§3 Universality

From now on assume:

A1 Either no $ \epsilon \Theta \in \text{Woodin} \text{ in an inner model or else } V_\Theta \text{ is closed under } \#.

A2 Let $ M \in V_\Theta $ be a 1-small premouse and $ I $ a normal iteration of $ M $ of length $ \Theta $. Then $ I $ has a cofinal branch.

A3 $ \Theta $ is a Mahlo cardinal.

Note: A1 subsumes our earlier A0. A1 is known to imply that $ K^c $ is iterable below $ \Theta $.

Note: A2 holds if $ V_\Theta \# $ exists or if $ \Theta $ is not Woodin in an inner model.

Note: A3 is assumed in order to insure a large supply of inaccessible $ \kappa \leq \Theta $ which can serve as critical points for the extenders in the $ K^c $ construction.
We prove:\n
Thm 5 \( K^c \) is universal wrt. 1-small premise in \( V_\Theta \) i.e. if \( \mathcal{N} \leq V_\Theta \) is a 1-small premise, then any coiteration of \( \mathcal{N}, K^c \) will terminate below \( \Theta \).

Note: A failure of well-foundedness is considered a termination. Hence the lemma would hold even without A1, which is used to ensure an iteration strategy for \( K^c \). The interesting case occurs when both sides have a normal iteration strategy, thus preventing a failure of well-foundedness.

We now prove Thm 5. Suppose not. Let \( \mathcal{Q} \in V_\Theta \) be a 1-small premise and let \( \langle \mathcal{Q}^Q, \mathcal{Q}^K \rangle \) be a coiteration of \( \mathcal{Q}, K^c \) of length \( \Theta \). Let:\n
\[
\gamma^Q = \langle \langle \mathcal{Q}^{i}, <\mathcal{K}^{i}_K >, <\mathcal{Q}^{i}_i >, <\pi^{i}_i >, T\mathcal{Q} > \\
\gamma^K = \langle \langle \mathcal{K}^{i}_K, <\mathcal{Q}^{i}_i >, <\mathcal{Q}^{i}_i >, <\pi^{i}_i >, T\mathcal{K} >.
\]

Making use of this we shall prove:
Lemma 5.1. There are a stationary \( \delta \subseteq \Theta \) and a commutative system \( \langle \tau_{\alpha \beta} \mid \alpha \leq \beta \text{ in } \delta \rangle \) s.t.

(i) \( \delta \) is inaccessible for \( \delta \in \Theta \)

(ii) Set \( \kappa_d = (\bigcup_{d} \kappa^c_{d} (\delta \in \Theta)) \).

Then \( \tau_{\alpha \beta} : \kappa_d \rightarrow \kappa_{\beta} \) cofinally; \( \tau_{\alpha \beta} \delta = \delta \); \( \tau_{\alpha \beta} (d) = \beta \) for \( \alpha \leq \beta \text{ in } \Theta. \)

Proof.

Let \( b \) be a cofinal branch in \( \Theta \). This has a transitive limit model \( \Theta_b \) and maps \( \tau_{\alpha} \Theta \) to \( \Theta_b \) for \( \alpha \in b \). There is \( \delta \in b \text{ s.t. no truncation occurs on } b \text{ above } \delta \). Hence \( \tau_{\alpha} \Theta \) is total on \( \Theta_b \) for \( \delta \leq \alpha \leq b \). For \( \delta \leq b \text{ s.t. } \delta \in b \text{ and } \kappa_{\delta} = \kappa_{\delta} \text{ as } \alpha \in (\tau_{\delta} \Theta) \).

A standard proof shows that there is a club \( C \subset b \setminus \delta \text{ s.t. if } \delta \subseteq C \), then \( \delta = \kappa_{\delta} \) and \( \tau_{\alpha \beta} (d) = \beta \) for \( \delta \leq \beta \subseteq C, \alpha \leq \beta \). It follows that \( \tau_{\alpha \beta} (d) = \Theta, \delta = \text{crit}(\tau_{\alpha \beta}) \text{ for } \delta \subseteq C. \)
Now let \( \kappa \in C \) be inaccessible. Then
\[ \kappa \in Q_{\kappa} \cap K_{\kappa}, \]
since otherwise \( K_{\kappa} \)
would be a segment of \( Q_{\kappa} \) and
the coiteration would terminate.

