§2 The Model $T^c$

In this section we provide more justification for the claim, made in §11, that weak iterability holds for the stager $N^c$, $M^c$ in the construction of $K^c$. We also discuss weakenings of the background condition used in that construction. Steel can weaken it considerably while still proving that each $N^c$, $M^c$ is a weak MS-mouse. Our form of weak iterability seems to require a bit more.

**Def** Let $F$ be an extender. By a background certificate for $F$ we mean a pair $<N, F^*>$ s.t.

(a) $N$ is a transitive $ZFC^-$ model with $\mathcal{V}_\kappa \subseteq N$, $\omega \subseteq N$ (where $\kappa = \text{crit}(F)$).

(b) $F^*$ is an extender at $\kappa$ on $N$ s.t.

\[ \mathcal{V}_{\lambda + 1} \subseteq \text{ Ult}(N, F^*) \] (where $\lambda = \text{lh}(F)$).

(c) $F \cap N = (F^* \upharpoonright \text{dom}(F))|\lambda$.

(\textit{Note} Then $\kappa = \mathcal{V}_\kappa$ and $\kappa > \omega$).
Def Let $F$ be an extender. $F$ fixes a set $U$ iff there is a map $h : U \lambda \rightarrow \kappa$ n.t. for all $X \subseteq U \lambda$, and all $x_1, \ldots, x_n \in U \lambda$ we have:

$\langle h(x_1), \ldots, h(x_n) \rangle \in X \leftrightarrow \langle x_1, \ldots, x_n \rangle \in F(X)$.

$F$ is $\omega$-complete iff $F$ fixes every countable $U$.

**Lemma 2** Let $\langle N, F^* \rangle$ be a background certificate. Then $F^*$ is $\omega$-complete.

**Proof.**

Set $U_1 = (\cup U \lambda \lambda \neq N). Then U_1 \subseteq N$ by $\omega$-closure. Let $\langle \delta^n \mid n < \omega \rangle$ enumerate $U \lambda \lambda$. Set $A =$ the set of $\langle \delta^n_0, \ldots, \delta^n_{i-1} \rangle \in \in \cap \omega$ n.t. for all $h \leq m$ and $X \subseteq U_1$:

$\langle \delta^n_0, \ldots, \delta^n_{i-1} \rangle \in X \leftrightarrow \langle \delta^n_0, \ldots, \delta^n_{i-1} \rangle \in F^*(X)$.

(1) $A \in N$

**Proof.**

For $u = \langle i_0, \ldots, i_{m-1} \rangle \in \omega^m$ let:

$F_u = \{ X \subseteq U_1 \mid \langle i_0, \ldots, i_{m-1} \rangle \in F^* (X) \}$.

Then $F_u \in N$ and $\langle F_u \mid u \in \omega^m \rangle \in N$ by $\omega$-closure. Hence $A \in N$.

$QED (1)$
Let \( R = \{ \langle u, v \rangle \mid u, v \in A \land u \not\leq v \} \). The claim reduces to:

(2) \( R \) is not well founded.

**Proof.** Let \( \pi : N \to \mathcal{P}(R) \).

It suffices to show: \( \pi'(R) \) is not well founded. By absoluteness of well-foundedness it is enough to show \( \pi(R) \) is not well founded. But this is trivial, since \( \langle \zeta | i < \omega \rangle \) is a branch through \( \pi(R) \). QED (Lemma 2)

**Corollary 2.1** Let \( \langle N, F^* \rangle \) be as above.

Let \( U = \Pi F_{\xi}(u) \mid N \in N \land \exists \xi. U_{\xi} < \omega \) and let \( U_0 = U \lambda \) be countable.

Then \( F^* \) fixes \( U \).

**Proof.** It suffices to show: \( A \in N \), since the rest of the proof is as before. \( F \in N \) for \( u \in \omega^{<\omega} \), since \( F \subseteq U \), where \( U \in N, U_{\xi} < \omega \) and \( \forall u \in N. \text{Hence } \langle F_{u \mid u \in \omega^{<\omega}} \rangle \subseteq N \) as before. Hence \( A \in N \). QED (2.1)
In place of §9 Lemma 4 we shall use:

Lemma 3. Let $\sigma : \overline{M} \rightarrow^\Sigma^* M \min (\bar{\rho}^*).$

Let $F$ be an extender on $M$ at $\nu$, $\lambda$, where $\nu$ is regular. Let:

$\langle \sigma, q \rangle : \langle \overline{M}, F \rangle \rightarrow^\ast \langle M ! \rho, F \rangle,$

where $F$ is weakly amenable (hence close to $\overline{M}$). Let $U_1 \subset \beta(\nu) \wedge M$ s.t.

