maximal tori using group cohomology is from these notes.

2 List of results proved in the course

Affine algebraic geometry

Proposition 1: Connected components of affine algebraic groups coincide with irreducible components. The identity component is a normal subgroup.

Proposition 2: $\mathrm{Mor}(X, Y) = \mathrm{Hom}(A(Y), A(X))$ for affine varieties X, Y. For any finitely generated, reduced K-algebra there is a unique affine variety X with $A(X) \cong A$.

Proposition 3: $A(X \times Y) \cong A(X) \otimes_K A(Y)$ for affine varieties X, Y.

Embedding theorem

Theorem: Any affine linear group G is a closed subgroup of some GL_n.

Chevalley Lemma: If $H \subset G$ is a closed subgroup of a linear algebraic group, then there exists a finite-dimensional representation $G \to \mathrm{GL}(V)$ and a 1-dimensional linear subspace $L \subseteq V$ such that $H = \mathrm{Stab}(L)$.

Proposition 4: For a normal subgroup $H \subseteq G$ of a linear algebraic group G, there is a finite-dimensional representation $G \to \mathrm{GL}(V)$ with kernel H.

Jordan decomposition

Theorem: Jordan decomposition for a linear algebraic group G.

Theorem (Kolchin): If $G \subset \mathrm{GL}_n$ is a unipotent group, then there exists a non-zero vector fixed by G.

Burnside Lemma: If V is a finite-dimensional K-vector space and $A \subseteq \mathrm{End}(V)$ a subalgebra without A-stable subspaces except 0 and V, then $A = \mathrm{End}(V)$.

Theorem (structure of commutative linear algebraic groups G): G_s, G_u are closed subgroups of G, and $G_s \times G_u \xrightarrow{\sim} G$ is an isomorphism of affine algebraic groups.

Actions and representations

Proposition 5: G diagonalisable $\iff$ G is isomorphic to a closed subgroup of D_n $\iff$ $G = G_s$ is commutative

Lemma: $X^*(\mathrm{D}_n) = \mathbb{Z}^n$ and $X^*(\mathrm{D}_n)$ is a basis of $A(\mathrm{D}_n)$.

Theorem (structure of diagonalisable groups G): G diagonalisable $\iff$ $X^*(G)$ is a finitely generated abelian group (without p-torsion if $\mathrm{char}(K) = p$) and $X^*(G)$ is a basis of $A(G)$ $\iff$ $G \cong \mu_{d_1} \times \cdots \times \mu_{d_r} \times \mathrm{D}_m$ with $p \nmid d_i$.

Proposition 6: G diagonalisable $\iff$ all finite-dimensional G-representations decompose into characters

Theorem (Weyl groups are finite): If G is a linear algebraic group and $T \subseteq G$ a torus, then the quotient $W(G, T) := N_G(T)/C_G(T)$ is finite.

Connected solvable groups

Theorem (Lie-Kolchin): If $G \subset \mathrm{GL}_n$ is a connected solvable subgroup, then there exists a complete flag of G-invariant subspaces in V, i.e. G can be conjugated into T_n.

Theorem (structure of connected nilpotent groups): If G is a connected nilpotent linear algebraic group, then G_u, G_s are closed, normal subgroups of G and $G_s \times G_u \xrightarrow{\sim} G$ is an isomorphism of affine algebraic groups, and G_s is a torus.

Tangent spaced and Lie algebras

Lemma: If X is an affine homogeneous space for a linear algebraic group G, then X is smooth.

Theorem: If G is a connected, 1-dimensional linear algebraic group, then either $G = \mathbb{G}_a$ or $G = \mathbb{G}_m$. [Only proved for $\mathrm{char}(K) = 0$.]

Lemma: If $G \subseteq \mathrm{GL}_n$ is a linear algebraic group, then the action of G on its Lie algebra $\mathfrak{g} = \{A \in \mathfrak{gl}_n \mid X_A f \in I(G)\, \forall f \in I(G)\}$ is given by $g \cdot A = gAg^{-1}$.

Proposition 7: For a linear algebraic group G, tangent space at the unit and Lie algebra coincide: $L(G) \cong T_e G$.

Quotients by normal subgroups

Theorem: If $H \subseteq G$ is a closed normal subgroup of a linear algebraic group, then G/H is a linear algebraic group.

Proposition 8: If $A \subset B$ are K-algebras, with B finitely generated over A, then the extension property holds: $\underset{\substack{b \in B \\ b \neq 0}}{\forall} \; \underset{\substack{a \in A \\ a \neq 0}}{\exists} \; \underset{\substack{\varphi\colon A \to K \\ \varphi(a) \neq 0}}{\forall} \; \underset{\substack{\Phi\colon B \to K \\ \Phi(b) \neq 0}}{\exists} \; : \; \Phi|_A = \varphi$.

Proposition 9: If $f\colon X \to Y$ is a morphism of affine varieties, then $f(X)$ contains an open subset of its closure $\overline{f(X)}$.

Corollary: If $f\colon G \to H$ is a morphism of affine algebraic groups, then the image $f(G)$ is a closed subgroup of H.

Quasi-projective varieties

Proposition 10: The Plücker map $p_d\colon \mathrm{Gr}(d, V) \to \mathbb{P}(\Lambda^d V)$ is a closed embedding.

Lemma: X proper, $Z \subseteq X$ closed $\Rightarrow$ Z proper.
X_1, X_2 proper $\Rightarrow$ $X_1 \times X_2$ proper.
X proper, $\varphi\colon X \to Y$ morphism $\Rightarrow$ $\varphi(X)$ closed, proper subvariety of Y.
X proper and affine $\Rightarrow$ X is a finite set.

Theorem: Projective varieties are proper.

Nakamaya Lemma: If A is a commutative ring, $M \subset A$ a maximal ideal and N a finitely generated A-module with $N = MN$, then there is an $f \in A \setminus M$ with $fN = 0$.

Homogeneous spaces and quotients

Theorem: If $H \subset G$ is a closed subgroup of a linear algebraic group, then G/H is a quasi-projective with a morphism $\pi\colon G \to G/H$ and if $\varphi\colon G \to X$ is any G-equivariant morphism of homogeneous G-spaces with $\varphi(H) = \varphi(1)$, then there exists a unique morphism $\psi\colon G/H \to X$ with $\varphi = \psi\pi$.
Moreover, if G acts on a quasi-projective variety Y such that $H \subseteq G_y$ for some $y \in Y$, then the natural map $G/H \to G \cdot y$ is a morphism.

Proposition 11: Let B be a finitely generated K-algebra without zero divisors and $A \subset B$ is a subalgebra. If there is $b \in B, b \neq 0$ such that all $\Phi\colon B \to K$ with $\Phi(b) \neq 0$ are uniquely determined by $\Phi|_A$, then $B \subseteq \mathrm{Quot}(A)$.

Proposition 12: Let $f\colon M \to N$ and $h\colon M \to P$ be regular, dominant morphisms of affine varieties such that there exists a non-empty open set $U \subseteq M$ with $f(m_1) = f(m_2) \Rightarrow h(m_1) = h(m_2)\ \forall m_1, m_2 \in U$, then there is a rational map $g\colon N \dashrightarrow P$ with $h = gf$.

Borel and parabolic subgroups

Orbit lemma: If a linear algebraic group G acts on a quasi-projective variety X, then (i) every orbit is open in its closure, (i) orbits of minimal dimension are closed and, in particular, (iii) closed orbits exist.

Borel Fixed Point Theorem: Any action of a connected solvable group G on a projective variety X has a fixed point.

Theorem: Any two Borel subgroups of a linear algebraic group are conjugate.

Proposition 13: Borel subgroups are parabolic.

Proposition 14: If $\varphi\colon G \twoheadrightarrow G'$ is a surjective morphism of affine algebraic groups and $H \subseteq G$ is a parabolic (or Borel) subgroup, then so is $\varphi(H) \subseteq G'$.

Maximal tori

Proposition 15: For a connected solvable group G, there exists a torus $T \subset G$ such that $T \hookrightarrow G \twoheadrightarrow G/G_u$ is an isomorphism.

Corollary: A connected solvable group G is a semi-direct product $G = G_u \rtimes T$, where $T \cong G/G_u$ is a maximal torus.

Theorem: Any maximal tori in a connected linear algebraic group are conjugate.

Structure results

Theorem: Let G be a connected linear algebraic group, $B \subseteq G$ a Borel subgroup and $T \subseteq G$ a maximal torus. Then G is covered by all Borel subgroups, and G_s is covered by all maximal tori: $G = \bigcup_{g\in G} gBg^{-1}$ and $G_s = \bigcup_{g \in G} gTg^{-1}$.

Proposition 16: Let G be a connected linear algebraic group and $T \subseteq G$ a torus. Then $C_G(T)$ is connected.

Theorem: Given $B \subseteq G$, a Borel subgroup of a connected linear algebraic group, then $N_G(B) = B$.

Proposition 17: There is a bijection between the generalised flag variety G/B and the set $\mathfrak{B}$ of all Borel subgroups of G.

3 Exercises (five problems each week)

1. Let N be the set of all matrices in $\mathrm{GL}_n(K)$ with exactly one non-zero entry in every row and every column. Show that N is a closed subgroup of $\mathrm{GL}_n(K)$, that its identity component $N^\circ = \mathrm{D}_n$ is the subgroup of diagonal matrices, that N has $n!$ connected components and that N is the normaliser of D_n.

2. Give examples of non-closed subgroups of $\mathrm{GL}_2(\mathbb{C})$ and compute their closures.

3. Describe the Hopf algebra structures on the coordinate rings of $\mathbb{G}_a$ and GL_n.

4. Prove that a T_0 topological group is already T_2. Show that an infinite linear algebraic group is always T_0 but never T_2. Explain the discrepancy!

5. Show that the product of irreducible affine K-varieties is again irreducible. This fails for non-algebraically closed fields K: exhibit zero divisors in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.

---

6. Prove that the group U_n of unipotent upper triangular matrices is nilpotent.

7. Prove that a group G is solvable if and only if it has a composition series with abelian factors, i.e. there is a chain of subgroups $G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_m = \{1\}$ such that each G_{i+1} is normal in G_i and all G_i/G_{i+1} are abelian.

8. What is the Jordan decomposition for a finite group? For $\mathbb{G}_a$?

9. Find a closed subgroup G of GL_2 such that G_s is not a closed subset.

10. Compute the centre C of $\mathrm{SL}_2(K)$, assuming $\mathrm{char}(K) \neq 2$. Show that the quotient group $\mathrm{PSL}_2(K) := \mathrm{SL}_2(K)/C$ is an affine algebraic group.

(Hint: embed $\mathrm{SL}_2 \subset \mathbb{A}^4$ as a Zariski-closed subset, then check that the action of C on SL_2 extends to an action of C on $\mathbb{A}^4$. Now map $\mathbb{A}^4/C$ to some affine space as a Zariski-closed subset.)

---

11. Prove that the group T_n of upper triangular matrices is solvable.

12. Show that $\mathbb{G}_a$ and $\mathbb{G}_m$ are not isomorphic as affine algebraic groups.

13. Let $\varphi\colon X \to Y$ be a morphism of affine varieties. Show that φ is dominant (i.e. the image of X is dense in Y) if and only if $\varphi^*\colon A(Y) \to A(X)$ is injective.

14. Let $H \subset \mathrm{GL}_n$ be an arbitrary subgroup. Show that the Zariski-closure $\overline{H}$ is a linear algebraic group. Moreover, prove that closure preserves the following properties: H commutative; H normal; H solvable; H unipotent.

15. Show that none of the following implications among properties of linear algebraic groups can be reversed:

$$\begin{array}{ccccccccc} & & & & & & \text{unipotent} & & \\ & & & & & & \Downarrow & & \\ \text{torus} & \Longrightarrow & \text{diagonalisable} & \Longrightarrow & \text{abelian} & \Longrightarrow & \text{nilpotent} & \Longrightarrow & \text{solvable} \end{array}$$

---

16. The group SL_3 naturally acts on K^3. Writing x_1, x_2, x_3 for the standard basis of K^3, this induces an action of SL_3 on polynomials $p(x_1, x_2, x_3) \in K[x_1, x_2, x_3]$ by $g \cdot p = pg$. Compute the weight spaces for the torus $T := \mathrm{D}_3 \cap \mathrm{SL}_3$ on the vector space $K[x_1, x_2, x_3]_2$ of polynomials of degree 2, and draw the weights.

17. Show that the normaliser of D_2 in GL_2 is solvable, but not conjugated to a subgroup of T_2.

18. Compute the Weyl group of GL_3 with respect to the torus D_3.

19. For any linear algebraic group G, let $H := \bigcap_{\chi \in X^*(G)} \ker(\chi)$. Show that H is a closed, normal subgroup of G and that G/H is diagonalisable. Also show $X^*(G) \cong X^*(G/H)$.

20. (Continuation of 19.) Compute H and G/H for $G = \mathrm{GL}_n$.

---

21. Assume $\mathrm{char}(K) \neq 2$ and let $\Gamma \in M(n, K)$ with associated bilinear form $K^n \times K^n, (x, y) \mapsto x^t \Gamma y$, and $\mathrm{O}(\Gamma)$ its isometry group. Show that the tangent space of $\mathrm{O}(\Gamma)$ at $I = I_n$ is $T_I \mathrm{O}(\Gamma) = \{A \in M(n, K) \mid A^t \Gamma + \Gamma A = 0\}$.

(Hint: use the Cayley transform $c(A) = (I + A)(I - A)^{-1}$ for $A \in M(n, K)$ with $\det(I - A) \neq 0$ and show that $c(A) \in \mathrm{O}(\Gamma)$ if and only if $A^t \Gamma + \Gamma A = 0$.)

22. Compute the dimensions of SO_n and Sp_n.

23. Assume $\mathrm{char}(K) = 0$. Let G be a linear algebraic group, all of whose elements have finite order. Show that G is finite.

24. For affine varieties X and Y, show that $\dim(X \times Y) = \dim(X) + \dim(Y)$.

25. If X is an irreducible, smooth affine variety and $Z \subsetneq X$ a closed subvariety, prove $\dim(Z) < \dim(X)$.

---

26. For a linear algebraic group G, use its comultiplication to define an associative, unital K-algebra structure on $A(G)^* = \mathrm{Hom}_K(A(G), K)$ such that the induced Lie algebra structure coincides with the one from left invariant vector fields.

27. Exhibit $\mathbb{G}_m$ as a closed subgroup of SO_2. Then find a two-dimensional torus $T \subset \mathrm{SO}_4$ and compute the weights for the action induced on the adjoint representation, $T \hookrightarrow \mathrm{SO}_4 \to \mathrm{GL}(\mathfrak{so}_4)$.

28. Let A be a finite-dimensional, associative and unital K-algebra. Show that the group of units is a linear algebraic group. What is its Lie algebra?

29. Show that the differential of the adjoint representation of a linear algebraic group G is given by $\mathrm{ad} := d(\mathrm{Ad})_e \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}),\ \mathrm{ad}(A)(B) = [A, B]$.

30. Show that a morphism $\varphi \colon G \to H$ of linear algebraic groups induces a homomorphism of Lie algebras $d\varphi_e \colon \mathfrak{g} \to \mathfrak{h}$.

31. Find a symmetric, non-degenerate bilinear form Γ on K^n such that $T := \mathrm{SO}(\Gamma) \cap \mathrm{D}_n$ is a torus of dimension m if $n = 2m$ or $n = 2m + 1$. Prove that T is a maximal torus in $\mathrm{SO}(\Gamma)$: if $H \subseteq \mathrm{SO}(\Gamma)$ is abelian with $T \subseteq H$, then $T = H$.

32. Compute the orbits of the natural actions of $\mathrm{GL}_n, \mathrm{T}_n, \mathrm{U}_n, \mathrm{D}_n$ on $\mathbb{A}^n$ and draw them for $n = 2$. Describe orbit closures as unions of orbits.