(1) \( \kappa \neq \prod_{i \in \kappa} \bar{\kappa} \) for any \( \bar{\kappa} < \kappa \).

Proof: Otherwise we could repeat
the proof that coiterations terminate
(§4 of [NFS]) to show that the
coiteration of \( Q, K \) must have
terminated below \( \kappa \). QED(1)

Hence:

(2) There is no truncation on the
branch \( \xi \mid i < \kappa \) \( \bar{\kappa} \). (Hence
\( \prod_{i \in \kappa} \bar{\kappa} \) is total for \( i < \kappa \).

Otherwise \( \kappa = \prod_{i \in \kappa} \bar{\kappa} \)
where \( \bar{\kappa} < \kappa \),
At follows easily that:

(3) \( \prod_{i \in \kappa} \bar{\kappa} (\bar{\kappa}) < \kappa \) for \( \bar{\kappa} < \kappa \),
hence:

(4) \( \prod_{i \in \kappa} \bar{\kappa} (\kappa) = \kappa \),
Let \( \tau = \kappa + K_{\kappa} \). Then
(5) \( \prod_{i \in \kappa} \bar{\kappa} (\tau) = \tau \),
Now let: \( \tilde{K}_\kappa = \bigcup E \kappa \) \( ^* \); \( K_\kappa = \pi^{* \kappa} (\tilde{K}_\kappa) \).

At follow easily that:

1. \( \tilde{K}_\kappa = \bigcup E \kappa = \bigcup E \kappa^{* \kappa} \).

(Note that \( E = \kappa + \omega \kappa \)).

Set: \( \tilde{\pi}_\kappa = \pi^{* \kappa} \mid \tilde{K}_\kappa \). Then

2. \( \tilde{\pi}_\kappa \colon \tilde{K}_\kappa \to \tilde{\mathcal{E}}_0 \kappa \).

But by (1):

3. \( \tilde{\pi}_\kappa^{* \kappa} \colon \tilde{K}_\kappa \to \tilde{K}_\kappa^{* \kappa} \).

for \( \kappa, \kappa' \in \mathcal{C}', \kappa \leq \kappa' \),

where \( \mathcal{C}' = \{ \kappa \in \mathcal{C} : \kappa \text{ is inaccessible in } \mathcal{C} \} \).

We wish to define a stationary set \( S \subseteq \mathcal{C}' \). Let \( \tilde{\pi}^{* \kappa} \tilde{\pi}_\kappa \subseteq \tilde{\pi}_\kappa \). for \( \kappa, \kappa' \in S, \kappa \leq \kappa' \). Recalling that

\( \tilde{\pi}_\kappa = h(\tilde{K}_\kappa) \) for \( \kappa \in \mathcal{C}' \), we set

\( \tilde{\pi}^{* \kappa}_\kappa = h(\tilde{K}_\kappa) \) and note that:

(4) \( c(\tilde{\pi}_\kappa) = c(\tilde{\pi}^{* \kappa}_\kappa) = c(\tilde{\pi}^{* \kappa}_\kappa) \)

for \( \kappa, \kappa' \in \mathcal{C}', \kappa \leq \kappa' \).

By (2), (3).
Let \( \mu_k \leq \kappa \) for all \( \kappa < \theta \).

Pick \( \delta \kappa \) monotonically and cofinal in \( \theta \) for \( \kappa < \theta \). Assume w.l.o.g. that \( \kappa > \delta \) for \( \kappa < \theta \).

Then there is \( \delta(\kappa) < \kappa \) s.t.

\[
\delta(\kappa) \leq \kappa \text{ for } \delta = \delta(\kappa), \text{ Hence by Feferman}
\]

there is a stationary \( S' \subset \theta \) s.t.

\( \delta(\kappa) = \delta \) in constant for \( \kappa < S' \).