$\text{rng}(f) \subset U_1$ whenever $f : \overline{\alpha} \rightarrow^\Sigma^* (\nu \wedge \lambda)$ is $M$-definable in parameters from

$\text{rng}(\sigma) \cup \{ \rho | i < \omega \}$ and $\beta < \kappa.$

Let $U_0 \subset \lambda$ s.t. $\text{rng}(q) \subset U_0.$ Suppose, moreover that $\overline{M}$ has cardinality $< \kappa.$

Let $F$ fix $U = U_0 \cup U_1$ as witnessed by $k : U_0 \rightarrow^\nu \kappa.$ Then:

(a) There is $\overline{\sigma} : \overline{M} \rightarrow^\Sigma^* \overline{M}.$

(b) There is $\sigma' : \overline{M} \rightarrow^\ast M \min (\bar{\rho}^*),\sigma'$ defined by $\sigma' (\overline{\sigma}(f)(x)) = \sigma(f)(kg(x))$

for $f \in \Gamma^*(\overline{\sigma}, \overline{M})$, $x < \lambda$ (where $\overline{F}$ is in $\alpha+\overline{\mu}, \overline{\lambda}$), (Hence $\sigma' \overline{\sigma} = \sigma$ and $\sigma' \overline{F} = kg.$)
Proof of Lemma 3:

Note first that each \( f \in \Gamma^*(\bar{\mu}, \bar{m}) \) has a uniquely defined \( \sigma(f) \mod (\bar{\rho}') \).

Let \( \langle \alpha, f \rangle \in D^* = D^*(\bar{m}, F) \). Let \( \bar{\rho} \) be \( \Sigma^m \) where \( \omega \bar{\rho}^m > \bar{\mu} \). Then

\[ \exists \bar{\xi} \mid \bar{m} = \bar{\rho}(f(\bar{\xi})) \exists \bar{\xi} \in \bar{F} \]

\[ \iff \exists \bar{\xi} \mid \bar{m} = \bar{\rho}(\sigma(f)(\bar{\xi})) \mod (\bar{\rho}') \exists \bar{\xi} \in \bar{F} \]

\[ \iff \exists \bar{\xi} \mid \bar{m} = \bar{\rho}(\sigma(f)(\bar{\xi})) \mod (\bar{\rho}') \exists \bar{\xi} \in \bar{F} \]

This verifies the existence of \( \bar{\tau} : \bar{m} \to \bar{m}' \) and of \( \sigma' : \bar{m}' \to \bar{m} \) defined as above. At this point, more over, that \( \sigma' : \bar{m}' \to \Sigma^m, \bar{m}' \to \bar{m} \) for \( \omega \bar{\rho}_m^m > \bar{\mu} \). We also note that

\[ \sigma' : \bar{m}' \to \Sigma^m, \bar{m} \mod (\bar{\rho}') \]

\[ \bar{\rho}' = \min (\bar{\rho}', \sigma', \bar{\rho}) \]

\[ = \min (\bar{\rho}', \sigma', \bar{\rho}) \]

Hence the theorem is proven if \( \omega \bar{\rho}_m^m > \bar{\mu} \). Now let:

1. \( \omega \bar{\rho}_m^{m+1} \leq \bar{\mu} < \omega \bar{\rho}_m^m \) in \( \bar{m} \).
By the regularity of $\kappa$, the minimal set of $\vec{\beta}$ and the fact that $\text{card}(\vec{M}) < \kappa$, we easily get:

$$(2) \quad \omega_{\vec{p}} < \kappa < \omega_{\vec{p}_{\vec{m}+1}}.$$ 

**Claim 1**

Let $\overline{R}(x, y) \in \Sigma^{(m)}_1(\vec{M})$ and $\overline{R}(x, y) \in \Sigma^{(m)}_1(M, \vec{\beta})$ by the same definition. Let $\bar{x} \in \vec{M}$, $\bar{x} = \pi'(\bar{x})$, Set:

$$\overline{P} = \{ \bar{s}^{m+1} \mid R(\bar{s}^{m+1}, x) \}, \quad P = \{ s^{m+1} \mid R(s^{m+1}, x) \}$$

Then $\overline{P} \in \Sigma^{(m)}_1(\vec{M})$ in a parameter $\bar{x}$ and $P \in \Sigma^{(m)}_1(M, \vec{\beta})$ in $x = \sigma(\bar{x})$ by the same definition.

**Proof.**

Let $\bar{x} = \pi(f \upharpoonright \kappa)$. Then $x = \sigma(f)(\kappa \bar{x}^{\kappa})$.

Suppose either $f = \overline{P}$ or $f$ is a good $\Sigma^{(m-1)}_1(\vec{M})$ function in the parameters $\overline{P}$ by a functionally absolute definition. Then $\sigma(f)$ has the same definition over $M \mod(\vec{\beta})$ in $P = \sigma(\bar{P})$.

We know:

$$<\sigma, q> : <\vec{M}, \vec{F}> \xrightarrow{\star \prec} <M_{1\bar{p}}, \vec{F}>.$$
Hence there are G, H s.t.

(i) $F, \overline{G}(\overline{u}) \cap \overline{M}$ are $\Sigma^m(\overline{M})$ in a parameter $q$ and $G, H$ are $\Sigma^m(M, \overline{m})$ in $q = \sigma(\overline{q})$ by the same definition.