33. Show that the adjoint representation of SL_2 has disconnected isotropy groups.

34. Let G be a unipotent linear algebraic group and X an affine G-variety. Show that all orbits of G in X are closed.

35. For a homogeneous ideal $I \subsetneq K[x_0, \ldots, x_n]$, geometrically compare the varieties $V(I) \subseteq \mathbb{A}^{n+1}$ and $V(I) \subseteq \mathbb{P}^n$. Prove the homogeneous Nullstellensatz.

---

36. Compute the orbits of the actions of GL_3 on $\mathbb{P}^2$, of GL_4 on $\mathrm{Gr}(2,4)$, and of GL_2 on $\mathbb{P}^1 \times \mathbb{P}^1$ (diagonal action). Also compute isotropy groups for all orbits.

37. Show that any non-degenerate conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. Deduce that homogeneous coordinate rings are not isomorphism invariants of projective varieties.

38. Show that $\mathbb{A}^1$ and $\mathbb{P}^1$ are homeomorphic, but $\mathbb{A}^2$ and $\mathbb{P}^2$ are not.

39. For an affine, irreducible variety X and a point $p \in X$, show that the local ring of X at p is given by the localisation of $A(X)$ at the maximal ideal M_p.

40. Show that $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$.

---

41. Show that varieties are compact in the Zariski topology: any open cover has a finite subcover.

42. Show that a locally compact Hausdorff space X is compact if and only if the projection $X \times Y \to Y$ is a closed map for all topological spaces Y.

43. Let $X = V(y^2 - x^3) \subset \mathbb{A}^2$ be the cuspidal cubic. Show that the map $\mathbb{A}^1 \to X, t \mapsto (t^2, t^3)$ is regular, birational and a homeomorphism, but not an isomorphism of varieties. Extend this to an example of a morphism of projective varieties with the same properties.

44. Let A be a commutative ring with unit and N a finitely generated A-module. Show that a surjective module endomorphism $N \to N$ is an isomorphism.

45. Find subgroups of SL_2 such that the quotient is respectively projective, affine or strictly quasi-affine.

---

46. Compute the dimensions of Grassmannians $\mathrm{Gr}(d,n)$ and flag manifolds $\mathrm{Fl}(n)$.

47. Show that two irreducible varieties X and Y are birational, i.e. their function fields are isomorphic: $K(X) \cong K(Y)$, if and only if there exist affine open subsets $U_X \subseteq X$ and $U_Y \subseteq Y$ which are isomorphic: $U_X \cong U_Y$.

48. Assume $\mathrm{char}(K) = p > 0$. Show that the map $\mathbb{G}_m \to \mathbb{G}_m, x \mapsto x^p$ is a bijective morphism of affine algebraic groups, but is not an isomorphism.

49. Let G be a linear algebraic group, acting on a quasi-projective variety X. Show that orbits of minimal dimension are closed; in particular, closed orbits exist.

(Hint: you can use the statement of exercise 25.)

50. Let G be a connected projective algebraic group. Show that G is commutative.

---

51. Find Borel subgroups in SO_4, Sp_4, T_4 and U_4.

52. Find all parabolic subgroups P with $\mathrm{T}_4 \subseteq P \subseteq \mathrm{GL}_4$.

53. Let the connected linear algebraic group G act on a quasi-projective variety X with finitely many orbits. Show that every irreducible closed G-invariant subset in X is the closure of a G-orbit. Find a counterexample for an action with infinitely many orbits.

54. Find a connected linear algebraic group G and a maximal solvable subgroup $U \subset G$ such that U is disconnected.

55. Classify all root systems in the Euclidean plane $E := \mathbb{R}^2$.

(Note: A *root system* in an Euclidean space $(E, (-,-))$ is a subset $\Phi \subset E$ such that
(RS1) Φ is finite, spans E and $0 \notin \Phi$;
(RS2) for any $\alpha \in \Phi$, $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$;
(RS3) for any $\alpha \in \Phi$, the reflection $s_\alpha \colon E \to E, x \mapsto x - \frac{2(x,\alpha)}{(\alpha,\alpha)}x$ preserves Φ;
(RS4) for any $\alpha, \beta \in \Phi$: $\langle \beta, \alpha \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.)

---

56. Compute the radicals $R(\mathrm{GL}_n), R(\mathrm{SL}_n), R(\mathrm{U}_n)$.

57. Let $\mathrm{char}(K) = 0$ and U a commutative, unipotent group. Show that $U \cong \mathbb{G}_a^r$ for some $r \in \mathbb{N}$.

58. Let G be a finite group and A a trivial G-module, i.e. an abelian group which has the trivial left G-action. Show that $H^1(G, A) = \mathrm{Hom}(G, A)$.

59. Let $\varphi \colon G \to H$ be a surjective morphism of linear algebraic groups, and $T \subseteq G$ a maximal torus (or a maximal connected normal unipotent subgroup, respectively). Show that $\varphi(T) \subseteq H$ has the same property.

60. Let (E, Φ) be a root system. A *base* of (E, Φ) is a subset $S \subseteq \Phi$ such that S is a basis for E and every root $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha \in S} m_\alpha \alpha$ with either all $m_\alpha \geq 0$ or all $m_\alpha \leq 0$. The elements of S are also called *simple roots*, and their non-negative linear combinations *positive roots* and denoted Φ^+.
Draw bases and positive roots for the root systems in $E = \mathbb{R}^2$, of exercise 55.

---

61. Let G be a connected algebraic group, $B \subseteq G$ a Borel subgroup and V a rational G-module. Show that the invariant subspaces coincide: $V^G = V^B$.

62. Let G be a linear algebraic group. Show that $R(G)_u = R_u(G)$, i.e. the unipotent part of the radical is the unipotent radical.

63. For a linear algebraic group G, show that $G/R(G)$ is semisimple and that $G/R_u(G)$ is reductive.

64. Let G be a group with subgroups $H, N \subseteq G$. Show that the following notions are equivalent — then $G = N \rtimes H$ is called a *semi-direct* product of H by N:
(1) There is a short exact sequence $1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} H \to 1$ admitting a section $\sigma\colon H \to G$, i.e. $\pi\sigma = \mathrm{id}_H$.
(2) N is normal in G, and $NH = G$ and $N \cap H = 1$.
(3) There is a homomorphism $\alpha\colon H \to \mathrm{Aut}(N)$ and G is isomorphic to the group $N \rtimes_\alpha H$ defined by $(n,h)\cdot(n',h') := (n\alpha(h)(n'), hh')$ on the set $N \times H$.
Also show that the existance of a retraction $\varrho\colon G \to N$, i.e. $\varrho\iota = \mathrm{id}_N$, is equivalent to a splitting $G \cong N \times H$ of G as a direct product.

65. Show that the following subsets define root systems of rank n:
$\{e_j - e_i \mid i,j \in \{1,\ldots,n+1\}, i \neq j\} \subset \mathbb{Q}^{n+1}$ (called type A_n)
$\{\pm e_j \pm e_i \mid i,j \in \{1,\ldots,n\}, i < j\} \subset \mathbb{Q}^n$ (called type D_n)

---

66. Show that centraliser of $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{C})$ is disconnected.

67. For $B \subseteq G$ a Borel subgroup of a connected linear algebraic group G, show that $Z(B) = Z(G)$.

68. Prove directly for $G = \mathrm{SL}_n$ and for $G = \mathrm{SO}_n$ (with the bilinear form from Problem 31) that G is covered by Borel subgroups and that maximal tori coincide with their centralisers.

69. Let G be a connected linear algebraic group with a maximal torus and $\mathfrak{B}$ be the set of Borel subgroups of G with its natural T-action. Show that there is a bijection between the fixed point set $\mathfrak{B}^T$ and the Weyl group $W(G)$.

70. Show that the following subsets define root systems of rank n:
$\{\pm e_i \mid i \in \{1,\ldots,n\}\} \cup \{\pm e_j \pm e_i \mid i,j \in \{1,\ldots,n\}, i < j\} \subset \mathbb{Q}^n$ (type B_n)
$\{\pm 2e_i \mid i \in \{1,\ldots,n\}\} \cup \{\pm e_j \pm e_i \mid i,j \in \{1,\ldots,n\}, i < j\} \subset \mathbb{Q}^n$ (type C_n)

4 Algebra II — solutions to exercise sheet 1

1. Let N be the set of all matrices in $\mathrm{GL}_n(K)$ with exactly one non-zero entry in every row and every column. Show that N is a closed subgroup of $\mathrm{GL}_n(K)$, that its identity component $N^\circ = \mathrm{D}_n$ is the subgroup of diagonal matrices, that N has $n!$ connected components and that N is the normaliser of D_n.

Solution: It is clear that $\mathrm{D}_n \subset N$ and that $\sigma \mathrm{D}_n \subset N$ for any permutation $\sigma \in S_n$ (letting permutations act on rows, say). It is also clear that $N = \bigcup_\sigma \sigma \mathrm{D}_n$ and that this union is disjoint. We observe that D_n is connected: it is an open subset of $\mathbb{A}^n$, hence irreducible (because $\mathbb{A}^n$ is irreducible — its affine coordinate ring is $K[x_1, \ldots, x_n]$ and an integral domain), hence connected. This shows all statements except that N is the normaliser of D_n. This is an easy result purely about groups.

2. Give examples of non-closed subgroups of $\mathrm{GL}_2(\mathbb{C})$ and compute their closures.

Solution: All of the following examples are closed in the norm topology of $GL_2(\mathbb{C})$ but not Zariski-closed.
$\mathrm{GL}_2(\mathbb{R})$. It is Zariski-dense in $\mathrm{GL}_2(\mathbb{C})$: denoting $G = \mathrm{GL}_2(\mathbb{R})$ and $X = \mathrm{GL}_2(\mathbb{C})$, then $G = \{x \in X \mid x = c(x)\}$ where $c\colon X \to X$ is complex conjugation (a bijection, but not a morphism of complex varieties). For any $f \in A(X)$ with $f(G) = 0$, we also have $c(f) \in A(X)$, and hence the two polynomials with real coefficients $f + c(f)$ and $(f - c(f))/i$. However, G is the real affine variety in affine (n^2+1)-sace with just one equation. Hence there can be no further polynomials vanishing on all of G, hence $I(G) = 0$ and $\overline{G} = V(I(G)) = V(0) = X$.

$\mathbb{Z}$, embedded via $n \mapsto \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. This subset is not closed, because it is discrete infinite. Its closure is $\mathrm{U}_2 \cong \mathbb{G}_a$.

S^1, the 1-dimensional real sphere, embedded via $S^1 \hookrightarrow \mathrm{GL}_2(\mathbb{C}), x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right)$. Its closure is $\mathbb{G}_m$ via $S^1 \subset \mathbb{C}^* = \mathbb{G}_m$ and the same embedding.

3. Describe the Hopf algebra structures on the coordinate rings of $\mathbb{G}_a$ and GL_n.

Solution: For $G = \mathbb{G}_a$ is $A(G) = K[x]$ with comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$ from $m(g,h) = g + h$. Moreover, coinverse $\iota(x) = -x$ and counit $\varepsilon(x) = 0$.
For $G = \mathrm{GL}_n$ is $A(G) = K[x_{11}, \ldots, x_{nn}, y]/(1 - y \cdot \det(x))$. Comultiplication $\Delta(x_{ij}) = \sum_l x_{il} \otimes x_{lj}$; coinverse $\iota(x_{ij}) = (-1)^{i+j} y \det(\hat{x}_{ij}$, where $\hat{x}_{ij}$ is the matrix obtained from $x = (x_{ij})$ by cutting row i and column j — this is precisely the i,j-entry of the inverse matrix to x, computed using the adjoint matrix via $X \cdot \mathrm{adj}(X) = \det(X) I_n$, note that $y = 1/\det(x) \in A(G)$; counit $\varepsilon(x_{ij}) = \delta_{ij}$.

4. Prove that a T_0 topological group is already T_2. Show that an infinite linear algebraic group is always T_0 but never T_2. Explain the discrepancy!

Solution: Let G be a T_0 topological group. We first show that the singleton $\{e\}$ is a closed subset, where $e \in G$ is the neutral element. Given any element $x \in G$, there is either an open neighbourhood U_x of x with $e \notin U$ or an open neighbourhood V of e with $x \notin e$. (This is the definition of T_0, applied to the two points x, e.) In the latter case, we may assume that $V = V^{-1}$ (if not, replace V with $V \cap V^{-1}$, this is still an open neighbourhood of e) and use the homeomorphism $f\colon G \to G, y \mapsto xy$, which maps $e \mapsto x$. Let $U_x := f(V)$, this is an open neighbourhood of $x = f(e)$ with $e \notin U_x$ (otherwise $e \in f(V) = xV$, hence $x^{-1} \in V$, contradicting $x \notin V$ and $V = V^{-1}$). We now take the union $U := \bigcup_{x \neq e} U_x$ over all these neighbourhoods. By construction, this is an open set with $U = G \setminus \{e\}$, so $\{e\}$ is indeed closed.
The map $G \times G \to G, (g,h) \mapsto gh^{-1}$ is continuous, since G is a topological group. Its preimage of the closed subset $\{e\}$ is the diagonal $\Delta_G = \{(g,g) \mid g \in G\}$, which is therefore closed. It is a general (and easy) fact that a topological space X is T_2 if and only if the diagonal Δ_X is a closed subset of $X \times X$.
The statements about linear algebraic groups follow from general properties of the Zariski topology: points of affine varieties are closed because they correspond on maximal ideals; hence varieties are T_0 (and even T_1 but note that schemes have more points and are only T_0 in general). Since any two non-empty open subsets of an irreducible variety meet, varieties are never T_2.
Explanation: this exercise shows that while linear algebraic groups are groups with a topology, they are not topological groups. The reason is that the topology on $G \times G$ for a variety is *not* the product topology.

5. Show that the product of irreducible affine K-varieties is again irreducible. This fails for non-algebraically closed fields K: exhibit zero divisors in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.

Solution: Let $A := A(X)$ and $B := A(Y)$ be the affine coordinate rings. We know that $A(X \times Y) \cong A \otimes_K B$. Also, X irreducible corresponds to $I(X)$ being a prime ideal or, equivalently, $A(X)$ being an integral domain. Therefore, we have to show that if A and B are integral domains, then so is $A \otimes_K B$.
Let $f = \sum_i a_i \otimes b_i \in A \otimes B$ be a zero divisor. We can assume that the b_i are linearly independent. If $x \in X$, then $f(x, -) = \sum_i a_i(x) b_j \in B$ is a zero divisor in B, hence zero. With the b_j linearly independent, we find $a_i(x) = 0$ for all i, thus $a_i = 0$ (Nullstellensatz!) and so $f = 0$.
In $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, one computes $(1 \otimes i + i \otimes 1)(1 \otimes 1 + i \otimes i) = 0$.

Algebra II — solutions to exercise sheet 2

6. Prove that the group U_n of unipotent upper triangular matrices is nilpotent.

Solution: This a straightforward computation with commutators of matrices. For example, $C^2(\mathrm{U}_n) = [\mathrm{U}_n, \mathrm{U}_n]$ consists of matrices with zeros on the secondary diagonal. In the central series, $C^n(\mathrm{U}_n)$ is trivial.

7. Prove that a group G is solvable if and only if it has a composition series with abelian factors, i.e. there is a chain of subgroups $G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_m = \{1\}$ such that each G_{i+1} is normal in G_i and all G_i/G_{i+1} are abelian.

Solution: We use the fact that for a subgroup $H \subseteq G$ holds: $H \supseteq D(G) = [G, G]$ if and only if H is normal in G and G/H is abelian.
If G is solvable, then by definition the derived series $D^n(G)$ trivialises. Put $G_i := D^i(G)$. Then by the fact, each $G_{i+1} = D(G_i)$ is normal in G_i with abelian quotient. The series terminates after finitely many steps because G is solvable.
If we are given the chain of subgroups, then invoking the fact in the reverse direction, we get $D(G_i) \subseteq G_{i+1}$. Thus inductively, $D^i(G) \subseteq G_i$ and therefore the derived series trivialises.