Set: \( h(\kappa) = \left< \prod_{\delta \kappa} \wedge_{\kappa} \frac{\beta}{\kappa} (\zeta \kappa) \right| \kappa < \theta > \).

Then \( h(\kappa) \in \Delta_\delta \). Hence there is stationary \( S' \subset S \) s.t. \( h \) is constant on \( S \). But then:

(5) \( \prod_{\kappa} \wedge_{\kappa} \frac{\beta}{\kappa} (\zeta \kappa) = \frac{\beta}{\kappa}(\zeta \kappa) \)

for \( \kappa \in S \). Hence \( \delta(\kappa) \)

(6) \( \frac{\beta}{\kappa} \leq \frac{\beta}{\kappa} \) for \( \kappa, \kappa' \in S, \kappa \leq \kappa' \).

Proof

\[
\frac{\beta}{\kappa} \leq \frac{\beta}{\kappa} \leq \frac{\beta}{\kappa} \text{ for } \kappa, \kappa' \in S, \kappa \leq \kappa'.
\]
Hence:

\[
\pi_k \subseteq \pi_{k'} \quad \text{for} \quad k, k' \in S, \quad a \leq k, k'.
\]

Let \( x \in \tilde{\kappa}_k \). Then \( x \in \bigcup_{i \leq k} E_i \) for some \( i \leq k \), where \( E = E_k \). Let \( f = f_k = \pi_k \). The \( \tilde{\kappa}_k \)-least \( f : k \longrightarrow \bigcup_{i \leq k} E_i \) for \( a = f(v) \) for a \( v \leq k \). We have:

\[
\bar{\pi}_k(f) = \text{the } \kappa^i_k \text{-least } f : k \longrightarrow \bigcup_i E_k^{\bar{i}} \quad \text{such that } \bar{f}(v) \in E_k^{\bar{i}},
\]

\[
\bar{\pi}_k(f) = \text{the } \kappa^i_k \text{-least } f' : k' \longrightarrow \bigcup_i E_k^{\bar{i}} \quad \text{such that } \bar{f}'(v) \in E_k^{\bar{i}}.
\]

\[
= \bar{\pi}_k'(f_k).
\]

Hence:

\[
\pi_{k'} \subseteq \bar{\pi}_k'(f_k) = \bar{\pi}_k'(f(v)) = \bar{\pi}_k'(f_k(v)) = \bar{\pi}_k'(f_k'(v)) = \bar{\pi}_k'(f_k(v)) \leq \pi_{k'}(v).
\]

(Note \( \bar{\pi}_k(v) = \pi_{k'}(v) \) by (6)). QED (7)

Now set: \( \pi_{k'} = \pi_{k'} \subseteq \pi_k \). Then:

\[
\pi_{k'} : \bar{\kappa}_k \longrightarrow \bar{\kappa}_k', \quad \text{cofinally. Moreover}, \quad \pi_{k'}(k) = id \text{ by (6)} \quad \text{and } \pi_{k'}(k) = k'.
\]

QED (Lemma 5, 1)
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Now let \( \tilde{\kappa} \), \( \langle \tilde{\pi}_a \mid a \in S \rangle = \lim_{\alpha \in \beta} (\tilde{\pi}_\alpha, \overline{\pi}_\alpha) \) for all \( \alpha \in S \). Clearly \( cf(\alpha^+ \omega^c) = \delta \) for all \( \alpha \in S \) where \( \delta > \omega \), \( \delta < \min(S) \).

Hence \( cf(h + (\tilde{\kappa})) = \delta < \theta \). Hence \( \tilde{\kappa} \in H_\theta \). Let \( \theta \in H < H_{\theta^+} \) s.t. 

\( \bar{H} = \theta \), \( H \) is transitive, \( \tilde{\kappa}, \langle \tilde{\pi}_a \mid a \in S \rangle \in H \).