(ii) $G \subseteq F, q(\overline{a})$

(iii) $H \subseteq \{ x \in \overline{\mathcal{Q}}(\overline{u}) \mid \Lambda i < u (x^i \in m \setminus x, i \in G) \}

We prove the claim for $\overline{z} = <\overline{u}, \overline{r}, \overline{q}>$, $\overline{z} = <\overline{u}, \overline{r}, \overline{q}>$. Let $R(s, x) \leftrightarrow V_{z_{\overline{m}}} \overline{Q}(z, s, x)$, where $\overline{Q}$ is $\Sigma^m(\overline{M})$. Let $\overline{H}$ have the same $\Sigma^m$ definition over $\overline{M}$. Then for $F \in H^m_{\overline{M}} = H^m_{\overline{M}}$, we have:

(1) $R(s, x) \leftrightarrow V_{z} H^m_{\overline{M}} V_{z \in \overline{\mathcal{Q}}(u)} \overline{Q}(z, s, x)$

$\overline{Q}$

$\overline{Q}$

$\overline{Q}$

Clearly it suffices to show:

Claim $\overline{R}(s, x) \leftrightarrow$

$\rightarrow V_{u} V_{z_{\overline{m}}} (u \in \overline{G} \setminus \overline{z} \in \overline{Q}(z, s, \sigma(\overline{f}(i)))$ $\land \sigma(\overline{G})$, where $\overline{Q}$ has
The same $\Sigma^m_0(M, \bar{\beta})$ def. as $Q$ over $M$. We first prove the easy direction:

(\leq) Let $u = \{ \delta < \kappa \mid \forall z \in u \, Q(z, \delta, \bar{\delta}(f)(\delta)) \in G \}$. Then $G \subseteq F_g(u)$, and hence $y(x) \in F(u)$. Hence $y(x) \in u$. Hence $Q(z, \delta, \bar{\delta}(f)(\delta) g(x))$ for $z \in H^m$, where $\bar{\delta}(f)(\delta) g(x) = x$. Hence $R(\delta, x)$.

QED

(\rightarrow) Assume $R(\delta, x)$. Then $\forall z \in u \, Q(z, \delta, x)$ for some $u \in H^m$. By the minimality of $\bar{\beta}$ we may assume w.l.o.g. that $u = h(\delta)$ for some $\delta < \omega_{\beta} < \kappa$, where $h$ is a $\Sigma^m_1(M, \bar{\beta})$ map in a parameter $x = \bar{\delta}(\bar{z})$ (since, by minimality, the set of such $u$ is cofinal in $H^m$).

Set:

$X(\delta, \bar{\delta}) = \{ \delta < \kappa \mid \forall z \in h(\delta) \, Q(z, \delta, \bar{\delta}(f)(\delta)) \}$ for $\delta < \omega_{\beta} < \kappa$. Set:

$X(\bar{z}) = \langle X(\bar{z}, \delta) \mid \delta < \kappa \rangle$. At is easily seen that $X$ is a partial
\[ \sum_{\alpha}^{(n)} (M, \rho^2) \text{ map from } \omega^m_{n+2} \text{ to } H^n \]

in the parameters \( r, \bar{r} \). Moreover, \( X(\bar{r}) \) is defined if \( h(\bar{r}) \) is defined. Let \( \bar{X} \) have the same functionally absolute \( \Sigma^{(m)}_{n} (M) \text{ definition in } \bar{r}, \bar{r} \). Then in \( \bar{M} \) we have:

\[ \exists \gamma^{m+2} \in \text{dom}(\bar{X}) \quad \bar{X}(\bar{r}) \in \bar{H}(\bar{a}) \]

This statement in \( \prod_1^{m+1}(\bar{M}) \) in \( \bar{r}, \bar{r} \) (using our \( \Sigma^{(m)}_{n} (M) \text{ def of } \bar{H} = \bar{H}(\bar{a}) \) in \( \bar{r} \)). Hence the corresponding \( \prod_1^{m+1}(M, \rho^2) \) statement holds in \( r, \bar{r} \):

\[ \exists \gamma^{m+2} \in \text{dom}(X) \quad X(5) \in H \]

Hence:

\[ X(S, \bar{s}) \in G \text{ or } (\bar{m} \cdot X(\bar{s}, \bar{s})) \in \bar{G} \]

But \( \bar{X} = \{ X(\bar{s}, \bar{s}) \mid \bar{s}, \bar{s} < \omega^m_{n+2} \} \) is \( M \)-definable in parameters from \( \Sigma^0_1 \) \( \cup \{ \rho_i \mid i < \omega \} \). Hence:

\[ \forall \bar{g}(x) \in X(\bar{s}, \bar{s}) \leftrightarrow X(\bar{s}, \bar{s}) \in \bar{F}_g(x) \]

for \( \bar{s}, \bar{s} < \omega^m_{n+2} \).
For our specific $\tilde{s}$, $\tilde{s}$ we have
$\delta g(x) \in X(\tilde{s}, \tilde{s})$, hence $X(\tilde{s}, \tilde{s}) \subseteq F_g(x)$. Hence $u \setminus X(\tilde{s}, \tilde{s}) \notin G \subseteq F_g(x)$. Hence $X(\tilde{s}, \tilde{s}) \in G$. Hence for $u = l(\tilde{s})$ we have $\mu \in H_n$ and
$\{z < u | \forall z \in u \in \mathcal{Q}(\tilde{s}, \tilde{s}) \}$ $\subseteq G$. QED ($\rightarrow$). QED (Claim 1).