8. What is the Jordan decomposition for a finite group? For $\mathbb{G}_a$?

Solution: If $\mathrm{char}(K) = 0$, then all elements of a finite group G (considered embedded in some GL_n) are semisimple. To see this, note that a unipotent matrix U has all eigenvalues 1. If $U \neq I_n$, then U has non-trivial Jordan blocks and $U^k \neq I_n$ for $k \geq 1$. Therefore U can never have finite order, i.e. belong to G. Note that this argument fails in finite characteristic: for example, the matrix $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ over a field K with $\mathrm{char}(K) = 2$ is unipotent with $U \neq I_2$ but $U^2 = I_2$.
$\mathbb{G}_a \cong \mathrm{U}_2$, so all elements of $\mathbb{G}_a$ are unipotent.

9. Find a closed subgroup G of GL_2 such that G_s is not a closed subset.

Solution: The subgroup T_2 of upper triangular matrices $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. The subset of semisimple elements is
$(\mathrm{T}_2)_s = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mid a, d \in K^*, b \in K \text{ with } a \neq d \text{ or } a = d, b = 0 \right\}$.
It is the complement of matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $a = d$, $b \neq 0$, and this subset is not Zariski-open: for example, its closure are the matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $a = d$ (and arbitrary b), which is not all of T_2.

10. Compute the centre C of $\mathrm{SL}_2(K)$, assuming $\mathrm{char}(K) \neq 2$. Show that the quotient group $\mathrm{PSL}_2(K) := \mathrm{SL}_2(K)/C$ is an affine algebraic group.

(Hint: embed $\mathrm{SL}_2 \subset \mathbb{A}^4$ as a Zariski-closed subset, then check that the action of C on SL_2 extends to an action of C on $\mathbb{A}^4$. Now map $\mathbb{A}^4/C$ to some affine space as a Zariski-closed subset.)

Solution: By direct computation, $C = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$ is the group with two elements. With $\mathrm{SL}_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{A}^4 \mid x_1x_4 - x_2x_3 = 1\}$, the group is a closed subset of $\mathbb{A}^4$. The group C acts by $\pm\mathrm{id}$, which obviously extends to $\mathbb{A}^4$. An element of $\mathbb{A}^4/C$ is a quadruple up to sign, and we can map

$$\mathbb{A}^4/C \to \mathbb{A}^6, \quad \pm(x_1, x_2, x_3, x_4) \mapsto (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4).$$

It is straightforward (and necessary) to check that this map is (a) well-defined and (b) cut out by the three polynomial equations $y_1y_6 = y_2y_5 = y_3y_4$, with coordinates $A(\mathbb{A}^4) = K[x_1, \ldots, x_4]$ and $A(\mathbb{A}^6) = K[y_1, \ldots, y_6]$. The map is *not* injective, but it is injective on the subset PSL_2.
This exhibits $\mathrm{PSL}_2 = \mathrm{SL}_2/C \subset \mathbb{A}^6$ as a Zariski-closed subset. (The equation $\det = 1$, i.e. $x_1x_4 - x_2x_3 = 1$, for SL_2 is invariant under C.) It remains to observe that multiplication $\mathrm{PSL}_2 \times \mathrm{PSL}_2 \to \mathrm{PSL}_2$ and inversion $\mathrm{PSL}_2 \to \mathrm{PSL}_2$ are given by polynomial maps.

Alternative: the map $\mathbb{A}^4/C \to \mathbb{A}^{10}$, $\pm(x_1, x_2, x_3, x_4) \mapsto (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4)$ is injective and can therefore also be used to embed PSL_2 in an affine space.

Note: This exercise shows that some linear algebraic groups do not come with a natural embedding into a linear group. For such groups, it is easier to check that they are affine algebraic groups.

Algebra II — solutions to exercise sheet 3

11. Prove that the group T_n of upper triangular matrices is solvable.

Solution: Start by checking $D(\mathrm{T}_n) = \mathrm{U}_n$ and then use the method of exercise 6. See Tauvel/Yu §10.8.

12. Show that $\mathbb{G}_a$ and $\mathbb{G}_m$ are not isomorphic as affine algebraic groups.

Solution: In fact, the underlying varieties are not isomorphic: $X := \mathbb{G}_a = K$ and $Y := \mathbb{G}_m = K^*$ have affine coordinate rings $A(X) = K[x]$ and $A(Y) = K[y, y^{-1}]$. From the equivalence between affine varieties and reduced, finitely generated algebras we know that $X \cong Y$ if and only if $K[x] \cong K[y, y^{-1}]$. One way to see that these rings are not isomorphic is by their groups of units: $A(X)^* = K[x]^* = K^*$ whereas $A(Y)^* = K[y, y^{-1}]^* = K^* \times \mathbb{Z}$, mapping $(\lambda, n) \mapsto \lambda y^n$. Note that the field K is fixed from the outset: in $A(X)$, every automorphism is a multiple of the identity, whereas this is not the case for $A(Y)$. (Abstractly, it may be hard to see that the groups K^* and $K^* \times \mathbb{Z}$ are non-isomorphic. There are many bizarre isomorphisms, for example $S^1 \cong \mathbb{C}^*$ as groups.) This solution is equivalent to comparing the automorphism groups as affine varieties: $\mathrm{Aut}(X) = A(X)^*$.

A different approach uses automorphisms of $\mathbb{G}_a$ and $\mathbb{G}_m$ as affine algebraic groups. The only automorphisms of $\mathbb{G}_m$ as an affine algebraic group are $\lambda \mapsto \lambda$ and $\lambda \mapsto \lambda^{-1}$. By contrast, $\mathbb{G}_a$ has no such automorphisms at all.

Note that the Lie group analogs of $\mathbb{G}_a$ and $\mathbb{G}_m$ are isomorphic via $\mathbb{R} \to \mathbb{R}_{>0}, t \mapsto e^t$.

13. Let $\varphi\colon X \to Y$ be a morphism of affine varieties. Show that φ is dominant (i.e. the image of X is dense in Y) if and only if $\varphi^*\colon A(Y) \to A(X)$ is injective.

Solution: Assume φ dominant and let $f \in A(Y)$ with $0 = \varphi^*(f) = f\varphi \in A(X)$. As a continuous function $Y \to K$, the vanishing locus of f is a closed subset of Y. However, f is zero on the image of φ, hence on a dense set. So the smallest possible vanishing locus is Y, hence $f = 0$.
Now assume that $\varphi^*\colon A(Y) \to A(X)$ is injective. Put $Z := \overline{\mathrm{im}(\varphi)}$, this is the Zariski-closure of the image of φ. If φ is not dominant, i.e. $Z \subsetneq Y$, then $I(Z) \supsetneq I(Y)$. Hence there is $f \in I(Z)$ with $f \notin I(Y)$; this is a function $f\colon Y \to K$ which is non-zero and vanishes on Z. Therefore, the composition $f\varphi = \varphi^*(f) = 0$, contradicting the injectivity of φ^*.

14. Let $H \subset \mathrm{GL}_n$ be an arbitrary subgroup. Show that the Zariski-closure $\overline{H}$ is a linear algebraic group. Moreover, prove that closure preserves the following properties: H commutative; H normal; H solvable; H unipotent.

Solution: (We repeatedly use: if $f\colon X \to Y$ is a continuous map with $f(A) \subseteq B$ for subsets $A \subseteq X, B \subseteq Y$, then $f(\overline{A}) \subseteq \overline{f(A)}$. In fact, this property (for all A) is equivalent to continuity.)
For any $h \in H$, left multiplication $l_h\colon \mathrm{GL}_n \to \mathrm{GL}_n$ is a homeomorphism preserving the subset H. Therefore, it induces a homeomorphism of closures, $l_h\colon \overline{H} \to \overline{H}$. As this works for all $h \in H$, we get $H \cdot \overline{H} \subseteq \overline{H}$. Now right multiplication by $h \in \overline{H}$ is a homeomorphism $\mathrm{GL}_n \to \mathrm{GL}_n$ restricting to $H \to \overline{H}$, hence mapping $\overline{H} \to \overline{H}$. We get $\overline{H} \cdot \overline{H} = \overline{H}$. Analogously, inversion $\mathrm{GL}_n \to \mathrm{GL}_n, g \mapsto g^{-1}$ is a homeomorphism preserving H, hence it provides a homeomorphism of $\overline{H}$. Therefore, $\overline{H}$ is a subgroup of GL_n, and thus a linear algebraic group.

If H is commutative, then for any $h \in H$, the map $[h, -]\colon H \to H$ has trivial image $\{1\}$. As the set $\{1\}$ is closed, the induced map on closures is $[h, -]\colon \overline{H} \to \{1\}$ and so $[h, g] = 1$ for all $h \in H, g \in \overline{H}$. Now for $g \in \overline{H}$, the map $[-, g]\colon H \to \{1\}$ induces $[-, g]\colon \overline{H} \to \{1\}$. Hence $\overline{H}$ is commutative as well.
For H normal, use the map $H \to H, h \mapsto ghg^{-1}$ for $g \in G$. Similar reasoning works for H solvable. For H unipotent, use that unipotency in GL_n is given by the equations $(I_n - g)^n = 0$.

15. Show that none of the following implications among properties of linear algebraic groups can be reversed:

$$\begin{array}{ccccccccc} & & & & & & \text{unipotent} & & \\ & & & & & & \Downarrow & & \\ \text{torus} & \Longrightarrow & \text{diagonalisable} & \Longrightarrow & \text{abelian} & \Longrightarrow & \text{nilpotent} & \Longrightarrow & \text{solvable} \end{array}$$

Solution: Any finite abelian subgroup of GL_2 is diagonalisable ($\mathrm{char}(K) = 0$) but not a torus.
$\mathbb{G}_a \cong \mathrm{U}_2$ is abelian but not diagonalisable (it is unipotent).
U_3 is nilpotent but not abelian.
T_2 is solvable but not nilpotent.
Any torus is nilpotent but not unipotent.
Wanted: an example of a nilpotent group which is neither abelian nor unipotent!

Algebra II — solutions to exercise sheet 4

16. The group SL_3 naturally acts on K^3. Writing x_1, x_2, x_3 for the standard basis of K^3, this induces an action of SL_3 on polynomials $p(x_1, x_2, x_3) \in K[x_1, x_2, x_3]$ by $g \cdot p = pg$. Compute the weight spaces for the torus $T := \mathrm{D}_3 \cap \mathrm{SL}_3$ on the vector space $K[x_1, x_2, x_3]_2$ of polynomials of degree 2, and draw the weights.

Solution: The vector space $V := K[x_1, x_2, x_3]_2$ has a basis $x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2$. The torus $T = \mathrm{D}_3 \cap \mathrm{SL}_3$ is isomorphic to D_2, and the basis of V also turns out to give all simultaneous eigenvectors for the T-module:

$$t = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & 1/t_1t_2 \end{pmatrix}, \quad t \cdot x_1^2 = t_1^2 x_1^2,\ t \cdot x_1x_2 = t_1t_2x_1x_2,\ t \cdot x_1x_3 = t_2^{-1}x_1x_3 \text{ etc.}$$

We have $X^*(T) = X^*(\mathrm{D}_2) = \mathbb{Z}^2$, and the character associated to $(a_1, a_2) \in \mathbb{Z}^2$ is $\chi_{(a_1,a_2)} \colon T \to \mathbb{G}_m,\ t \mapsto t_1^{a_1} t_2^{a_2}$. Therefore, we get the following six weight spaces (this is the full set due to $\dim(V) = 6$)

$$V_{\chi_{(2,0)}} = Kx_1^2,\ V_{\chi_{(0,2)}} = Kx_2^2,\ V_{\chi_{(-1,-1)}} = Kx_3^2,\ V_{\chi_{(1,1)}} = Kx_1x_2,\ V_{\chi_{(0,-1)}} = Kx_1x_3,\ V_{\chi_{(-1,0)}} = Kx_2x_3.$$

The set of weights is $\{(2,0), (0,2), (1,1), (-1,-1), (0,-1), (-1,0)\} \subset \mathbb{Z}^2$.

17. Show that the normaliser of D_2 in GL_2 is solvable, but not conjugated to a subgroup of T_2.

Solution: The normaliser already appeared in Exercise 1: $N_2 = \mathrm{D}_2 \coprod A_2$ (disjoint union), where $A_2 = \{\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)\}$ is the subset of matrices with zeroes on the diagonal. The group N_2 is solvable because $1 \subset \mathrm{D}_2 \subset N_2$ is a composition series with abelian factors ($N_2/\mathrm{D}_2 \cong \mathbb{Z}/2\mathbb{Z}$).
We show that there are no invariant subspaces for the action of N_2 on K^2: as a D_2-module, K^2 has the invariant subspaces $\binom{*}{0}$ and $\binom{0}{*}$ but neither of these is A_2-invariant. Therefore, N_2 cannot embed into T_2.
This exercise shows that both assumptions of *connected, solvable* are necessary in the Lie-Kolchin theorem.

18. Compute the Weyl group of GL_3 with respect to the torus D_3.

Solution: Put $G := \mathrm{GL}_3$ and $T := \mathrm{D}_3$. A direct computation shows that the centraliser $C_G(T) = T$.
Let $t \in T$ be a diagonal matrix and $g \in G$. We want to see when gtg^{-1} is diagonal. Replacing g by $(\det(g))^{-1/3}g$, if necessary, we can assume $\det(g) = 1$. (This simplifies the formula for g^{-1} via the adjoint matrix a bit.) A lengthy computation by hand then shows that g has exactly one non-zero entry in each row, and each column. Therefore g is obtained from t via a permutation matrix, and $W(\mathrm{D}_3, \mathrm{GL}_3) = S_3$, the symmetric group on 3 letters.

19. For any linear algebraic group G, let $H := \bigcap_{\chi \in X^*(G)} \ker(\chi)$. Show that H is a closed, normal subgroup of G and that G/H is diagonalisable. Also show $X^*(G) \cong X^*(G/H)$.

Solution: Each $\ker(\chi) = \chi^{-1}(1)$ is a closed subgroup, hence their intersection H is closed. Also kernels are normal, so again their intersection H is normal.
In order to show that G/H is diagonalisable, we will prove that it is commutative and consists of semisimple elements. For any $g_1, g_2 \in G$ and $\chi \in X^*(G)$, we have $\chi([g_1, g_2]) = \chi(g_1)\chi(g_2)\chi(g_1)^{-1}\chi(g_2)^{-1} = 1$, hence $[G, G] \subseteq \ker(\chi)$, and so $[G, G] \subseteq H$. This implies G/H commutative.
Denoting by $\pi \colon G \to G/H$ the projection, any element of G/H is of the form $\pi(g)$. From the properties of Jordan decomposition we know that $\pi(g) = \pi(g)_s\pi(g)_u = \pi(g_s)\pi(g_u)$. We claim that $\chi(u) = 1$ for any $u \in G_u$. This will imply $u \in H$, and thus $\pi(g) = \pi(g)_s$. Now a character is an action on a 1-dimensional vector space, so it has to be linearisable — but then the eigenvalues of a unipotent element are all 1, hence $\chi(u) = 1$.
The homomorphism $X^*(\pi) = \pi^* \colon X^*(G/H) \to X^*(G)$ is injective, because π is surjective. Moreover, $X^*(\pi)$ is also surjective: if $\chi \in X^*(G)$, then $\chi\iota = 1$ by the definition of H, where $\iota \colon H \to G$ denotes the embedding. Hence χ gives a well-defined character $G/H \to \mathbb{G}_m$.

20. (Continuation of 19.) Compute H and G/H for $G = \mathrm{GL}_n$.

Solution: Any character of G is trivial on $D(G) = [G, G]$. For GL_n, the commutator subgroup is $D(\mathrm{GL}_n) = \mathrm{SL}_n$. (This requires a computation.) Hence $H \supseteq \mathrm{SL}_n$. Since SL_n is normal in GL_n with quotient $\mathrm{GL}_n/\mathrm{SL}_n \cong \mathbb{G}_m$ and there is the non-trivial character $\det \colon \mathrm{GL}_n \to \mathbb{G}_m$, we find $H = \mathrm{SL}_n$.