Let \( f : \theta \rightarrow H \) and set \( i : X_\theta = f^{-1} \) for \( a \leq \theta \). Set \( i : C = \text{the set of} \)

\( a \leq \theta \) s.t. \( \tilde{\kappa}, \langle \tilde{\pi}_a \mid a \in S \rangle \in X_\theta \). Then \( C \) is \( c \in i \) \( \Theta \) 

Pick \( \kappa \in C \cap S \), Let \( \sigma : N \rightarrow X_\kappa \). 

Set \( : F^x = \sigma^{-1} \mathcal{P}(\kappa) \). Then \( F^x \) is an extender on \( N \) and \( \sigma : N \rightarrow F^x \), \( \mathcal{H}. \)

(Hence \( F^x \in H = \text{Ult}(N, F^x) \).) Clearly \( N = \mathcal{F} C \) and \( \mathcal{N} \subseteq N \).

At it easily seen that 

\( \sigma (\langle \tilde{\pi}_a \mid a < \kappa \in S \rangle) = \langle \tilde{\pi}_a \mid a < \kappa \in S \rangle \), 

Hence since \( \tilde{\kappa}, \langle \tilde{\pi}_a \mid a \in S \cap \kappa > \rangle = \lim_{\alpha \in \beta} \langle \tilde{\pi}_\alpha, \overline{\pi}_\alpha \rangle \), we have .
(9) \( \sigma(\tilde{\pi}_x) = \tilde{\pi}_x \) (\( x \in \kappa \times S \)). Hence:

(10) \( \sigma(\tilde{\pi}_x) = \tilde{\pi}_x \), since \( \sigma(\tilde{\pi}_x(\kappa)) = \tilde{\pi}_x(\kappa) = \tilde{\pi}_x(\kappa) \).

Set \( F' = F \star \tilde{\pi}_x = \tilde{\pi}_x \star \tilde{\pi}_x \). Then \( \langle \tilde{\pi}_x, F' \rangle \) satisfies all premouse conditions except the initial segment condition.

We shorten \( F' \) so as to satisfy this condition: Let \( \lambda < \Theta \) be least such that \( \sigma(f)(\lambda) < \lambda \) whenever \( f : \kappa \rightarrow \kappa \), \( f \in \tilde{\pi}_x \) and \( \lambda < \Theta \). Set \( F = F' \star \lambda \).

Let \( \tilde{\pi}_x \rightarrow \tilde{\pi}_x \rightarrow \kappa \), \( \langle \tilde{\pi}_x, F \rangle \) is easily seen to be a 1-small premouse.

(11) \( \tilde{\pi}_x \) is an initial segment of \( \tilde{\pi}_x \) (hence of \( K^c \)).

Proof:

Let \( k : \tilde{\pi}_x \rightarrow \tilde{\pi}_x \) be defined by:

\[ k(\tilde{\pi}(f)(\alpha)) = \tilde{\pi}_x(f)(\alpha). \]

Then \( k \) is a cofinal \( \subseteq \) preserving map and \( k \star \lambda = \text{id} \), \( k(\lambda) = k(\tilde{\pi}_x(\kappa)) = \tilde{\pi}_x(\kappa) = \Theta \).

Let \( \lambda < \delta < \text{ht}(\tilde{\pi}_x) \) and \( \omega \subseteq \delta \). Then:

\( \delta \) and \( \tilde{\pi}_x \) are cofinal in \( \text{ht}(\tilde{\pi}_x) \).

Set \( Q = J^{\tilde{\pi}_x}_\delta \), \( \bar{Q} = \tilde{\pi}_x(\bar{Q}) = J^{\tilde{\pi}_x}_\delta \), \( k' = k(\tilde{\pi}_x) \). Then \( \bar{Q}, \bar{Q} \) are round.
and $k': Q \xrightarrow{e_k} Q$ where $wp\overline{Q} = \text{crit}(k')$.

We apply §8 Lemma 4 of [NES].

(a) is impossible since $k \neq i Q + \alpha$ in round

(b) is impossible since $wp\overline{Q} \geq \text{crit}(k')$.