By this we easily get:

Claim 2 \( \{z^{m+1} | R(z^{m+1}, x) \} \subseteq \Sigma_{n}^{-m}(C) \) in some $\tilde{s}$ and \( \{z^{m+1} | R(z^{m+1}, x) \} \subseteq \Sigma_{n}^{-m}(M, \rho) \) in $\tilde{s} = \sigma(\tilde{s})$ by the same definition.

But $\rho_{i}^{x} = \rho_{j}^{x}$ for $i > m$. It follows easily that:

Claim 3 Let $i > m$. Let $R$ be $\Sigma_{n}^{-i}(C')$ and $R$ be $\Sigma_{n}^{-i}(M)$ by the same definition. Then $\{z^{i} \mid R(z^{i}, x) \} \subseteq \Sigma_{n}^{-i}(C')$ in some $\tilde{s}$ and $\{z^{i} \mid R(z^{i}, x) \} \subseteq \Sigma_{n}^{-i}(M, \rho')$ in $\tilde{s} = \sigma(\tilde{s})$ by the same definition.
Now let $\varphi$ be a $\Sigma^i_1$ formula ($i \geq m$).
Let $\bar{x} \in \bar{M}'$, $x = \sigma'(\bar{x})$. The statement
$\bar{M}' \models \varphi(\bar{x}) \iff \Sigma^i_1(\bar{M})$ in a parameter $\bar{x}$
and $M \models \varphi(x) \mod(p') \iff \Sigma^i_1(M, p')$
in $\varphi = \sigma'(\bar{x})$. By the same definition.
Hence: $\bar{M}' \models \varphi(\bar{x}) \iff M \models \varphi(x) \mod(p')$,
since $\sigma': \bar{M} \to \Sigma^i_1 M \mod(p')$.
\textbf{QED (Lemma 3)}

We assume $\Theta$ to be inaccessible and
define $N_3, M_3$ ($3 < \Theta \leq \Theta$) as in §10
except that in Case 2.1 we relax the condition that $F$ be the restriction
of an extender $F^*$ on $V$. We instead require:

\begin{enumerate}
\item[(\textit{*})] \textbf{Let $s = \text{crit}(F)$. Whenever $3 < \kappa$ and $f : \delta \to \text{dom}(F)$, then $F$ has a background certificate $\langle N, F^* \rangle$ at $f \in N$.}
\end{enumerate}

(Note \textit{(*)} implies that $\kappa$ is regular, hence inaccessible.)
An order to define $\mathcal{K}^c = N = N_\Theta$ we show inductively that $N_3$ is defined and satisfies (a), (b) of §10 (p. 8). As before, (b) follows from (a). We must show:

**Lemma 4** $N_3$ is a weak mouse (i.e. if $\sigma : \mathcal{P} \rightarrow 2^{\Sigma^* N_3}$ and $\mathcal{P}$ is countable, then $\mathcal{P}$ is countably iterable).

(\textit{Note:} We of course assume AO of the previous section.)

As before this reduces to:

**Lemma 4.1** Let $\sigma : \mathcal{P} \rightarrow 2^{\Sigma^* N_3}$, where $\mathcal{P}$ is countable. Then $\mathcal{P}$ has a countable normal iteration strategy $S$.

Moreover if $\mathcal{J} = \langle \langle \mathcal{P}_i \rangle, \langle \chi_i \rangle, \langle \gamma_i \rangle, \langle \pi_i \rangle, \mathcal{T} \rangle$ is a countable normal $S$-iteration of length $\Theta + 1$, then:

(i) There is $\sigma' : \mathcal{P} \rightarrow 2^{\Sigma^* N_3}$, $\min(\mathcal{P})$ for $\delta \leq \delta < \bar{s}$.

(ii) If $\mathcal{T}_0 \sigma$ is not total, then $\delta < \bar{s}$.

(iii) If $\mathcal{T}_0 \sigma$ is total, then $\delta = \bar{s}$, $\bar{s}' = \bar{s}'$.

(\textit{Lemma 4 follows from Lemma 4.1 as Corollary 2 follows from Lemma 1 in §10.})
At we only need to prove that $\mathcal{N}_\gamma$ is a weak MS-mouse, then a weaker version of Lemma 4.1 will do. Steel sketches a proof of this in [5] for the case $\mathcal{L}(\gamma) = \omega$, making several other simplifying assumptions. We now show how Steel's sketch can be modified to give our result. Our simplifying assumptions are:

1. $\mathcal{L}(\gamma) = \omega$ and $\gamma$ has no truncations.