Algebra II — solutions to exercise sheet 5

21. Assume $\mathrm{char}(K) \neq 2$ and let $\Gamma \in M(n,K)$ with associated bilinear form $K^n \times K^n, (x,y) \mapsto x^t\Gamma y$, and $\mathrm{O}(\Gamma)$ its isometry group. Show that the tangent space of $\mathrm{O}(\Gamma)$ at $I = I_n$ is $T_I\mathrm{O}(\Gamma) = \{A \in M(n,K) \mid A^t\Gamma + \Gamma A = 0\}$.

(Hint: use the Cayley transform $c(A) = (I+A)(I-A)^{-1}$ for $A \in M(n,K)$ with $\det(I-A) \neq 0$ and show that $c(A) \in \mathrm{O}(\Gamma)$ if and only if $A^t\Gamma + \Gamma A = 0$.)

Solution: [Goodman/Wallach, exercise 1.4.5.5] An element $g \in \mathrm{GL}_n$ is a Γ-isometry if and only if $(gx)^t\Gamma(gy)$ for all $x, y \in K^n$. This is equivalent to $g^t\Gamma g = \Gamma$. We compute (where the last step uses $2 \neq 0$ in K)

$$\begin{aligned} c(A) \in \mathrm{O}(\Gamma) \quad &\iff c(A)^t\Gamma c(A) = \Gamma \iff (I-A)^{-t}(I+A)^t\Gamma(I+A)(I-A)^{-1} = \Gamma \\ &\iff (I+A)^t\Gamma(I+A) = (I-A)^t\Gamma(I-A) \iff \Gamma + A^t\Gamma A + A^t\Gamma + \Gamma = \Gamma + A^t\Gamma A - A^t\Gamma - \Gamma \\ &\iff 2(A^t\Gamma + \Gamma A) = 0 \iff A^t\Gamma + \Gamma A = 0. \end{aligned}$$

Let now $A \in M(n,K)$ with $A^t\Gamma + \Gamma A = 0$. The set of $t \in \mathbb{A}^1$ such that $\det(I - tA) = 0$ is finite; let $U \subset \mathbb{A}^1$ be its open complement. (Note that U is also an affine variety, similarly to how $\mathbb{G}_m \subset \mathbb{A}^1$ is.) We consider the morphism of affine varieties, $\gamma\colon U \to \mathrm{GL}_n$, $t \mapsto c(tA)$. Obviously $\gamma(0) = I$ and by the above, $\gamma(t) \in \mathrm{O}(\Gamma)$ for all $t \in U$. Its differential at 0 is a map $d\gamma_0\colon T_0U = K \to T_I\mathrm{O}(\Gamma)$. We will show that $d\gamma_0(1) = d\gamma(t)/dt(0) = 2A$; this will imply $T_I\mathrm{O}(\Gamma) \supseteq \{A \in M(n,K) \mid A^t\Gamma + \Gamma A = 0\}$. For the computation, apply the usual rules for differentiating products and powers: $d/dt|_{t=0}((I+tA)\cdot(I-tA)^{-1}) = d/dt|_{t=0}(I+tA)\cdot I + I\cdot d/dt|_{t=0}(I-tA)^{-1} = A - (I-0\cdot A)^{-2}(-A) = 2A$; this is possible since A and I commute. (Over $K = \mathbb{C}$, one can also use the geometric series for $(I-tA)^{-1}$.)
For $B \in M(n,K)$, consider the regular function $\psi_B\colon \mathrm{GL}_n \to \mathbb{A}^1$, $g \mapsto \mathrm{tr}((g^t\Gamma g - \Gamma)B)$. Note $g^t\Gamma g - \Gamma = 0 \iff \mathrm{tr}((g^t\Gamma g - \Gamma = 0)B) = 0\ \forall B$; this follows from the non-degeneracy of the trace bilinear form on $M(n,K)$. Via ψ_B, we restrict to deriving scalar-valued functions. For $A \in M(n,K)$, we have the directional derivative $D_A = \sum_{ij} A_{ij}\,\partial/\partial x_{ij}$, and $D_A\psi_B(I) = \mathrm{tr}((A^t\Gamma + \Gamma A)B)$. This can be seen in matrix coordinates: writing $\Gamma = (\Gamma_{ij})$ etc., we have $\mathrm{tr}((g^t\Gamma g - \Gamma = 0)B) = \sum_{ijkl}(g_{ki}\Gamma_{kl}g_{lj} - \Gamma_{kl})b_{ji}$. Thus $T_I\mathrm{O}(\Gamma) \subseteq \{A \in M(n,K) \mid A^t\Gamma + \Gamma A = 0\}$.

22. Compute the dimensions of SO_n and Sp_n.

Solution: We apply the previous exercise to compute the tangent spaces $T_I\mathrm{SO}_n$ and $T_I\mathrm{Sp}_n$. The dimensions of these tangent spaces coincide with the dimensions of the groups, since algebraic groups are smooth.
For the orthogonal group O_n, the bilinear form is given by $\Gamma = I_n$, hence $T_I\mathrm{SO}_n = \{A \in M(n,K) \mid A^t + A = 0\}$. The condition $A^t + A = 0$ is equivalent to A skew-symmetric. Hence the diagonal entries are zero, and A is determined by its upper triangular part. Therefore $\dim(T_I\mathrm{O}_n) = (n-1) + (n-2) + \cdots + 1 = \frac{1}{2}n(n-1)$. Moreover, O_n and SO_n have the same tangent space at I, because SO_n is the identity component of O_n.
For the symplectic group Sp_n, the bilinear form on K^{2n} is given by $\Gamma = J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, where I_n is the identity matrix in $M(n,K)$. We get $T_I\mathrm{Sp}_n = \{A \in M(2n,K) \mid A^tJ + JA = 0\}$. (Here $I = I_{2n}$.) Writing $A = \left(\begin{smallmatrix} B & C \\ D & E \end{smallmatrix}\right)$ as a block matrix, we find the condition $A^tJ + JA = 0$ turns out to be equivalent to $D = D^t, C = C^t, B = -E^t$. Hence C and D are symmetric $n \times n$-matrices, B is an arbitrary $n \times n$-matrix and E is completely determined by B. The dimension therefore is $\dim(T_I\mathrm{Sp}_n) = 2 \cdot \frac{1}{2}(n+1)n + n^2 = n(2n+1)$.

23. Assume $\mathrm{char}(K) = 0$. Let G be a linear algebraic group, all of whose elements have finite order. Show that G is finite.

Solution: We can assume that G is connected because for any linear algebraic group, G/G° is a finite group (G° connected component of the neutral element). Put $G_t := \{g \in G \mid g^t = 1\}$. This is a closed subset of G and by assumption $G = \bigcup_t G_t$. We deduce $G = G_t$ for some fixed t from the following general fact:
A union of countably many subvarieties $Z_i \subsetneq X$ does not equal X: we can assume that $Z_i = V(f_i)$ are hypersurfaces and we do induction on $\dim(X)$. The case $\dim(X) = 1$ is obvious (K algebraically closed and $\mathrm{char}(K) = 0$ implies K uncountable). Now take two non-proportional functions $h_1, h_2 \in A(X)$ and consider the pencil of hypersurfaces $L_t := V(h_1 + th_2$ with $t \in \mathbb{A}^1$. These are uncountably many pairwise different hypersurfaces, so not all Z_i can be among the L_t. This gives us points lying in X but not on any Z_i.
The map $f\colon G \to G, g \mapsto g^t$ is a morphism of affine algebraic groups; in particular, it is continuous. We have $f^{-1}(1) = G$, due to $G = G_t$. However, the fibre $f^{-1}(1)$ is finite, as can be seen from embedding $G \subseteq \mathrm{GL}_n$ — here we use $\mathrm{char}(K) = 0$. Hence G is finite (and with the assumption G connected, even $G = \{1\}$).

24. For affine varieties X and Y, show that $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Solution: This boils down to showing $T_{(x,y)}(X \times Y) \cong T_xX \oplus T_yY$ which is easy in either of the definitions of tangent space. Now let $U \subseteq X$ and $V \subseteq Y$ be the open, dense subsets where the dimensions of tangent spaces are minimal. It is then clear that $\dim(X) + \dim(Y)$ is the minimum attained on the open subset $U \times V \subseteq X \times Y$.

25. If X is an irreducible, smooth affine variety and $Z \subsetneq X$ a closed subvariety, prove $\dim(Z) < \dim(X)$.

Solution: Denote the inclusion morphism $\iota\colon Z \hookrightarrow X$; its induced morphism $\iota^*\colon A(X) \twoheadrightarrow A(Z)$ is then surjective. We get induced surjections $M_{X,P} \twoheadrightarrow M_{Z,P}$ of maximal ideals and also $M_{X,P}^2 \twoheadrightarrow M_{Z,P}^2$. Hence, we obtain a surjection $M_{X,P}/M_{X,P}^2 \twoheadrightarrow M_{Z,P}/M_{Z,P}^2$ (of cotangent spaces at P). There exist regular functions $f \in M_{X,P}$ which don't vanish on Z (as X is irreducible). Choose such an f of minimal degree. Then f cannot be in $M_{X,P}^2$, so $M_{X,P}/M_{X,P}^2 \twoheadrightarrow M_{Z,P}/M_{Z,P}^2$ is not an isomorphism, so $\dim(Z) \leq \dim(T_pZ) < \dim(T_pX) = \dim(X)$ for $p \in Z$.

Algebra II — solutions to exercise sheet 6

26. For a linear algebraic group G, use its comultiplication to define an associative, unital K-algebra structure on $A(G)^* = \mathrm{Hom}_K(A(G), K)$ such that the induced Lie algebra structure coincides with the one from left invariant vector fields.

Solution: For two functionals $\psi_1, \psi_2 \in A(G)$, we define their tensor product to be $\psi_1 \otimes \psi_2 \in (A(G) \otimes A(G))^*$ with $(\psi_1 \otimes \psi_2)(f_1 \otimes f_2) := \psi_1(f_1) \cdot \psi_2(f_2) \in K$ for any $f_1, f_2 \in A(G)$. Using the comultiplication $\Delta = \mu^* \colon A(G) \to A(G) \otimes A(G)$, we define $\psi_1 \cdot \psi_2 := (\psi_1 \otimes \psi_2)\Delta \in A(G)^*$.
This product is associative: coassociativity of Δ means $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta \colon A(G) \to A(G)^{\otimes 3}$. Composing $\psi_1 \otimes \psi_2 \otimes \psi_3 \colon A(G)^{\otimes 3} \to K$ gives $(\psi_1 \cdot \psi_2) \cdot \psi_3 = ((\psi_1 \otimes \psi_2)\Delta \otimes \psi_3)\Delta = (\psi_1 \otimes (\psi_2 \otimes \psi_3)\Delta)\Delta = \psi_1 \cdot (\psi_2 \cdot \psi_3)$.
Now we use the isomorphisms $\theta \colon \mathrm{Der}_K(A(G), K(e)) \rightleftarrows \mathrm{Der}_K(A(G), A(G))^{\lambda(G)} : \varepsilon$ where $\varepsilon(d)(f) = d(f)(e)$ and $\theta(v)(f) = D_v(f)$ with $D_v(f)(g) = v(\lambda_g f)$ for $f \in A(G)$ and $g \in G$. Let $\psi_1, \psi_2 \in \mathrm{Der}_K(A(G), K(e)) \subset A(G)^*$, and $f \in A(G)$ with $\Delta(f) = a_1 \otimes b_1 + \cdots + a_r \otimes b_r$. Then $D_{\psi_1}(f) = \sum_i \psi_1(b_i) \cdot a_i$ and $D_{\psi_2} D_{\psi_1}(f)(e) = \sum_i \psi_1(b_i) \cdot \psi_2(a_i)$:

$$D_{\psi_1}(f)(g) = \psi_1(\lambda_g f) = \psi_1\big(\textstyle\sum_i a_i(g) \cdot b_i\big) = \sum_i a_i(g) \cdot \psi_1(b_i) \text{ and}$$
$$D_{\psi_2} D_{\psi_1}(f)(e) = \psi_2(D_{\psi_1}(f)) = \psi_2\big(\textstyle\sum_i \psi_1(b_i) \cdot a_i\big) = \sum_i \psi_1(b_i) \cdot \psi_2(a_i).$$

Hence $\varepsilon[D_{\psi_2}, D_{\psi_1}] = \psi_2 \cdot \psi_1 - \psi_1 \cdot \psi_2 = [\psi_2, \psi_1]$.

27. Exhibit $\mathbb{G}_m$ as a closed subgroup of SO_2. Then find a two-dimensional torus $T \subset \mathrm{SO}_4$ and compute the weights for the action induced on the adjoint representation, $T \hookrightarrow \mathrm{SO}_4 \to \mathrm{GL}(\mathfrak{so}_4)$.

Solution: First, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{O}_2 \iff a = d, b = -c$. Solving for a linear combination of $\mathbb{G}_m \to \mathbb{G}_m$, $x \mapsto x$ and $x \mapsto x^{-1}$ with determinant 1, we find $a = x/2 + 1/2x$ and $b = -ix/2 + i/2x$, where $i \in K$ is a fixed root of -1; hence that $D(x) := \left(\begin{smallmatrix} x/2+1/2x & -ix/2+i/2x \\ ix/2-i/2x & x/2+1/2x \end{smallmatrix}\right) \in \mathrm{SO}_2$.
Using block matrices, we obtain $T := \mathrm{D}_2 = (\mathbb{G}_m)^2 \hookrightarrow \mathrm{SO}_4$, $(x, y) \mapsto \left(\begin{smallmatrix} D(x) & 0 \\ 0 & D(y) \end{smallmatrix}\right)$. As T is diagonalisable, the representation $T \hookrightarrow \mathrm{SO}_4 \to \mathrm{GL}(\mathfrak{so}_4)$, $t \cdot M = tMt^{-1}$ for $t = (x, y) \in T$ and $M \in \mathfrak{so}_4 = \{M \in M_4(K) \mid M^t = -M\}$ has exactly two non-trivial eigenspaces. The weights are $(1, 0), (0, 1) \in X^*(T) = \mathbb{Z}^2$ and the weight space for $\chi_{(1,0)}$ is $\{\left(\begin{smallmatrix} A & B \\ -B & 0 \end{smallmatrix}\right) \mid A = \left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right) \in \mathfrak{so}_2, B = \left(\begin{smallmatrix} b_1 & b_2 \\ ib_1 & ib_2 \end{smallmatrix}\right)\}$, by explicit computation (note $D(x)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)D(x^{-1}) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$):

$$(x, y) \cdot M = \left(\begin{smallmatrix} D(x) & 0 \\ 0 & D(y) \end{smallmatrix}\right)\left(\begin{smallmatrix} A & B \\ -B & C \end{smallmatrix}\right)\left(\begin{smallmatrix} D(x^{-1}) & 0 \\ 0 & D(y^{-1}) \end{smallmatrix}\right) \stackrel{!}{=} \chi_{1,0}(x, y)M = xM$$

28. Let A be a finite-dimensional, associative and unital K-algebra. Show that the group of units is a linear algebraic group. What is its Lie algebra?

Solution: For any unit $a \in A$, the map $L_a \colon A \to A$ given by left multiplication with a is an isomorphism. Hence we get an injection $A^* \hookrightarrow \mathrm{GL}(A), a \mapsto L_a$. In order to see that the image is closed, we consider $T \in \mathrm{End}(A)$ and show that $TR_b = R_bT$ for all $b \in A$ is equivalent to $T = L_a$ for some $a \in A$. The implication $(\Longleftarrow)$ is trivial, so assume that T commutes with all R_b and put $a := T(1)$; then $T(b) = TR_b(1) = R_bT(1) = ab = L_a(b)$. Hence $A^* \subset \mathrm{GL}(A)$ is the zero set of finitely many commutator equations $[-, R_b] = 0$, which are polynomial.
The Lie algebra of A^* is A with Lie bracket from the algebra commutator: $[a_1, a_2] = a_1a_2 - a_2a_1$, since for any $a \in A$, because derivating the equation $gR_bg^{-1} = R_b$ gives $GR_b + R_bG = 0$ for $G \in \mathrm{End}(A)$.