Hence (b) holds i.e. $\overline{Q}$ is a sequent of $Q$, hence of $\overline{R}$. QED

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We note that $\lambda$ is a limit cardinal in $\overline{R}$ by the definition of $\lambda$. Hence:

\begin{align*}
12: \quad \Sigma^*_{\lambda} &= \int_{\lambda} E^{1', \lambda}, \quad \text{where } \lambda \text{ is a limit cardinal in } R.
\end{align*}

13: $\langle N, F^* \rangle$ is a certificate for $\langle \overline{R}, F \rangle$.

Moreover, if $f : \lambda \rightarrow \text{dom}(E)$, $\lambda < \kappa$, then $f \in N$.

Proof:

$\langle N, F^* \rangle$ is trivially a certificate.

Since $\overline{\kappa} \in N$, there is $n \in N$ such that $\overline{\kappa} = \overline{\kappa}_n$. Set $\overline{f} = g^{-1}f$.

Then $\overline{f} \in \overline{V}_\overline{\kappa} \subset N$, since $\kappa$ is regular.

Hence $f = g\overline{f} \in N$. QED (13)
By §11 Lemma 2.1, 2.4 it follows from (13) that for \( \delta = \delta(\lambda) = \sup \{ \xi \mid \text{\( \lambda^\xi < \lambda \)} \}
 we have:

\[
(14) \quad M_\delta = N_\delta = \langle J_\lambda^{E^K}, \varphi \rangle \quad \text{and} \quad \text{\( \tilde{\lambda}_\delta = \lambda \).}
\]

Now let \( \text{\( \text{\( \tilde{K} = \bigcup \text{\( J_\lambda^{E^K} = J_\lambda^{E^K} \) \)}} \)} \). Since \( \lambda \) is the largest cardinal in \( \text{\( \text{\( \tilde{K} \) \)}} \), \( \lambda \) is cardinally absolute in \( \text{\( \text{\( \tilde{K} \) \)}} \). By §11 Lemma 2.2 we conclude that for \( \delta = \delta(\lambda) \) we have:

\[
(15) \quad M_{\delta', \lambda} = N_{\delta', \lambda} = \langle \text{\( \tilde{K} \), \( \varphi \) \rangle}.
\]

Hence by (13) \( N_{\delta'+1} = \langle \text{\( \text{\( \tilde{K} \), \( \varphi \) \rangle} \rangle \rangle \). Hence \( \text{\( \wp^\omega = \lambda \).} \) This is a contradiction.

\[\text{Corollary 6.1} \quad \text{Let} \ Q \in V_\theta \ \text{be a normally iterable premouse in} \ V_\theta. \text{Then} \ Q \ \text{is iterable in} \ V_\theta.\]
proof of Corollary 6.1

Consider $Q, K^c$ to $Q', K'$. Since $K^c$ is pseudo-die, the usual proofs tell us that one side of the iteration must be simple and that the simple side must be a segment of the non-simple side, if this occurs. $K'$ cannot be a proper segment of $Q'$, since the $K'$-side would then have to be non-simple and $K'$ would be unbounded. Thus $Q'$ is a segment of $K'$. Hence $Q'$ is a moose. Hence $Q$ is a moose, since $\pi : Q \rightarrow \Sigma \times Q'$, where $\pi$ is the iteration map. QED (6.11)

Corollary 6.2 If $Q$ is an $A$-small MS moose in $T\beta$, then $Q$ is a moose in $T\beta$.

We now prove a refinement of Corollary 6.1.
(Since otherwise Q coiterates out to a segment of a mouse and is therefore a mouse in W.)

Proof. Suppose not. Then Q is not a tail, i.e. it is not definable in W. Hence there must be a failure of the Θ rule on the Θ rule.

On the Θ-rule we employ the "economical" strategy: At time \( x \) there is no continued to length \( x + 1 \) if possible. The \( K \)-rule, we suppose \( K \) is definable in W. Hence there must be a failure of the Θ rule on the Θ rule.