Since no truncations occur, we can constrained to find a cofinal branch $\mathcal{N}_\gamma$ of $\mathcal{N}_\gamma$. Following Steel we devise, for any map $\mathcal{I} : \mathcal{P} \to \mathcal{Q} \bmod(\mathcal{P}^d)$ a tree $\mathcal{U} = \Sigma \times \mathcal{Q} \bmod(\mathcal{P}^d)$ such that any branch through $\mathcal{U}$ gives a cofinal branch $\gamma$ and a $\sigma : \mathcal{P}_\gamma \to \mathcal{Q} \bmod(\mathcal{P}^d)$ s.t. $\sigma \mathcal{P}_\gamma = \mathcal{I}$. The definition is the same as in [5] (p. 10) except that in (b) the condition $\langle \mathcal{P}_\gamma, x_0, \ldots, x_k \rangle \equiv \langle \mathcal{Q}_\gamma, y_0, \ldots, y_k \rangle$ is interpreted in the $\Sigma^\ast$-language with
reference to $\Sigma^*$ formulas only, which in the case of $\Omega$ we interpret mod $(p^2)$.

4) $U = U(\bar{e}, \bar{Q}, p^2)$ and $\sigma: P_i \rightarrow \Sigma^* Q \min_p$

s.t. $\tau = \sigma P_i^c$, we define $P(c, \sigma, \tau, Q, p)$
exactly as Steel does.

We assume that no such branch $b$ and map $\sigma'$ exist. This means that $U = U(\bar{e}, N^b, p^2)$ is well founded.

We derive a contradiction. Following Steel we define $<\sigma_i^c, q_i^c, P_i^c, \bar{P}_i^c>$

s.t.

(1) $P_i = <P_i, e_i, \delta_i>$ is a coarse premouse
     in the sense of [5].

(2) $\sigma_i^c: P_i \rightarrow \Sigma^* Q_i \min (\bar{P}_i)$ and $Q_i$
     is a model on the sequence $\langle N^P_i \rangle$
     defined in $\bar{P}_i$ like $\bar{N}$ in $(\bar{N}, *)$.

We recall from §10 that if $\gamma \leq \text{ht}(N_\gamma)$
and $E_\gamma^{N_\gamma} \neq \emptyset$, then there exist
s.t.

and a canonical $k_\gamma: N_\gamma \mathbf{N} \rightarrow N_\delta^\gamma$.

If $\gamma = \text{ht}(N_\gamma)$, then $\delta = \gamma$ and

$k_\gamma = \text{id}$. Otherwise we still have:

$k_\gamma \upharpoonright \beta = \text{id}$ for all $\beta < \text{ht}(N_\gamma)$.
(We also write: 

\[ k_\nu = k_\nu, \gamma (\overline{N}) \]) Now set: 

\[ k^*_i = \left( k_{\sigma_i (\nu)}, \gamma (\overline{N}) \right) \overline{R}_i \] where \( Q_i = N^{R_i} \).

Set \( X^*_i = k^*_i \sigma_i (\lambda_i) \). We also require:

(3) \( \forall h \leq i \), \( X^*_i + 1 = \overline{V}_h \lambda^*_h + 1 \) and \( X^*_h < \lambda^*_h \).

(4) \( \sigma_i \cap X^*_h = k^*_h \sigma_h \cap \lambda^*_h \) for \( h \leq i \).

(Hence \( j < h \leq i \rightarrow \sigma_h \cap \lambda_j = \sigma_h \cap \lambda_j \).

Since \( \sigma_h (\lambda_j) \) is a cardinal in \( Q^*_h \), hence \( \omega^h \geq \sigma_h (\lambda_j) \) for \( \beta < ht (Q^*_h) \) and

\( k^*_h \cap \sigma_h (\lambda_j) = \{ \lambda_j \} \).

(5) Let \( U = U (\sigma_i \overline{P}_i, Q_i, \beta^i) \) and \( P = P (i, \sigma_i, \overline{P}_i, Q_i, \beta^i) \). Then 

\( U \) is well founded and the order type of the set of cutoff points of \( R_i \) is at least \( 1 \) plus \( U \).

(6) \( \overline{R}_i \in \overline{R}_{i-1} \) for \( i > 0 \).
As in [5] we let

\[ P_0 = \langle \theta^+ \cdot \delta, \theta \rangle \]  
where \( \delta \geq \text{H} \), \( U = U(\sigma, N, \theta) \),  
\( \sigma = \sigma^+ \), \( \theta_0 = N \), \( \theta^+ = \theta \).

Following [5] closely, we now define

\[ P_{i+1}, (\sigma_{i+1}), (\theta_{i+1}), \theta^{i+1} \].

We know that  
\( \sigma_i \cdot (\theta_i) \) is a cardinal in \( P_i \), \( (\sigma_i, \theta_i) \) is a cardinal in \( P_i \), where \( j = T(i+1) \), since there is no truncation. If \( i = j \) we are done. If not, \( \sigma_i \) is a cardinal in \( J \mathcal{E} P_i = J \mathcal{E} P_i \), and \( \lambda_i \) is a cardinal in \( P_i \). Hence \( \sigma_i, (\theta_i) \) is a cardinal in \( Q_i \). Hence \( k_i^* \cdot (\theta_i) = \lambda_i \), and a

\[ \omega^\beta \leq \sigma_i (\theta_i) \text{ for } \beta < \text{ht}(Q_i) \]