29. Show that the differential of the adjoint representation of a linear algebraic group G is given by $\mathrm{ad} := d(\mathrm{Ad})_e \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $\mathrm{ad}(A)(B) = [A, B]$.

Solution: Wallach/Goodman, Theorem 1.5.7

30. Show that a morphism $\varphi \colon G \to H$ of linear algebraic groups induces a homomorphism of Lie algebras $d\varphi_e \colon \mathfrak{g} \to \mathfrak{h}$.

Solution: Embedding $H \hookrightarrow \mathrm{GL}(V)$ for some finite-dimensional vector space V, we obtain a rational representation $\varphi \colon G \to \mathrm{GL}(V)$ and want to show that $d\varphi_e \colon \mathfrak{g} \to \mathrm{End}(V)$ has image in $\mathfrak{h}$. Let $v \in \mathfrak{g}$, $h \in H$ and $f \in I(H) \subset A(G)$. Then $(D_{d\varphi(v)}f)(h) = d\varphi_e(v)(\lambda_h f) = v(\varphi^*(\lambda hf)) = 0$, because $(\lambda hf)(\varphi(g)) = f(h\varphi(g))$ and f vanishes on H. Hence $D_{d\varphi(v)}(I(H)) = 0$, which means that $d\varrho_e(v) \in \mathfrak{h}$.

Algebra II — solutions to exercise sheet 7

31. Find a symmetric, non-degenerate bilinear form Γ on K^n such that $T := \mathrm{SO}(\Gamma) \cap \mathrm{D}_n$ is a torus of dimension m if $n = 2m$ or $n = 2m+1$. Prove that T is a maximal torus in $\mathrm{SO}(\Gamma)$: if $H \subseteq \mathrm{SO}(\Gamma)$ is abelian with $T \subseteq H$, then $T = H$.

Solution: Let Γ have entries 1 on the skew diagonal and 0 else; e.g. $\Gamma = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $n = 2$. Then Γ is obviously symmetric and non-degenerate. For $n = 2m$ even, $\mathrm{SO}(\Gamma) \cap \mathrm{D}_{2m}$ contains the diagonal matrices $\mathrm{diag}(x_1, \dots, x_m, x_1^{-1}, \dots, x_m^{-1})$. If $n = 2m+1$ is odd, then $\mathrm{SO}(\Gamma) \cap \mathrm{D}_{2m+1}$ contains the diagonal matrices $\mathrm{diag}(x_1, \dots, x_m, 1, x_1^{-1}, \dots, x_m^{-1})$. In either case, these form a torus D_m in $\mathrm{SO}(\Gamma)$.
Now let $g \in \mathrm{SO}(\Gamma)$ such that $gh = hg$ for all $h \in \mathrm{SO}(\Gamma) \cap \mathrm{D}_n$. Each h acts on the standard basis vectors $e_i \in K^n$ by a character χ_i, i.e. $h(e_i) = \chi_i(h) \cdot e_i$. These characters are

$$\begin{aligned} n = 2m: \quad & \chi_1 = x_1, \dots, \chi_m = x_m, \chi_{m+1} = x_1^{-1}, \dots, \chi_{2m} = x_m^{-1}, \\ n = 2m+1: \quad & \chi_1 = x_1, \dots, \chi_m = x_m, \chi_{m+1} = 1, \chi_{m+2} = x_1^{-1}, \dots, \chi_{2m+1} = x_m^{-1}. \end{aligned}$$

In either case, the characters are pairwise distinct. Hence, all weight spaces are one-dimensional. As g centralises $\mathrm{SO}(\Gamma) \cap \mathrm{D}_n$, it must preserves all weight spaces, thus acts diagonally itself. But then $g \in \mathrm{D}_n$.

32. Compute the orbits of the natural actions of $\mathrm{GL}_n, \mathrm{T}_n, \mathrm{U}_n, \mathrm{D}_n$ on $\mathbb{A}^n$ and draw them for $n = 2$. Describe orbit closures as unions of orbits.

Solution: GL_n acts on $\mathbb{A}^n$ with two orbits: 0 is a fixed point, i.e. a closed orbit; $\mathbb{A}^n \setminus \{0\}$ is an open orbit.
There are $n+1$ orbits for the T_n-action whose closures are nested: $O_0 = \{0\}$ and $O_i = \mathbb{A}^i \times \{0\} \setminus O_{i-1}$ for $i > 0$; so that $\overline{O}_i = O_0 \cup \ldots \cup O_i$.
The U_n-orbit of a vector $v = (v_1, \dots, v_n)$ is $(*, \dots, *, v_i, 0, \dots, 0)$, where $v_i \neq 0$ and $v_{i+1} = \ldots = v_n = 0$. In particular, all orbits are of the form $\mathbb{A}^{i-1} \times \{(x, 0, \dots, 0)\}$ and hence closed.
D_n acts on $\mathbb{A}^n$ with 2^n orbits: any map $t\colon \{1, \dots, n\} \to \{0, 1\}$ determines a type of vectors $v \in \mathbb{N}^n$ by $v_i = 0 \iff t(i) = 0$. These types are preserved by the D_n-action and classify orbits. The closed orbit is $\{0\}$, corresponding to $t = (0, \dots, 0)$; the orbit of type $(1, \dots, 1)$, i.e. of vectors without zero components is open and dense. Given two orbits O_t and O_s of types t and s, then $O_t \subseteq \overline{O}_s$ if and only if $s(i) = 0 \Rightarrow t(i) = 0$ for all $i = 1, \dots, n$.

33. Show that the adjoint representation of SL_2 has disconnected isotropy groups.

Solution: [The point is that *some* (not all) isotropy groups are disconnected. Note that both SL_2 and $\mathfrak{sl}_2$ are irreducible; this exercise shows that these properties are not enough to ensure irreducible isotropy.]
We consider the adjoint representation $\mathrm{SL}_2 \to \mathrm{GL}(\mathfrak{sl}_2)$, which is given by $g \cdot M = gMg^{-1}$ for $g \in \mathrm{SL}_2$ and $M \in \mathfrak{sl}_2$, i.e. a traceless matrix. The isotropy group of $M = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is $F = \{g \in \mathrm{SL}_2 \mid gM = Mg\}$. A quick matrix computation gives $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in F \iff a = d, c = 0$. Together with $\det(g) = 1$, we get $F = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in K, a^2 = 1\}$. This set decomposes into the two components $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & * \\ 0 & -1 \end{smallmatrix}\right)$.

34. Let G be a unipotent linear algebraic group and X an affine G-variety. Show that all orbits of G in X are closed.

Solution: This is the theorem of Kostant–Rosenlicht (1961). The property even characterises unipotent groups.
Let $O = G \cdot x$ be an orbit. By replacing X with $\overline{O}$, we can assume that O is dense in X. Next, O is also open in X: the image of the map $G \to X$, $g \mapsto g \cdot x$ contains an open subset $U \subseteq X$ (this is a general property of morphisms between affine varieties) and thus $G \cdot x = \bigcup_{g \in G} g \cdot U$ is open.
Denote by $Z := X \setminus O$ the closed complement. Then G acts on Z, hence on the ideal $I(Z) \subseteq A(X)$. The latter action is locally finite, i.e. there exists a finite-dimensional, G-invariant subspace $V \subset I(Z)$. As G is unipotent, the representation $G \to \mathrm{GL}(V)$ has a fixed vector $0 \neq f \in V^G$ (Kolchin). This means $\lambda_g(f) = f$, i.e. $f(g \cdot x) = f(x)$ for all $g \in G$. Hence f is constant on O, and thus constant on $\overline{O} = X$. However, $f(Z) = 0$, so $f = 0$, a contradicton.

35. For a homogeneous ideal $I \subseteq K[x_0, \dots, x_n]$, geometrically compare the varieties $V(I) \subseteq \mathbb{A}^{n+1}$ and $V(I) \subseteq \mathbb{P}^n$. Prove the homogeneous Nullstellensatz.

Solution: Denote $\pi\colon \mathbb{A}^{n+1} \to \mathbb{P}^n, (a_0, \dots, a_n) \mapsto (a_0 : \ldots : a_n)$, also $X := V(I) \subseteq \mathbb{A}^{n+1}$ and $Y := V(I) \subseteq \mathbb{P}^n$. The crucial observation: $x \in \mathbb{A}^{n+1}$ satisfies $f(x) = 0$ for all $f \in I$ (i.e. $x \in X$) if and only if $f(\pi(x)) = f([x]) = 0$ for all homogeneous $f \in I$, i.e. $[x] \in Y$. Hence X consists of lines through the origin, and all fibres of $\pi\colon X \setminus \{0\} \to Y$ are K^*. In particular, $\pi(X \setminus \{0\}) = Y$ and $\pi^{-1}(Y) \cup \{0\} = X$. [This is why X is called the *affine cone* of Y.]

For the Nullstellensatz, let $I \subset K[x_0, \dots, x_n]$ be a radical ideal with $(x_0, \dots, x_n) \not\subseteq I$. We first consider its affine variety $X = V(I) \subseteq \mathbb{A}^{n+1}$: the affine Nullstellensatz yields that X contains points different from the origin (I is contained in maximal ideals, and it cannot coincide with the homogeneous maximal ideal by assumption). From the above, we then see that the projective variety $Y = V(I) \subseteq \mathbb{P}^n$ contains points.

Algebra II — solutions to exercise sheet 8

36. Compute the orbits of the actions of GL_3 on $\mathbb{P}^2$, of GL_4 on $\mathrm{Gr}(2,4)$, and of GL_2 on $\mathbb{P}^1 \times \mathbb{P}^1$ (diagonal action). Also compute isotropy groups for all orbits.

Solution: GL_3 acts transitively on $\mathbb{A}^3 \setminus \{0\}$, hence it also acts transitively on $\mathbb{P}^2$. The isotropy group of the point $(1:0:0)$ consists of the block matrices of type $\left(\begin{smallmatrix} *&*&* \\ \circ&*&* \\ \circ&*&* \end{smallmatrix}\right)$.

For the action of GL_4 on $\mathrm{Gr}(2,4)$, we consider an arbitrary subspace $V \subset K^4$ with $\dim(V) = 2$. If v_1, v_2 is a basis for V, then we can apply an automorphism of K^4, which maps v_1 to $e_1 = (1,0,0,0)$. Hence we can assume that the basis is e_1, v for some $v \in K^4$. The subgroup $\mathrm{T}_4 \subset \mathrm{GL}_4$ preserves e_1, up to scalars. Using an appropriate triangular matrix, we can fix e_1 and map v to a vector of the form $(0,*,*,*)$. At this point we use the subgroup $\mathrm{GL}_1 \times \mathrm{GL}_3 \subset \mathrm{GL}_4$, and with GL_3 acting transitively on $\mathbb{A}^3 \setminus \{0\}$, we can map $(0,*,*,*) \mapsto (0,1,0,0) = e_2$. Hence GL_4 acts transitively on $\mathrm{Gr}(2,4)$. The isotropy group of the subspace $V = (*,*,0,0)$ consists of matrices whose lower left 2×2-block is zero.

The diagonal action of GL_2 on $\mathbb{P}^1 \times \mathbb{P}^2$ has two orbits: one is the diagonal $\Delta := \{(p,p) \mid p \in \mathbb{P}^1\}$ (this is a closed orbit), the other is the open complement $U := \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$. It is obvious that Δ is GL_2-invariant. Moreover, $\Delta \cong \mathbb{P}^1$ and GL_2 acts transitively on $\mathbb{P}^1$; hence Δ is an orbit. For a point $((x_0 : x_1),(y_0 : y_1)) \notin \Delta$, we have $x_0 y_1 - x_1 y_0 \neq 0$, and applying an appropriate matrix multiplication, we get

$\left(\begin{smallmatrix} y_1 & -y_0 \\ -x_1 & x_0 \end{smallmatrix}\right) \cdot ((x_0 : x_1),(y_0 : y_1)) = ((y_1 x_0 - y_0 x_1 : 0),(0 : y_1 x_0 - y_0 x_1)) = ((1:0),(0:1)) \in \mathbb{P}^1 \times \mathbb{P}^1.$

The isotropy group of any point in $\mathbb{P}^1 \times \mathbb{P}^1$ is D_2.

37. Show that any non-degenerate conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. Deduce that homogeneous coordinate rings are not isomorphism invariants of projective varieties.

Solution: A conic in $\mathbb{P}^2$ is, by definition, the vanishing locus of some homogeneous polynomial of degree two: $a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{12}x_1x_2 = 0$ with $a_{ij} \in K$. This zero set can be also described in matrix form $x^t M x = 0$, where the symmetric bilinear form M can be diagonalised to

$$M = \begin{pmatrix} a_{00} & a_{01}/2 & a_{02}/2 \\ a_{01}/2 & a_{11} & a_{12}/2 \\ a_{02}/2 & a_{12}/2 & a_{22} \end{pmatrix} \rightsquigarrow \begin{pmatrix} c_0 & 0 & 0 \\ 0 & c_1 & 0 \\ 0 & 0 & c_2 \end{pmatrix}$$

with c_i either 0 or 1 (K is algebraically closed). This is given by a linear coordinate change, i.e. an automorphism of $\mathbb{P}^2$. However, if one of the $c_i = 0$, then the quadric is degenerate. Hence every non-degenerate quadric in $\mathbb{P}^2$ is isomorphic to $V(x_0^2 + x_1^2 + x_2^2)$. Moreover, $\mathbb{P}^1 \to \mathbb{P}^2, (y_0 : y_1) \mapsto (y_0^2 : y_0 y_1 : y_1^2)$ is a morphism which is injective and whose image is cut out by $V(x_0^2 - x_1^2 + x_2^2)$. Hence $\mathbb{P}^1 \cong V(x_0^2 - x_1^2 + x_2^2) \cong V(x_0^2 + x_1^2 + x_2^2)$ using the coordinate change $x_1 \mapsto i x_1$, and all non-degenerate conics are isomorphic to $\mathbb{P}^1$.

For the second claim, note that the homogeneous coordinate ring of $\mathbb{P}^1$ is $K[x,y]$, the polynomial ring in two variables (with its standard grading). However, the homogeneous coordinate ring of a quadric is e.g. $S = K[x,y,z]/(x^2+y^2+z^2)$. These rings are not isomorphic, for example because the associated affine varieties aren't ($V(x^2+y^2+z^2) \subset \mathbb{A}^3$ is singular).

38. Show that $\mathbb{A}^1$ and $\mathbb{P}^1$ are homeomorphic, but $\mathbb{A}^2$ and $\mathbb{P}^2$ are not.

Solution: As topological spaces, both $\mathbb{A}^1$ and $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ are sets of the same cardinality as K, with the cofinite topology. Hence they are homeomorphic.

When comparing $\mathbb{A}^2$ and $\mathbb{P}^2$, let a *curve* be a closed, irreducible subset which is neither a point nor the whole space. We claim that for any curve $C \subset \mathbb{A}^2$, there exists another curve $C' \subset \mathbb{A}^2$ with $C \cap C' = \varnothing$. (If $C = V(f)$, then $C' = V(f+1)$ will work.) However, in $\mathbb{P}^2$ there are curves which intersect all other curves. (For example, a line $V(x_0)$ will work: if $V(g)$ is a curve given by a homogeneous polynomial $g \in K[x_0,x_1,x_2]$, then either $g \in (x_0)$ and $V(g) \cap V(x_0)$ is even infinite, or $g(0,x_1,x_2) \neq 0$ and defines a finite set of points in $\mathbb{P}^1$ — this set is non-empty because K is algebraically closed.) These are topological properties, so $\mathbb{A}^2$ and $\mathbb{P}^2$ cannot be homeomorphic.