Theorem 7. Let \( Q \) be a countable A-normal
\[ \langle y^q, y^K \rangle \in X < V_\theta, \bar{X} = \omega. \]

Let \( \sigma : \bar{V} \rightarrow X \), where \( \bar{V} \) is transitive. Let \( \sigma(\langle y^q, y^K \rangle) = \langle y^q, y^K \rangle \). Let:

\[ y\bar{q} = \langle \langle \bar{q}_i \rangle, \langle \bar{v}_i \rangle, \ldots, \bar{t} \bar{q} \rangle, \]

\[ y\bar{K} = \langle \langle \bar{K}_i \rangle, \langle \bar{v}_i \rangle, \ldots, \bar{t} \bar{K} \rangle, \]

Let \( \sigma(\bar{X}) = \lambda \).

\[ \langle y\bar{q}, y\bar{K} \rangle \] is then a countable coiteration.

If \( \delta < \bar{X} \) is a limit ordinal, then we know that \( \{ i : i \in \bar{q} \delta \} \) is the unique cofinal well-founded branch through \( \bar{q} \delta \) in the sense of \( \bar{V} \), but not necessarily in \( V \).

Set: \( \bar{X} = \) the least \( \bar{X} \leq \bar{X} \) such that for all \( \delta < \bar{X} \), \( \{ i : i \in \bar{q} \delta \} \) is the unique cofinal well-founded branch in \( \bar{q} \delta \). Then

(1) \( \lim (\bar{X}) \)

Mf. Suppose not. Let \( \bar{X} = \bar{h} + 1 \). Then \( \bar{X} = \bar{X} \) and \( \bar{V} \) thinks that the coiteration cannot be continued. Since \( \bar{X} \) is countable and we have followed the unique iteration strategy for \( \bar{q} \), and \( \bar{q} \) is countably normally iterable, then it can be continued. Hence \( \bar{V} \) thinks so by an easy absoluteness
argument. QED (1)

\( T \) must have a well founded cofinal branch \( b \) since \( Q \) is countably normal iterable. Choose \( b \) s.t.
the limit model \( Q_b \) has minimal height.

**Case 1** \( \text{ht}(Q_b) \in \overline{V} \).

**Case 1.1** \( b \) is the unique cofinal branch.

Then \( \overline{x} = x \) and \( \overline{y}^Q \) has no well founded cofinal branch. Hence \( \overline{V} \) thinks that \( \overline{y}^Q \) has no such branch. We derive a contradiction. Let \( \overline{z} = \overline{z} \in \overline{V} \) s.t.

\( L_S[\overline{y}^Q] \) is admissible and \( \text{ht}(Q_b) \in \overline{S} \).

Let \( \overline{z} = \text{ht}(Q_b) \), \( \overline{z} = L_S, \overline{z} = L_S, \overline{z} = \overline{L_S, \overline{z}} = \overline{L_S, \overline{z}, \overline{y}^Q} \) be the following infinitary language on \( L_S[\overline{y}^Q] \):

- Predicates: \( \epsilon, = \), Constant: \( x \) \( (x \in L_S[\overline{y}^Q]) \), \( b \).

- Axioms: \( \exists \overline{z} \neg \); \( \forall \overline{z}\overline{x}( \overline{z} \epsilon \overline{x} \iff \overline{z} \epsilon \overline{x} \) \)

\( (x \in L_S[\overline{y}^Q]) \); \( b \) is a cofinal branch in \( \overline{y}^Q \) \( b \) has a transitive limit model \( Q_b \) s.t. \( \text{ht}(Q_b) = \overline{z} \).
Clearly $L$ is consistent. Every model $M$ of $L$ is isomorphic to one which is good in the sense that its well-founded core is a transitive $\varepsilon$-structure. Hence we may work only with good models. We then have $x^{M} = x$ for $x \in L_{b}[y_{1}]$. But then $b^{M}$ is really a well-founded cofinal branch in $y_{1}$; hence $b^{M} = b$.

By the completeness theorem for countable admissible it follows that:

$\forall \, b \in b \implies L = \{ x \in b \}$. Hence $b \in \bigcup \bar{V}$.

Contr! QED (Case 1.1)

Case 1.2 $y_{1}$ has another well founded cofinal branch.