at \( Q_i \) if \( \beta \) follows that \( k_i^* \cdot (\theta_i) = \sigma_i (\theta_i) \). (To see this let \( \rho = \text{the minimal } \rho = \omega^\beta \)  
for a \( \beta < \text{ht}(Q_i) \). By the construction of \( k_i^* \) in §10 (as a composition of core maps) and by §8 Lemma 5 it follow that \( \rho \) is a cardinal in \( N_i \). But then \( k_i^* (\rho) = \rho \) if \( \rho = \sigma_i (\theta_i) \), hence \( \sigma_i (\theta_i) \) is a successor cardinal in \( Q_i \). )
Since \( \omega_\beta \geq \tilde{\sigma}_i(\tilde{\gamma}_i) \) for \( \beta < h t (Q_i) \),

the same argument shows:

\[
\tilde{k}_i \sigma_i \upharpoonright (\tilde{\gamma}_i + 1) = \sigma_i \upharpoonright (\tilde{\gamma}_i + 1),
\]

then:

\[
\sigma_i \upharpoonright (\tilde{\gamma}_i + 1) = \tilde{k}_i \sigma_i \upharpoonright (\tilde{\gamma}_i + 1) = \sigma_i \upharpoonright (\tilde{\gamma}_i + 1).
\]

Let \( \tilde{k}_i : Q_i \upharpoonright \sigma_i(\tilde{\gamma}_i) \rightarrow \tilde{\gamma}_i \) such \(\tilde{\gamma}_i \leq \gamma_i\). (Recall \( k_i = \text{id} \) if \( \tilde{\gamma}_i = \gamma_i \).)

Let \( \langle N, F \rangle \) be a background certificate for \( N_{\tilde{\gamma}_i} = \langle J_i^{E_{N_{\tilde{\gamma}_i}}} \upharpoonright \tilde{E}_i \rangle \)

in \( R_i \), chosen s.t. \( f \in N \) for every \( f: S \rightarrow Q_i \upharpoonright \sigma_i(\tilde{\gamma}_i) \) s.t. \( S \leq \sigma_i(\tilde{\gamma}_i) \) and \( f \in \sigma_i \)-definable in parameters from \( \text{rng}(\sigma_i) \cup \{ \beta \} \).

(There are only countably many.)

Set \( \tilde{\gamma}_i^* = k_i \sigma_i \). Let \( \pi_i : N \rightarrow \tilde{N}_i \).

Then \( \tilde{\lambda}_i, \tilde{\sigma}_i^* \in \tilde{N}_i \). Pick \( \beta < h t (R_i) \) s.t.

\[
\tilde{\sigma}_i^* = \pi_i(\tilde{\sigma}_i)(\beta), \quad \tilde{\chi}_i = \pi_i(\tilde{\chi})(\beta),
\]

where \( \tilde{\sigma}_i, \tilde{\chi} \) map \( \sigma_i(\tilde{\gamma}_i) \) into \( \tilde{\varphi}_i R_i \).

Following [5], we assert:
Claim: For $\mathcal{F}$ many $\xi$, there are \( \check{\nu}^N_{\xi_i}(\xi_i) \): A coarse promise \( R \)

in the sense of \([5,7]\), a model \( Q \) in the sense of \( \check{N}^R \), \( \sigma: \Gamma \rightarrow \mathbb{Z} \times \min(\check{\nu}^R) \)

such that:

(i) \( \check{\nu}^R = \check{\nu}^N \)

(ii) \( \sigma \upharpoonright \lambda_i = \check{\nu}^R(\lambda_i) \)

(iii) Let \( U = U(\sigma, \overline{\nu_o}, i+\nu, Q, \check{\nu}^R) \),

\( P = P(i+\nu, \sigma, \sigma, \overline{\nu_o}, i+\nu, Q, \check{\nu}^R) \). Then \( U \) is well founded and there are in order-type at least \( |P| \) many cutoff points of \( R \).

Before proving the Claim, we recall that in addition to (1)-(6), we need another property:

(7) \( \langle \sigma_i, \sigma_i \upharpoonright \lambda_i \rangle \cdot \langle P_i, E^{P_i} \rangle \xrightarrow{**} \langle Q_i, E_{\check{i}}^{Q_i} \rangle \)

Hence:

(7') \( \langle \sigma_i, \sigma_i \upharpoonright \lambda_i \rangle \cdot \langle P_i, E^{P_i} \rangle \xrightarrow{**} \langle Q_i, E_{\check{i}}^{*} \rangle \)

We pass over the verification.
of (67) which must be handled inductively in the manner of §10 Lemma 4.