39. For an affine, irreducible variety X and a point $p \in X$, show that the local ring of X at p is given by the localisation of $A(X)$ at the maximal ideal M_p.

40. Show that $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$.

Solution: We already know that GL_2 acts on $\mathbb{P}^1$ with kernel $Z(\mathrm{GL}_2) \cong \mathbb{G}_m$. Hence PGL_2 acts faithfully on $\mathbb{P}^1$, and we have to show that all automorphisms of the projective line come from linear maps.

Let $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ be an arbitrary automorphism. Put $(x_0 : x_1) := \varphi(1:0)$. If $(x_0 : x_1) \neq (1:0)$, then $x_1 \neq 0$, and then matrix $\psi := \left(\begin{smallmatrix} 0 & 1/x_1 \\ x_1 & -x_0 \end{smallmatrix}\right)$ satisfies $\psi\varphi(1:0) = (1:0)$. Hence $\psi\varphi$ induces an automorphism of $\mathbb{P}^1 \setminus \{(1:0)\} = \mathbb{A}^1$. As $\mathbb{A}^1$ is an affine variety, we have $\mathrm{End}(\mathbb{A}^1) = \mathrm{End}(A(\mathbb{A}^1)) = \mathrm{End}(K[x])$, so the automorphism must be of the form $\psi\varphi = ax + b$, i.e. a linear polynomial. Then $\theta := \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ maps $(x:1) \mapsto (ax+b:1) = \psi\varphi(x:1)$ and $(1:0) \mapsto (a:0) = (1:0)$. Hence $\theta = \psi\varphi$ and $\varphi = \psi^{-1}\theta \in \mathrm{PGL}_2$.

[It is also true that $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}$ but the proof requires considerably harder methods.]

Algebra II — solutions to exercise sheet 9

41. Show that varieties are compact in the Zariski topology: any open cover has a finite subcover.

Solution: This holds true for any noetherian topological space, i.e. a space X where descending chains of closed subsets stabilise or, equivalently, ascending chains of open subsets stabilise. If $X = \bigcup_{i \in I} U_i$ is an open cover, then choose a well-ordering on I and define $V_j := \bigcup_{i \leq j} U_i$. Then the V_j form an ascending chain of open subsets. [In fact, a topological space is noetherian if and only if every open subset is compact.]
Affine varieties are noetherian topological spaces because their affine coordinate rings are noetherian, so that asscending chains of ideals stabilise or, equivalently, descending chains of Zariski-closed subsets stabilise. The same reasoning works for projective varieties. For quasi-affine or quasi-projective varieties, we observe that a subspace of a noetherian topological space is still noetherian.

42. Show that a locally compact Hausdorff space X is compact if and only if the projection $X \times Y \to Y$ is a closed map for all topological spaces Y.

Solution: Let X be compact and take a closed subset $Z \subseteq X \times Y$. We have to show that $\pi(Z) \subseteq Y$ is closed, where $\pi\colon X \times Y \to Y$ is the projection. Let $y \in Y \setminus \pi(Z)$. Then $(x, y) \notin Z$ for all $x \in X$. By the definition of the product topology, for each $x \in X$, there exist open neighbourhoods $x \in U_x \subseteq X$ and $y \in V_x \subseteq Y$ such that $U_x \times V_x \subseteq X \times Y \setminus Z$. In particular, $\bigcup_{x \in X} U_x = X$. By compactness of X, there are finitely many points $x_1, \ldots, x_r$ such that $X = U_{x_1} \cup \cdots \cup U_{x_r}$. Let $V := V_{x_1} \cap \cdots \cap V_{x_r}$, then $y \in V \subseteq Y$ open. By construction, $V \subset Y \setminus \pi(Z)$, so that $Y \setminus \pi(Z)$ is open, hence $\pi(Z)$ closed.
For the reverse implication, assume that X is not compact. Then we consider its one-point compactification $Y := \hat{X} = X \cup \{\infty\}$ — this is a compact space; Y is Hausdorff space since X is locally compact and Hausdorff. Moreover, let $Z := \{(x, x) \mid x \in X\} \subset X \times \hat{X}$ be the diagonal. As X is Hausdorff, $Z \subset X \times X$ is closed; thus X is also closed in $X \times \hat{X}$. Then $\pi(Z) = X \subset \hat{X}$, a closed subset in a compact Hausdorff space, hence compact itself.

The statement actually holds for arbitrary topological spaces X: for the reverse implication, let $X = \bigcup_{i \in I} U_i$ be an open covering without any finite subcovering. Let now $Y := X \cup \{\omega\}$, topologised with the following open sets: all subsets of X; all sets containing $\{\omega\} \cup (X \setminus U)$ where U is a finite union of the U_i. This is indeed closed under arbitrary unions and finite intersections. Moreover, by our assumption on the covering U_i, the subset $X \subset Y$ is not closed, i.e. $\{\omega\} \subset Y$ is not open. Let Z be the closure of the diagonal $\Delta := \{(x, x) \mid x \in X\}$ in $X \times Y$. By hypothesis, its closure $\pi(Z)$ is closed in Y, hence $\pi(Z) = Y$. Hence $(x, \omega) \in Z$ for some $x \in X$. By the definitions of closure, of the topology on Y and of the product topology, $\big(V \times (\{\omega\} \cup (X \setminus U_i))\big) \cap \Delta \neq \varnothing$ for all $i \in I$ and open sets $x \in V \subseteq X$. Hence $V \cap (X \setminus U_i) \neq \varnothing$, i.e. $V \not\subseteq U_i$ for all V and i. But this would imply $x \notin U_i$ for all $i \in I$, contradicting that the U_i cover X.

43. Let $X = V(y^2 - x^3) \subset \mathbb{A}^2$ be the cuspidal cubic. Show that the map $\mathbb{A}^1 \to X, t \mapsto (t^2, t^3)$ is regular, birational and a homeomorphism, but not an isomorphism of varieties. Extend this to an example of a morphism of projective varieties with the same properties.

Solution: The map is given by polynomials, so it is regular. It is bijective, because for any point $(x, y) \in X$, we have $y^2 = x^3$ and (with K algebraically closed), there are two square roots of x, call them t and $-t$. Moreover, $t^6 = x^3 = y^2$, regardless of sign. But exactly one of the two roots satisfies $t^3 = y$. Next, $\mathbb{A}^1$ and X are infinite sets with the cofinite topology, hence every bijection between them is automatically a homeomorphism.
Finally, the two varieties are birational: $K(\mathbb{A}^1) = \mathrm{Quot}(K[t]) = K(t)$, i.e. the function field of the affine line is the field of rational functions in one variable. But $A(X) = K[x, y]/(y^2 - x^3) = K[t^2, t^3])$ has the same quotient field: $\mathrm{Quot}(K[t^2, t^3]) = K(t)$, as $t^3/t^2 = t$ in $\mathrm{Quot}(A(X))$.
The two curves are not isomorphic because their affine coordinate rings aren't. For example, the tangent space $T_0X = (M_0/M_0^2)^*$ is 2-dimensional, whereas all tangent spaces of $\mathbb{A}^2$ are 1-dimensional.
By homogenisation, we get an example of projective curves: $\mathbb{P}^1 \to V(x_0x_2^2 - x_1^3)$, $(t_0 : t_1) \mapsto (t_0^3 : t_0t_1^2 : t_1^3)$.

44. Let A be a commutative ring with unit and N a finitely generated A-module. Show that a surjective module endomorphism $N \to N$ is an isomorphism.

Solution: The theorem of Vasconcelos.
Write $f\colon N \to N$ and use it to consider N as an $A[X]$-module, with $X \cdot n := f(n)$. Then $XN = N$ by the assumption that f is surjective. Applying the Nakayama lemma (with $I = (X) \subset A[X]$) yields an element $Y \in A[X]$ such that $YN = 0$ and $Y \in 1 + (X)$, i.e. $Y = 1 + XZ$ for some $Z \in A[X]$. Let $n \in \ker(f)$, then $X \cdot n = 0$ and hence $0 = (1 + XZ)n = n + ZXn = n$, so that f is indeed injective.

45. Find subgroups of SL_2 such that the quotient is respectively projective, affine or strictly quasi-affine.

Solution: The quotient $\mathrm{SL}_2/(\mathrm{T}_2 \cap \mathrm{SL}_2) \cong \mathbb{P}^1$ is a projective variety; $\mathrm{SL}_2/(\mathrm{D}_2 \cap \mathrm{SL}_2) \cong \mathrm{PSL}_2$ is an affine variety; $\mathrm{SL}_2/\mathrm{U}_2 \cong \mathbb{A}^2 \setminus \{0\}$ is quasi-affine but not affine.

Algebra II — solutions to exercise sheet 10

46. Compute the dimensions of Grassmannians $\mathrm{Gr}(d,n)$ and flag manifolds $\mathrm{Fl}(n)$.

Solution: We already know that $\mathrm{Fl}(n) = \mathrm{GL}_n/\mathrm{T}_n$ is a homogeneous space. This implies that tangent spaces at the flag manifold are given by quotients $T_e\mathrm{GL}_n/T_e\mathrm{T}_n$, hence $\dim(\mathrm{Fl}(n)) = \dim(\mathfrak{gl}_n) - \dim(\mathfrak{t}_n) = n^2 - n(n+1)/2 = n(n-1)/2$, where $\mathfrak{t}_n \subset \mathfrak{gl}_n$ is the subset of all upper triangular matrices.
The Grassmannian $\mathrm{Gr}(d,n)$ has a transitive action by GL_n, so $\mathrm{Gr}(d,n) = \mathrm{GL}_n/H$ where H is the stabiliser of some d-dimensional subspace $U \subset K^n$. Taking U to be the span of the first d standard basis vectors of K^n, we find $H = \{\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \mid A \in \mathrm{GL}_d, B \in M(d \times n-d), D \in \mathrm{GL}_{n-d}\}$. Again, we compute the dimension as $\dim(\mathrm{Gr}(d,n)) = \dim(\mathfrak{gl}_n) - \dim(T_eH) = n^2 - (d^2 + d(n-d) + (n-d)^2) = nd - d^2 = d(n-d)$.

47. Show that two irreducible varieties X and Y are birational, i.e. their function fields are isomorphic: $K(X) \cong K(Y)$, if and only if there exist affine open subsets $U_X \subseteq X$ and $U_Y \subseteq Y$ which are isomorphic: $U_X \cong U_Y$.

Solution: For any open affine subset $\varnothing \neq U \subseteq X$, we have $\mathrm{Quot}(A(U)) = K(X)$. Hence $U_X \subseteq X$ and $U_Y \subseteq Y$ with $U_X \cong U_Y$ implies $K(X) \cong \mathrm{Quot}(A(U_X)) \cong \mathrm{Quot}(A(U_Y)) \cong K(Y)$.
Now, assume $K(X) \cong K(Y)$. Pick an open affine subset $V \subseteq X$; this gives $K(X) \cong K(V)$. Choose an isomorphism $\varphi\colon K(V) \to K(Y)$. Let $f_1, \ldots, f_r \in A(V)$ be generators of the affine coordinate ring of Y. Then the $\varphi(f_i)$ are rational functions on Y, and there is an open subset $U \subseteq Y$ on which all $\varphi(f_i)$ are regular. Shrink U, if necessary, to ensure it is affine. Then φ induces a morphism $\varphi\colon A(V) \to A(U)$. This map of K-algebras must be injective, since the induced map of quotient fields is. Hence we obtain a dominant morphism $\alpha\colon U \to V$ of affine varieties. Repeating this argument for $\varphi^{-1}\colon K(Y) \to K(X)$ yields a dominant morphism $\beta\colon V \to U'$ for some open, affine subset $U' \subseteq Y$. The morphisms α and β are inverse on an open subset.

48. Assume $\mathrm{char}(K) = p > 0$. Show that the map $\mathbb{G}_m \to \mathbb{G}_m, x \mapsto x^p$ is a bijective morphism of affine algebraic groups, but is not an isomorphism.

Solution: The map is obviously a group homomorphism, due to characteristic p. It is a morphism of varieties, since it is evidently given by a polynomial. It is surjective, because K is algebraically closed (every element of the field has roots of any order). It is injective because of $x^p = y^p \iff (x-y)^p = 0 \iff x - y = 0$.
However, it is not an isomorphism: variety morphisms $\mathbb{G}_m \to \mathbb{G}_m$ are the same as endomorphisms of the K-algebra $A(\mathbb{G}_m) = K[T, T^{-1}]$. The map $F\colon \mathbb{G}_m \to \mathbb{G}_m, F(x) = x^p$ induces $F^*\colon K[T,T^{-1}] \to K[T, T^{-1}], f(T) \mapsto f(T^p)$. This ring homomorphism is not surjective!

49. Let G be a linear algebraic group, acting on a quasi-projective variety X. Show that orbits of minimal dimension are closed; in particular, closed orbits exist.

(Hint: you can use the statement of exercise 25.)

Solution: Let $x \in X$ and $Z := \overline{G \cdot x}$ be the closure of the orbit of x. Then Z is a G-invariant, closed subset of X. We already know that $O := G \cdot x \subseteq Z$ is open. Assume the orbit is not closed. Then $Z \setminus O$ is closed in Z and G-invariant, hence quasi-projective and a union of orbits. Therefore, $\dim(Z \setminus O) < \dim(Z)$ and since dimensions are non-negative and finite, orbits of minimal dimension must be closed.

50. Let G be a connected projective algebraic group. Show that G is commutative.

Solution: We consider the two morphisms $f, p\colon G \times G \to G$ with $f(g,h) = ghg^{-1}h^{-1}$ and $p(g,h) = g$. Let $U \subset G$ be an open affine subset of G such that $e \in U$. Then $p(G \times G \setminus f^{-1}(U)) \subset G$ is a closed subset not containing e. Thus there is an open subset $V \subset G$ such that $p^{-1}(V) \subseteq f^{-1}(U)$, hence $f(p^{-1}(V)) \subseteq U$. Let $g \in V$; then $f(\{g\} \times G) = f(p^{-1}(g)) \subseteq U$. However, with G proper and connected, this set is the singleton $\{e\}$ since $f(g,e) = e$. Thus $f(p^{-1}(V)) = \{e\}$, but $p^{-1}(V) \subset G \times G$ is a dense subset, and so $f(G \times G) = \{e\}$, i.e. G is commutative.

Sketch of an alternative proof if $\mathrm{char}(K) = 0$: for any $g \in G$, let $c_g\colon G \to G, h \mapsto ghg^{-1}$. Its derivative at e gives an automorphism $\mathrm{Ad}(g) := dc_e\colon T_eG \to T_eG$. In particular, we get a morphism $\mathrm{Ad}\colon G \to \mathrm{GL}(T_eG)$. With G proper and connected, its image must be the identity of T_eG. Like in the affine case, T_eG is a Lie algebra and we now know that it is abelian. G connected and $\mathrm{char}(K) = 0$ imply that G is commutative.

Algebra II — solutions to exercise sheet 11

51. Find Borel subgroups in SO_4, Sp_4, T_4 and U_4.

Solution: Recall that $\mathrm{GL}_4/TT_4 \cong \mathrm{Fl}(K^4)$, the variety of full flags in K^4. Given the subgroup $\mathrm{SO}_4 \subset \mathrm{GL}_4$, we consider the subset of *isotropic flags* $\mathrm{Fl}_0(K^n,\beta) = \{V^0 \subsetneq V^1 \subsetneq V^2 \mid \dim(V^i) = i, \beta|_{V^i \times V^i} = 0\}$ where $\beta\colon K^4 \times K^4 \to K$ is the bilinear form. This is a closed subset, so that $\mathrm{Fl}_0(K^n,\beta)$ is a projective variety. SO_4 acts transitively on isotropic flags (similar proof as with GL_n and full flags). Using the bilinear form β from Exercise 31, a particular isotropic flag is given by $V^0 = 0 \subset V^1 = Ke^1 \subset V^2 = Ke^1 + Ke^3$. From $g \in \mathrm{GL}_4$ with $ge^1 = e^1$, $ge^3 = e^3$ and $g^t\beta g = \beta$, we get

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & * & 1 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and in particular the stabiliser group in SO_4 is solvable.
T_4 and U_4 are solvable and connected, hence their Borel groups are the full group in each case.