We now prove the Claim. Suppose not; let $X$ be the set of $\beta$ for which the claim holds. Then $X \notin F$. We derive a contradiction. The main step is devising an appropriate map

$$\sigma : P \rightarrow \mathbb{Q}, \min \beta$$

Set $W_0 = \{ \beta \mid \exists \gamma \in \mathbb{Q}, \min \beta \}

W_1 = \text{The union of all run}_1 \{ f \}

\text{such that } f : \tilde{S} \rightarrow \mathbb{Q} \land \exists (\tilde{\sigma}, \tilde{\eta}, \tilde{\rho}) \text{ for } a \tilde{S} = \tilde{\sigma}_1(\tilde{\rho}, \tilde{\eta}) \text{ and } f \in \mathbb{Q} - \text{definable form run}_1(\tilde{\eta}) \cup \{ \rho : m < \omega \}

Then $W_0$ is countable, $W_1 \subset \mathbb{Q}$ and $W_1 \in \mathbb{N}$. Moreover $W_1 \subset \tilde{\sigma}_1(\tilde{\rho})$ in $\mathbb{N}$. Let $W_2$ be the set of $\gamma \subset \tilde{\sigma}_1(\tilde{\rho})$ which are $\mathbb{N}$-definable in $\mathbb{Q}$, $X$ and parameters from $T(C(X))$. Then $W_2 \in \mathbb{N} \mathbb{R}(\tilde{\sigma}_1(\tilde{\rho}))$. And
$X \subseteq W_2$ and $W_2$ is countable. Set $W = \mathcal{B} \cup W_0 \cup W_1 \cup W_2$. Let $\sigma : \mathcal{B} \cup W_0 \to \Sigma^{\omega}$ fix $W$ in the sense of our earlier definition.

Then:

$<r(b)'> \in X \iff <a'> \in F(X)$

for $a_1, \ldots, a_n \in \mathcal{B} \cup W_0$ and $X \subseteq W_1 \cup W_2$.

Clearly $\mathcal{B} \cap W_0$ fixes $W_0 \cup W_1$ w.r.t. $E_i$. Hence, by (7') we may define:

$\sigma' : \prod_{i+1} Q_i \times \Sigma^*$

by:

$\sigma'(\prod_{i+1} (f)(x)) = \sigma(f)(\sigma_1^*(x)).$

Now let $\mathcal{B} = \mathcal{S}(b)$. Let $U = U(\sigma, \mathcal{P}_0, Q_i, \mathcal{P}_i')$ and $P = P(1, \sigma, \mathcal{P}_0, Q_i, \mathcal{P}_i')$. Then $P_i$ has at least $\mathfrak{p} \cdot \mathfrak{m}$ many cutoff points. Set $\varphi = P(1+1, \sigma, \mathcal{P}_0, Q_i, \mathcal{P}_i')$. Then $\sigma_1' = \sigma_i'_{1, i+1}$, and it follows
That if extends \( \rho \) in \( U \). Hence 
\[ 1q^1U < 1p^1U \] and there is \( \sigma \in \mathbb{R}_1 \), 
act \( \sigma = \) the \( 1q^1U - \text{th} \) cutoff point of \( \mathbb{R}_1 \). Pick \( Y < \mathbb{V}^R \) act.
\[ \mathbb{V}^R \bigcap \{ \xi_j \} < Y \] and \( \bar{Y} < \sigma^*(\xi_j) \)
and \( Y \) is \( \infty \)-closed in \( \mathbb{R}_1 \).

Let \( S: \mathbb{R} \leftrightarrow Y \) where \( \mathbb{R} \) is \( \mathbb{R}_1 \) 
transitive. Set: \( \mathcal{Q} = S^{-1}(\mathcal{Q}_1) \), 
\( \sigma = S^{-1}(\mathcal{Q}_1) \), \( \mathcal{P} = S^{-1}(\mathcal{P}_1) \).
Then \( \langle \mathcal{Q}, \mathcal{P}, \mathcal{Q}_1, \mathcal{P}_1 \rangle \in V^R_{\sigma^*}(\xi_j) = 
= V^N_{\sigma^*(\xi_j)}. \) Moreover \( V^N_{\sigma^*(\xi_j)} = V^R_{\bar{\lambda}(\beta) + 1} = V^R_{\bar{\lambda}(\beta)} \)
= \( V^R_{\lambda_j} \). For \( \lambda < \lambda_j \) we have
\[ \sigma^*(\lambda) = \sigma^*(\lambda_j) \). \]
But \( \sigma^*(\lambda_j) < \lambda_j \)
\[ T(\lambda_j)(\beta) \) and hence \( \langle \sigma^*(\lambda), \beta \rangle \in \mathbb{C} \) 
and \( \mathbb{C} = \mathbb{C}(Z) \) where \( Z = \{ \lambda \sigma^*(\lambda) \} \).
Hence \( \sigma^*(\lambda) < \lambda_j^*(\beta) \).
Clearly \( \delta^* \bar{\lambda}(\beta) = id \). Hence
\[ \sigma' (x) = \delta \sigma_0 (x) = \mathcal{O}(x). \text{ Hence } \sigma' (\chi_c) = \mathcal{O}(x). \text{ But for } x < \chi_c, \quad \mathcal{O} = \sigma' (x) = \sigma_0 (\pi) (x) \text{ we have:} \]

\[ \langle x, y \rangle \in \mathcal{O}(\beta) \implies \langle x, \sigma_0 (x) \rangle \in \mathcal{O}(\pi) (\beta) \]

\[ \implies \langle x, y \rangle \in \mathcal{O}(\chi_c) \implies \langle x, \sigma_0 (x) \rangle \in \mathcal{O}(\chi_c) \]

Hence \[ \mathcal{O}(\beta) = \mathcal{O}(\chi_c) = \mathcal{O}(\chi_0). \]

Finally we note that since \( \mathcal{O}(\chi_0) \)
has \( \mathcal{O} = \rho (\pi, \chi_0, \sigma_0 \pi_0, \chi_0, \mathcal{O}) \)
much cutoff points, then the same statement holds in \( \mathcal{O}(\chi_0) \)
has \( \rho (\pi, \chi_0, \sigma_0 \pi_0, \chi_0, \mathcal{O}) \) many
cutoff points and \( \mathcal{O}(\chi_0) \) is well founded. Thus we have\[ \chi_0 \in \mathcal{O}(\chi_0) \]

Hence \[ \beta \in \mathcal{O}(\chi_0), \text{ since } \beta = \mathcal{O}(\beta) \]. Centers!

This proves the Claim. An

N choose \[ \mathcal{O}(\chi_0) \rightarrow \langle \mathcal{O}(\chi_0), \mathcal{O}(\chi_0), \mathcal{O}(\chi_0), \mathcal{O}(\chi_0) \rangle \]
with the above properties for \( \mathcal{O}(\chi_0) \)
Set: \( \tau_{\chi_0} = \mathcal{O}(\chi_0), \mathcal{O}(\chi_0), \mathcal{O}(\chi_0), \mathcal{O}(\chi_0) \). etc.
The verifications are straightforward. This completes the construction of \(< P_i, Q_i, \xi, \beta >\), giving us our contradiction. Thus the Claim is proven. We believe - but haven't checked - that the rest of Steel's proof can be modified in the same way. In this sense Lemma 4 is more "proven".

We now consider the problem of constructing a version of \( V^c \) which segments need only be weak MS-mice. This is a somewhat easier problem and we can get by with a weaker background condition for the \( N_3 \). In place of Lemma 4.1 it suffices to prove:
Lemma 4.1 Let \( \sigma : P \rightarrow \Xi_k \mathcal{N} \eta \) where \( P \) is countable. Then \( P \) has a countable normal \( k \)-iteration strategy \( S \).

Moreover, if \( \tau = \langle \langle \tau_i \rangle_i \rangle, \ldots, T \rangle \) is a countable normal \( S \)-iteration of length \( \theta + 1 \), then there is \( \sigma' : P_0 \rightarrow \mathcal{N}_\eta \) s.t.

i) \( \Phi \) is total, then \( \tau = \underline{\gamma} \), \( \sigma_{\Phi_{\theta} \mathcal{N}} = \sigma \), and \( \sigma' \in \Sigma_{\mathcal{N}}^{(\Delta)} \) preserving whenever \( m \leq k \) and \( \omega P_0^m > \chi_i \) for all \( i < \Theta \).

ii) \( \phi \) is not total, then \( \tau < \underline{\gamma} \) and \( \sigma' \in \Sigma_{\mathcal{N}}^{(\Delta)} \) preserving whenever \( \omega P_0^m > \chi_i \) for all \( i < \Theta \).

From this follow:

Lemma 4.4 Let \( \sigma : P \rightarrow \Xi \mathcal{N} \eta \) where \( P \) is countable. Then \( P \) is countably \( MS \)-iterable.

To obtain this it suffices to use a weaker background condition in Case 3.1 of the def. of \( N_3 \); we require only that, whenever \( Z \in \text{dom}(F) \) is countable, there be a background certificate \( \langle N, F^* \rangle \) s.t. \( Z \subseteq N \). Hence
\( \kappa = \text{crit}(F) \) need no longer be regular, although we still have \( \kappa = \text{V}^*_\kappa \) and \( \text{cf}(\kappa) > \omega \). The proof of Lemma 4.1' for the special case that \( Y \) satisfies \((\text{**})\) is much as before. In the decisive step we construct \( \sigma': P_{i+1} \rightarrow Q \) as follows. We choose a background certificate for \( E_i = \text{Ext}(\Sigma^*_c, \text{V}^*_\kappa, \text{V}^*_\kappa) \) for \( P_{i+1} \rightarrow Q \) as before, and we may \( \sigma^*_c \in \Sigma^*_c \) and \( \text{rg} \sigma^*_c \subset \kappa \). Define \( W_0, W_2 \) as before and let \( W_1 = \text{rg} \sigma^*_c \cup \Sigma^*_c \) and \( W_1 = W_0 \cup W_1 \cup W_2 \) and proceed as before: Let \( r \) fix \( W \) and define \( \sigma' \) by \( \sigma'(P^*_i, \text{V}^*_\tau(f)(\kappa)) = \sigma_r(f)(\sigma_\kappa^* \circ \kappa) \). Then \( \sigma' \) is defined and \( \sigma' \in \Sigma^*_c \) - preserving whenever \( m \leq k \) and \( \omega P^m > \kappa \). (Hence \( \sigma' \in \Sigma^*_c \) - preserving for \( m \leq k \) and \( \omega P^m > \kappa \).) The proof of this is contained in the initial part of the proof of Lemma 3.'