52. Find all parabolic subgroups P with $\mathrm{T}_4 \subseteq P \subseteq \mathrm{GL}_4$.

Solution: There are 8 such parabolic subgroups, there are given by block matrices of the formats

$$\mathrm{GL}_4 = \begin{pmatrix} **** \\ **** \\ **** \\ **** \end{pmatrix}, \begin{pmatrix} **** \\ **** \\ ** \\ * \end{pmatrix}, \begin{pmatrix} **** \\ *** \\ *** \\ * \end{pmatrix}, \begin{pmatrix} **** \\ *** \\ ** \\ ** \end{pmatrix}, \begin{pmatrix} **** \\ *** \\ *** \\ *** \end{pmatrix}, \begin{pmatrix} **** \\ **** \\ ** \\ ** \end{pmatrix}, \begin{pmatrix} **** \\ **** \\ *** \\ * \end{pmatrix}, \begin{pmatrix} **** \\ *** \\ ** \\ * \end{pmatrix} = \mathrm{T}_4.$$

53. Let the connected linear algebraic group G act on a quasi-projective variety X with finitely many orbits. Show that every irreducible G-invariant subset in X is the closure of a G-orbit. Find a counterexample for an action with infinitely many orbits.

Solution: We know that orbits in general are quasi-projective and open in their closure. An irreducible G-invariant subset $Z \subseteq X$ is a union of orbits. Since there are only finitely many orbits, Z contains an orbit O as an open subset; the closure $\overline{O}$ must then be Z.
Counterexamples abound. (For a trivial one, take $G = 1$, $Z = X = \mathbb{A}^1$.)

54. Find a connected linear algebraic group G and a maximal solvable subgroup $U \subset G$ such that U is disconnected.

Solution: This is problem 17 in disguise: we take $G = \mathrm{GL}_2$ and $U := N(\mathrm{D}_2) = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right) \cup \left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$. Then U is obviously disconnected, it is solvable by problem 17. It is a matrix calculation to show that the only subgroup of G strictly containing U is G.

55. Classify all root systems in the Euclidean plane $E := \mathbb{R}^2$.

Algebra II — solutions to exercise sheet 12

56. Compute the radicals $R(\mathrm{GL}_n), R(\mathrm{SL}_n), R(\mathrm{U}_n)$.

Solution: On the one hand, $R(\mathrm{GL}_n)$ is the identity component of the intersection over all Borel subgroups. With the Borel subgroups $B := \mathrm{T}_n$ and B^- of upper/lower triangular matrices, we find $R(\mathrm{GL}_n) \subseteq B \cap B^- = \mathrm{D}_n$. On the other hand, $R(\mathrm{GL}_n)$ is the maximal normal, solvable, connected subgroup: we get $R(\mathrm{GL}_n) = K^* \cdot I_n$ from $R(\mathrm{GL}_n)$ normal: for $n = 2$, we have $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} b & a-b \\ 0 & a \end{smallmatrix}\right) \in R(\mathrm{GL}_2) \subseteq \mathrm{D}_2$, hence $a = b$.
The same computation for SL_n shows $R(\mathrm{SL}_n) = 1$, i.e. SL_n is semisimple.
$R(\mathrm{U}_n) = \mathrm{U}_n$ because U_n is unipotent, hence solvable; connected; normal in itself.

57. Let $\mathrm{char}(K) = 0$ and U a commutative, unipotent group. Show that $U \cong \mathbb{G}_a^r$ for some $r \in \mathbb{N}$.

Solution: One way is to use exponential and logarithm for matrices (introduced in the classification of connected, 1-dimensional groups). Consider $U \subseteq \mathrm{U}_n$ for some n (Kolchin). The map $\log\colon U \to \mathfrak{n}$ is then polynomial, i.e. a morphism of affine varieties, where $\mathfrak{n}$ is an additive subgroup of nilpotent matrices. As U is commutative, the logaritm is even a group homomorphism, and its inverse is exponential. Now $\mathfrak{n}$ is a K-vector space, so in particular isomorphic to $\mathbb{G}_a^r$ for some $r \in \mathbb{N}$.

58. Let G be a finite group and A a trivial G-module, i.e. an abelian group which has the trivial left G-action. Show that $H^1(G, A) = \mathrm{Hom}(G, A)$.

Solution: This is an obvious application of the definitions: note that trivial G-action implies $B^1(G, A) = 0$, so that $H^1(G, A) = Z^1(G, A) = \mathrm{Hom}(G, A)$, since the 1-cocycle condition collapses $\varphi(g_1 g_2) = \varphi(g_1) + g_1(\varphi(g_2)) = \varphi(g_1) + \varphi(g_2)$.

59. Let $\varphi\colon G \to H$ be a surjective morphism of linear algebraic groups, and $T \subseteq G$ a maximal torus (or a maximal connected normal unipotent subgroup, respectively). Show that $\varphi(T) \subseteq H$ has the same property.

Solution: Humphreys §21.3

60. Let (E, Φ) be a root system. A *base* of (E, Φ) is a subset $S \subseteq \Phi$ such that S is a basis for E and every root $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha \in S} m_\alpha \alpha$ with either all $m_\alpha \geq 0$ or all $m_\alpha \leq 0$. The elements of S are also called *simple roots*, and their non-negative linear combinations *positive roots* and denoted Φ^+.
Draw bases and positive roots for the root systems in $E = \mathbb{R}^2$, of exercise 55.

Algebra II — solutions to exercise sheet 13

61. Let G be a connected algebraic group, $B \subseteq G$ a Borel subgroup and V a rational G-module. Show that the invariant subspaces coincide: $V^G = V^B$.

Solution: Trivially, $V^G \subseteq V^B$. Let $v \in V^B$. We look at the morphism $G \to G \cdot v, g \mapsto gv$. Because of $v \in V^B$, this map factors through $G/B \to G \cdot v$. However, G/B is projective and the orbit $G \cdot v$ is affine; both varieties are connected. Hence, $G/B \to G \cdot v$ is constant, and $v \in V^G$.

62. Let G be a linear algebraic group. Show that $R(G)_u = R_u(G)$, i.e. the unipotent part of the radical is the unipotent radical.

Solution: Springer 7.6.3

63. For a linear algebraic group G, show that $G/R(G)$ is semisimple and that $G/R_u(G)$ is reductive.

Solution: We know that the surjective group homomorphism $\pi\colon G \to G/R(G)$ maps Borel subgroups to Borel subgroups. Then $R(G) = (\bigcap_B B)^\circ$, by a characterisation of the radical. Hence $G/R(G)$ has trivial radical by

$$\Big(\bigcap_{\pi(B)} \pi(B)\Big)^\circ \subset \pi(R(G)).$$

For the statement about $G/R_u(G)$, show that if an extension $1 \to G' \to G \to G'' \to 1$ the outer groups G' and G'' are unipotent, then so is the middle one G. (This follows from Jordan decomposition.) Now the preimage of unipotent and normal subgroup $\bar{U} \subseteq G/R_u(G)$ in G is $R_u(G) \subseteq U \subseteq G$. Then U is normal in G and also unipotent (from the extension $1 \to R_u(G) \to U \to U/R_u(G) \to 1$, hence $R_u(G) = U$.

64. Let G be a group with subgroups $H, N \subseteq G$. Show that the following notions are equivalent — then $G = N \rtimes H$ is called a *semi-direct* product of H by N:

(1) There is a short exact sequence $1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} H \to 1$ admitting a section $\sigma\colon H \to G$, i.e. $\pi\sigma = \mathrm{id}_H$.

(2) N is normal in G, and $NH = G$ and $N \cap H = 1$.

(3) There is a homomorphism $\alpha\colon H \to \mathrm{Aut}(N)$ and G is isomorphic to the group $N \rtimes_\alpha H$ defined by $(n,h)\cdot(n',h') := (n\alpha(h)(n'), hh')$ on the set $N \times H$.

Also show that the existance of a retraction $\varrho\colon G \to N$, i.e. $\varrho\iota = \mathrm{id}_N$, is equivalent to a splitting $G \cong N \times H$ of G as a direct product.

Solution: (1) $\iff$ (2): Given the split extension, then $N = \ker(\iota$, hence is normal in G. Given $g \in G$, then $\pi\sigma\pi(g) = \pi(g)$, so $\sigma\pi(g) = gn$ for some $n \in N$; altogether $g = n^{-1}\sigma\pi(g) \in NH$. Finally, if $g \in N \cap \sigma(H)$, then $g = \sigma(h)$, hence $1 = \pi(g) = \pi(\sigma(h)) = h$; so $N \cap \sigma(H) = 1$. The reverse implication is proved similarly.
(1) $\Rightarrow$ (3): The homomorphism $\alpha\colon H \to \mathrm{Aut}(N)$ is defined as follows: for $h \in H$ and $n \in N$, let $\alpha(h)(n) = \sigma(h)n\sigma(h)^{-1}$. Then $\pi(\alpha(h)(n)) = \pi\sigma(h)\cdot\pi(n)\cdot\pi\sigma(h)^{-1} = h \cdot 1 \cdot h^{-1} = 1$, hence $\alpha(h)(n) \in N$.

65. Show that the following subsets define root systems of rank n:

$\{e_j - e_i \mid i,j \in \{1,\dots,n+1\}, i \neq j\} \subset \mathbb{Q}^{n+1}$ (called type A_n)

$\{\pm e_j \pm e_i \mid i,j \in \{1,\dots,n\}, i < j\} \subset \mathbb{Q}^n$ (called type D_n)

Algebra II — solutions to exercise sheet 14

66. Show that centraliser of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C})$ is disconnected.

Solution: In fact, the centraliser of $A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in SL_2 is disconnected (th exercise follows from this). Now $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad+bc & -2ab \\ 2cd & -(ad+bc) \end{pmatrix}$, so $M = A \iff ad+bc = 1, ab = cd = 0$, leading to

$$C_A(\mathrm{SL}_2) = \{\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}\} \cup \{\begin{pmatrix} 0 & b \\ -1/b & 0 \end{pmatrix}\},$$

which is a disjoint union.

67. For $B \subseteq G$ a Borel subgroup of a connected linear algebraic group G, show that $Z(B) = Z(G)$.

Solution: Trivially, $Z(B) = C_B(B) \subseteq C_G(B) \subseteq C_G(G) = Z(G)$. Let $g \in Z(G)$. Then $g \in B'$ for some Borel subgroup $B' \subseteq G$ (because Borel groups cover G). Since Borel subgroups are conjugated, there is an $h \in G$ with $B' = hBh^{-1}$. As g is central, we have $g = h^{-1}gh \in B$.

68. Prove directly for $G = \mathrm{SL}_n$ and for $G = \mathrm{SO}_n$ (with the bilinear form from Problem 31) that G is covered by Borel subgroups and that maximal tori coincide with their centralisers.

Solution: $G = \mathrm{SL}_n$ with $B = \mathrm{T}_n \cap \mathrm{SL}_n$ (upper triangular matrices of determinant 1). If $M \in \mathrm{SL}_n$ is any matrix, then there is a base change matrix $g \in \mathrm{GL}_n$ such that gMg^{-1} is upper triangular (this is a weak form of the Gauss algorithm). Moreover, we can assume that $\det(B) = 1$ (because K is algebraically closed). Hence $gMg^{-1} \in B$ or, equivalently, $M \in g^{-1}Bg$, a Borel subgroup of G.
$T = \mathrm{D}_n \cap \mathrm{SL}_n$ (diagonal matrices of determinant 1) is a maximal torus in G. Let $d = \mathrm{diag}(d_1, \dots, d_n) \in T$ and $g \in G$. Then gd is obtained from g by multiplying the i-th row of g by d_i, whereas in dg the i-th column of g is multiplied by d_i. If $g_{ij} \neq 0$, then we get $d_i g_{ij} = d_j g_{ij}$, hence $g_{ij} = 0$ unless $d_i = d_j$. If $g \in C_G(T)$, then we have $dg = gd$ for all $d \in T$, hence g is a diagonal matrix.

$G = \mathrm{SO}_n$ with $B = \mathrm{T}_n \cap \mathrm{SO}_n$ a Borel subgroup. Here, any matrix can be brought into upper triangular form using the Gram-Schmidt algorithm. Hence Borel subgroups again cover G.
The argument for $C_G(T) = T$ proceeds as above (we can work with diagonal tori by choosing an appropriate symmetric bilinear form).

69. Let G be a connected linear algebraic group with a maximal torus and $\mathfrak{B}$ be the set of Borel subgroups of G with its natural T-action. Show that there is a bijection between the fixed point set $\mathfrak{B}^T$ and the Weyl group $W(G)$.

Solution: We let $W = N_G(T)/C_G(T)$ act on $\mathfrak{B}^T$ by $n \cdot B := nBn^{-1}$. In order to show that this is well-defined, we first prove that $C_G(T) \subseteq \bigcap_{B \in \mathfrak{B}^T} B$. As $C_G(T)$ is connected (all torus centralisers are) and nilpotent (because it has the unique maximal torus T), it is solvable and hence contained in some $B \in \mathfrak{B}^T$. For another $B' = gBg^{-1} \in \mathfrak{B}^T$, we have maximal tori $T, gTg^{-1} \subseteq B'$. Hence these are conjugated to each other under some $b' \in B'$. Then $C_G(T) = C(b'gTg^{-1}b'^{-1}) = b'gC_G(T)g^{-1}b'^{-1} \subseteq B'$.
Let $B, B' \in \mathfrak{B}^T$, i.e. $T \subseteq B, B' \subseteq G$. By conjugacy of Borel subgroups, there is $g \in G$ with $B = gB'g^{-1}$. Then $T \subseteq B$ and $gTg^{-1} \subseteq B$ are maximal tori, so by conjugacy of maximal tori in B, there is $b \in B$ with $bgTg^{-1}b^{-1} = T$. Hence $n := bg \in N_G(T)$. Then also $nB'n^{-1} = B$, so $\overline{n} \in W$ maps $B' \mapsto B$. Hence W acts transitively on $\mathfrak{B}^T$.
Assume now that some $B \in \mathfrak{B}^T$ is fixed under W. So we want to show that $nBn^{-1} = B$ implies $n \in C_G(T)$. But $nBn^{-1} = B$ implies $n \in N_G(B) = B$ by the theorem about normalisers of Borel subgroups, so $n \in B \cap N_G(T) = N_B(T) = C_B(T) \subseteq C_G(T)$, where the last equality follows from the semi-direct decomposition $B = B_u \rtimes T$ and Jordan decomposition.

70. Show that the following subsets define root systems of rank n:

$$\{\pm e_i \mid i \in \{1, \dots, n\}\} \cup \{\pm e_j \pm e_i \mid i, j \in \{1, \dots, n\}, i < j\} \subset \mathbb{Q}^n \text{ (type } B_n)$$
$$\{\pm 2e_i \mid i \in \{1, \dots, n\}\} \cup \{\pm e_j \pm e_i \mid i, j \in \{1, \dots, n\}, i < j\} \subset \mathbb{Q}^n \text{ (type } C_n)$$

5 About the exam

Topics in descending importance.

Calculations with matrices and matrix groups

For example, for any explicitly defined matrix group G, you could check that it is a linear algebraic group, compute is centre, commutator subgroup, Lie algebra etc. Or check for properties such as solvable, unipotent, Borel etc.

Exercises along these lines have been 6, 11, 16, 17, 18, 32, 36, 51, 52, 68.

Properties of linear algebraic groups, orbits and homogeneous spaces

There have been some recurring arguments that you should be able to apply as well: embedding a group in GL_n; Chevalley lemma; Jordan decomposition; the structure of diagonalisable and connected solvable groups; Borel fixed point theorem (and a map from a connected, projective variety to an affine variety is constant); conjugacy of Borel subgroups and of maximal tori.

Exercises 14, 15, 46, 49, 53, 61, 67, 69.

Basic algebraic geometry

Affine geometry: Zariski topology, singular points, tangent spaces.
Projective geometry: projective space, Grassmannians, proper morphisms.

Exercises: 13, 35, 43, 47.

Lie algebras and adjoint representation

You should know the definitions of the classical groups $(\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_n)$ and of their Lie groups. If the non-standard bilinear forms are needed, I will list them; you don't have to memorise these.

Exercises 22, 27, 30.

Matriculation number

1	2	3	4	5	6	7	8	$\sum$

Exam Algebra II

K is an algebraically closed field. All algebras and varieties are defined over K.

1. An *affine transformation* of $\mathbb{A}^2$ is a bijection $\mathbb{A}^2 \xrightarrow{\sim} \mathbb{A}^2$ mapping affine lines to affine lines.
a) Show that the set G of affine transformations of $\mathbb{A}^2$ is a linear algebraic group.
b) Describe a maximal torus in G.

2. Prove that a connected linear algebraic group consisting of semisimple elements is a torus. Show by example that the connectedness assumption is necessary.

3. Let G be a connected solvable linear algebraic group. Does G have a unique maximal torus? (Proof or counterexample.)

4. Compute the Lie algebra $\mathfrak{sl}_2$ of SL_2 from first principles, choose a basis and compute the matrix of $\mathrm{ad}(g) \colon \mathfrak{sl}_2 \to \mathfrak{sl}_2$ for arbitrary $g \in \mathrm{SL}_2$ in the chosen basis.

5. Let B be a non-degenerate symmetric bilinear form on K^{2n}. Show that the set of full isotropic flags $V^0 \subset V^1 \subset V^2 \subset \ldots \subset K^{2n}$ (i.e. $\dim(V^i) = i$ and $B|_{V^i \times V^i} = 0$ for all i) is a projective variety.

6. Compute the roots of the group Sp_4. You can use that this group has rank 2.

7. Let $B \subseteq G$ be a Borel subgroup of a connected linear algebraic group G. Show that the centres coincide: $Z(B) = Z(G)$.

8. Let $G = \mathrm{GL}_n$ and $B \subset G$ a Borel subgroup. Let W be the Weyl group of G.
a) Show $G = \bigcup_{w \in W} BwB$. (Explain first that BwB is a well-defined subset of G.)
b) Show that the union is disjoint.

Solutions for exam

1. An *affine transformation* of $\mathbb{A}^2$ is a bijection $\mathbb{A}^2 \xrightarrow{\sim} \mathbb{A}^2$ mapping affine lines to affine lines.
a) Show that the set G of affine transformations of $\mathbb{A}^2$ is a linear algebraic group.
b) Describe a maximal torus in G.

Solution: a) First solution: It is clear that G is a group. Let us check first that $G \subset \operatorname{Aut}(\mathbb{A}^2)$ (automorphisms as an affine variety). For this, we have to check $g^*(f) = fg \in A(\mathbb{A}^2) = K[x,y]$. If $\deg(f) = 1$, i.e. $f = ax+by+c$ for some $a,b,c \in K$, then we know $g^*(f)$ is again a polynomial of degree 1 — this is condition that G preserves affines lines (i.e. zero sets $V(ax+by+c)$). Since g^* is obviously a ring automorphism (of the ring of all K-valued functions on $\mathbb{A}^2$, for example), we see that g^* preserves $A(\mathbb{A}^2)$, as every polynomial is generated from linear forms. In particular, we have $g^* \in \operatorname{Aut}(\mathbb{A}^2) = \operatorname{Aut}(K[x,y])$, but the above argument shows even more: $g^* \in \operatorname{Aut}(K + Kx + Ky) = \mathrm{GL}_3$.

Second solution: If $\varphi \in G$ is arbitrary with $\varphi(O) \neq O$ (where $O = (0,0) \in \mathbb{A}^2$ is a fixed origin), then we consider the translation $v\colon \mathbb{A}^2 \to \mathbb{A}^2, x \mapsto x - \varphi(O)$. Translations are affine transformations, so $v\varphi \in G$ with $v\varphi(O) = O$. Hence $v\varphi$ is an affine transformation of $\mathbb{A}^2$ fixing O, hence it is a linear automorphism of K^2, i.e. $v\varphi \in \mathrm{GL}_2$.
There are two natural subgroups in G: translations $\mathbb{G}_a^2 \cong V \subset G$ and linear transformations $\mathrm{GL}_2 = G^O \subset G$ (the subgroup fixing O). Then $V \cap \mathrm{GL}_2 = 1$ and $VG^O = G$ by the above argument. An immediate computation shows that G^O is a normal subgroup of G: for $g \in G$ and $v(x) = x + v$ the translation along $v \in \mathbb{A}^2$, we have $g^{-1}vg(x) = g^{-1}(g(x)+v) = x + g^{-1}(v)$, which is again a translation.
Altogether we see that the abstract group G has a semi-direct product decomposition $G = G^0 \rtimes V$. Since $G^O \cong \mathrm{GL}_2$ and $v \cong \mathbb{G}_a^2$ are affine varieties,so is their product (which is the underlying set of G). The composition is given by linear functions, so in particular is polynomial.

b) Here we use the semi-direct decomposition from above. Since G_a is unipotent, so is $V \cong \mathbb{G}_a^2$. Tori consist of semisimple elements, hence any torus in G has to be disjoint from V. We find that a maximal torus in G is a maximal torus in GL_2, for example D_2.

2. Prove that a connected linear algebraic group consisting of semisimple elements is a torus. Show by example that the connectedness assumption is necessary.

Solution: Let $G = G_s$ be connected. We have to show that G is commutative (because a torus is, by definition, a connected commutative group all of whose elements are semisimple).
Let $B \subseteq G$ be a Borel subgroup. Then B is connected solvable, so has a semi-direct product decomposition $B = B_u \ltimes T$ where $T \subseteq B \subseteq G$ is a maximal torus. From the assumption $G = G_s$ we know $B_u = 1$, so $B = T$. In particular, B is commutative and hence nilpotent. This forces $G = B$. (In the course, we have seen that if a Borel subgroup is nilpotent (or normal), then it is the whole group.)
Connectedness is necessary: any finite group consists of semisimple elements (this uses $\operatorname{char}(K) = 0$ and Jordan decomposition).

3. Let G be a connected solvable linear algebraic group. Does G have a unique maximal torus? (Proof or counterexample.)

Solution: G does not necessarily have only one maximal torus. (In fact, we have shown that *connected nilpotent* groups are characterised by having a unique maximal torus.)
A counterexample can be found in $B := \mathrm{T}_2 \cap \mathrm{SL}_2$: this is a Borel subgroup of SL_2, so is conncted solvable. It has the standard maximal torus $T := \mathrm{D}_2 \cap \mathrm{SL}_2 = \{\left(\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right)\}$ but another torus is $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) T \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) = \{\left(\begin{smallmatrix} x & 1/x - x \\ 0 & 1/x \end{smallmatrix}\right)\}$.

4. Compute the Lie algebra $\mathfrak{sl}_2$ of SL_2 from first principles, choose a basis and compute the matrix of $\operatorname{ad}(g)\colon \mathfrak{sl}_2 \to \mathfrak{sl}_2$ for arbitrary $g \in \mathrm{SL}_2$ in the chosen basis.

Solution: By definition, $\mathrm{SL}_2 = V(x_1x_4 - x_2x_3 - 1) \subset \mathbb{A}^4$. We want to compute the tangent space at the point $e = (1,0,0,1)$. For a variety cut out by a single polynomial f and a point p, the tangent space at p is given by the vanishing locus of $\sum_i \partial f/\partial x_i(p)\, x_i = 0$. In our situation, we get the linear equation $x_1 + x_4 = 0$, i.e. the Lie algebra $T_e\mathrm{SL}_2 = \mathfrak{sl}_2 = \{M \in M(2,K) \mid \operatorname{tr}(A) = M\}$ is given by traceless matrices.
A (very typical) basis for $\mathfrak{sl}_2$ consists of the three matrices $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.
A group element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts on a tangent vector $M \in \mathfrak{sl}_2$ by $\operatorname{ad}(g)(M) = gMg^{-1}$. Hence we get

$$\begin{aligned}
\operatorname{ad}(g)(H) &= \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) &= \left(\begin{smallmatrix} ad+bc & -2ab \\ -2cd & -bc-ad \end{smallmatrix}\right) &= (ad+bc)H - 2abX - 2cdY; \\
\operatorname{ad}(g)(X) &= \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) &= \left(\begin{smallmatrix} -ac & a^2 \\ -c^2 & ac \end{smallmatrix}\right) &= -acH + a^2X - c^2Y; \\
\operatorname{ad}(g)(Y) &= \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) &= \left(\begin{smallmatrix} bd & -b^2 \\ d^2 & -bd \end{smallmatrix}\right) &= bdH - b^2X + d^2Y.
\end{aligned}$$

Hence the matrix of $\mathrm{ad}(g)$ in the basis (H, X, Y) is

$$\begin{pmatrix} ad+bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ -2cd & -c^2 & d^2 \end{pmatrix}.$$

5. Let B be a non-degenerate symmetric bilinear form on K^{2n}.
Show that the set of full isotropic flags $V^0 \subset V^1 \subset V^2 \subset \ldots \subset K^{2n}$ (i.e. $\dim(V^i) = i$ and $B|_{V^i \times V^i} = 0$ for all i) is a projective variety.

Solution: We write F_n for the set of isotropic flags in K^{2n}. Because B is non-degenerate, the maximal dimension of an isotropic subspace in K^{2n} is n. Now a flag $V^\cdot \in F_n$ is partial flag (of ordinary subspaces) in K^{2n}; the latter is given by the partial flag manifold $X_n := \mathrm{GL}_{2n}/P$ where the parabolic subgroup has block form $P = \left(\begin{smallmatrix} \mathrm{T}_n & * \\ 0 & * \end{smallmatrix}\right)$. In particular, X_n is a projective variety. The subset $F_n \subset X_n$ is closed: B is a bilinear form, so isotropicity is a polynomial condition. Hence, F_n is projective.

6. Compute the roots of the group Sp_4. You can use that this group has rank 2.

Solution: With $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, the symplectic group is defined by $\mathrm{Sp}_4 = \{g \in \mathrm{GL}_4 \mid g^t J g = J\}$.
It is an easy computation that $\mathrm{diag}(x, y, x^{-1}, y^{-1}) \in \mathrm{Sp}_4$, hence $T := \mathrm{D}_4 \cap \mathrm{Sp}_4$ is a 2-dimensional torus, so is a maximal torus (because we can use that Sp_4 has rank 2). As Sp_4 is given by a bilinear form, its Lie algebra is

$$\mathfrak{sp}_4 = \{X \in M(4, K) \mid X^t J + JX = 0\} = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \mid B^t = B, C^t = C, D = -A^t\}.$$

In particular, $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right), D = \left(\begin{smallmatrix} 0 & d \\ -d & 0 \end{smallmatrix}\right), B = \left(\begin{smallmatrix} b_1 & b_2 \\ b_2 & b_3 \end{smallmatrix}\right), C = \left(\begin{smallmatrix} c_1 & c_2 \\ c_2 & c_3 \end{smallmatrix}\right)$.
The adjoint action of Sp_4 (and hence that of T) on $\mathfrak{sp}_4$ is given by $\mathrm{ad}(g)(X) = gXg^{-1}$. We explicitly compute

$$\begin{aligned}
\mathrm{ad}(t)(X) &= \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & x^{-1} & 0 \\ 0 & 0 & 0 & y^{-1} \end{pmatrix} \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_2 & b_3 \\ c_1 & c_2 & -a_1 & -a_3 \\ c_2 & c_3 & -a_2 & -a_4 \end{pmatrix} \begin{pmatrix} x^{-1} & 0 & 0 & 0 \\ 0 & y^{-1} & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix} \\
&= \begin{pmatrix} a_1 & xy^{-1}a_2 & x^2 b_1 & xyb_2 \\ x^{-1}ya_3 & a_4 & xyb_2 & y^2 b_3 \\ x^{-2}c_1 & x^{-1}y^{-1}c_2 & -a_1 & -x^{-1}ya_3 \\ x^{-1}y^{-1}c_2 & y^{-2}c_3 & -xy^{-1}a_2 & -a_4 \end{pmatrix}
\end{aligned}$$

and we are now looking for 8 eigenvectors relative to characters (i.e. functions $x^m y^n$). They are easy to see:

character	weight	eigenvector (all other matrix entries 0)
1	0	$a_1 = 0$ or $a_4 = 0$
xy^{-1}	$e_1 - e_2$	$a_2 = 1$
$x^{-1}y$	$e_2 - e_1$	$a_3 = 1$
x^2	$2e_1$	$b_1 = 1$
xy	$e_1 + e_2$	$b_2 = 1$
y^2	$2e_2$	$b_3 = 1$
x^{-2}	$-2e_1$	$c_1 = 1$
$x^{-1}y^{-1}$	$-e_1 - e_2$	$c_2 = 1$
y^{-2}	$-2e_2$	$c_3 = 1$

Therefore, the roots (i.e. the non-zero weights) are $\pm 2e_1, \pm 2e_2, \pm(e_1 - e_2), \pm(e_1 + e_2)$.

7. Let $B \subseteq G$ be a Borel subgroup of a connected linear algebraic group G. Show that the centres coincide: $Z(B) = Z(G)$.

Solution: This is exercise 67.

8. Let $G = \mathrm{GL}_n$ and $B \subset G$ a Borel subgroup. Let W be the Weyl group of G.
a) Show $G = \bigcup_{w \in W} BwB$. (Explain first that BwB is a well-defined subset of G.)
b) Show that the union is disjoint.

Solution: a) We choose the standard torus and Borel subgroup: $T = \mathrm{D}_n \subset B = \mathrm{T}_n \subset \mathrm{GL}_n$. The Weyl group of GL_n is the permutation group $W = N_G(T)/C_G(T) = N_G(T)/T \cong S_n$. In particular, from $C_G(T) = T$ (which

has been an exercise for $T = \mathrm{D}_n \subset \mathrm{GL}_n$ and is easy to check again) and $T \subset B$, we see that $nB = ntB$ for $n \in N_G(T)$ and $t \in T$. Note that W is the subgroup of permutation matrices in GL_n. (This is a special feature of GL_n — for other reductive groups, the Weyl group is not necessarily a subgroup in a canonical way!)
One way to see $\mathrm{GL}_n = BWB$ is via the Gauss algorithm: bringing an arbitrary matrix $g \in \mathrm{GL}_n$ in row echelon (hence upper triangular) form starts with $(g \mid I_n)$ and uses row transformations to end in $(U \mid L)$ with $Lg = U$ where U is an upper triangular matrix and L is a lower triangular matrix, i.e. $U, L^t \in B$. However, in general one has to apply a permutation (e.g. if the top left entry of g is zero); i.e. there is a permutation matrix $w_1 \in W$ such that the row algorithm works on $w_1 g$ (using $w_1 g$ means that we permute rows of g). We record that for any $g \in G$, there are $w_1 \in W, L \in B^t, U \in B$ with $wg = L^{-1}U$.
We arrive at $g = w_1^{-1}L^{-1}U$. Now there is a unique permutation matrix $w_2 \in W$ such that $w_1^{-1}L^{-1}w_2 \in B$ — i.e. we undo the row permutations of w_1^{-1} by appropriate column permutations of w_2. Eventually, we get $g = w_1^{-1}L^{-1}w_2w_2^{-1}U = U'wU \in BwB$ where $U' := w_1^{-1}L^{-1}w_2 \in B$ by construction and $w := w_2^{-1}$.

b) If $BwB \cap Bw'B \neq \varnothing$, then there exist $b_1, b_2, b_3, b_4 \in B$ such that $b_1wb_2 = b_3w'b_4$, hence $b_3^{-1}b_1wb_2b_4^{-1} = w'$ and $w' \in BwB$. This immediately implies $Bw'B \subseteq BwB$. The other inclusion follows by symmetry.