By 56 and the minimality of $h^{+}(Q_{b})$, $Q_{b}$ has the form $J_{\delta}^{E}$, where $N = J_{\delta}^{E}$, $\delta = \sup_{i \leq \delta} y_{i}$ and $N = \bigcup_{i < \delta} J_{\delta}^{E}$. Moreover $\delta$ is Woodin in $Q_{b}$ if $\delta > \delta$.

Claim 1 $\bar{R}_{X}$ is not a proper segment of $Q_{b}$.

Suppose not. Then there is no truncation on the main branch to $\bar{R}_{X}$, since
otherwise \( \overline{R}_x \) would be uncounted. Hence \( h^+ (\overline{R}_x) = \text{ann} \cap \overline{V} > h^+ (Q_b) \), Contrad!

**Claim 2** \( Q_b \) is a segment of \( \overline{R}_x \).

Suppose not. There is \( n \) s.t. \( 5 < n \leq 2 \) and \( E_{\overline{R}_x} \neq \emptyset \), where \( 5 \) is round in \( \text{ann} \cap \overline{V} \). At follows easily that \( \overline{R}_x \) is not \( 1 \)-small. Contrad! \( \Box \) QED (Claim 2).

But then \( Q_b \) is a monotone in \( V_\delta \). \( Q_b \) is a simple iterate of \( Q \), since otherwise \( Q_b \) could not be a proper segment of \( \overline{R}_x \). Hence \( \overline{R}_x \) is not a simple iterate of \( \overline{R} \). Contradiction! Hence \( \overline{x}_b^Q : Q \xrightarrow{\delta} Q_b \) and \( Q \) is a monotone in \( V \). Contrad! \( \Box \) QED (Case 1.2)

**Case 2** Case 1 fails. (Hence \( \overline{x} = \overline{x} \).)

We repeat an argument of Woodin. Let \( F = \{ \gamma \in \overline{V} \text{ which are admissible in } \gamma \} \). For \( \gamma \in F \).

Let \( L_\gamma = L_\gamma \gamma \). \( \overline{y}_\gamma \) be the following theory in the infinitary language of \( L_\gamma [ \gamma \overline{q} ] \).
Predicates:  $\varepsilon_1 = \cdot$, Constants $\bar{x} \in L_\bar{\theta}[\bar{YQ}]$.

and $b^*$. Axioms $ZF^-$, $\forall \bar{\sigma} \sigma (\exists \bar{x} \bar{\sigma} \rightarrow \forall \bar{z} \bar{\sigma} \exists \bar{z} \bar{\sigma})$

($x \in L_\bar{\theta}[\bar{YQ}]$), $b^*$ is a cofinal well-founded branch through $\bar{YQ}$, $\bar{z} \in Qb^* (\exists < \bar{x})$.

Clearly $L_{\bar{\theta}}$ is consistent. By Ville's Lemma $L_{\bar{\theta}}$ has a good model $\bar{Q}$

where well-founded core has rank

exactly $\bar{x}$. (As above, good means

that the well-founded core is a transitive

$\in$-model.) Let $b_\bar{y} = b^\bar{Q}$. Then

$b_\bar{y}$ is a cofinal branch through $\bar{YQ}$

and $\bar{Q}_\bar{y} = (Qb^*)^{\bar{Q}}$ is a good limit

model whose well-founded core has

rank exactly $\bar{y}$. But then $b_\bar{y} \neq b_{\bar{y}}$, for $\bar{x}, \bar{y} \in \bar{\Gamma}$, $\bar{x} < \bar{y}$.

Repeating some arguments from §6 we see

that if $\bar{\sigma} = \sup \bar{\nu}_i$ and $N = \bar{\sigma}_\bar{E} =

\bigcup_{\bar{i} < \bar{x}} \bar{E} \bar{\nu}_i$, then $\bar{\sigma}$ is Woodin

in $\bar{E}^\bar{Q}$. But this holds for arbi-

trarily large $\bar{\nu} \in \bar{\nu}$. Hence

$\bar{\sigma}$ is Woodin in $L_{\bar{\theta}}^{\bar{E}^\bar{Q}}$, where $\bar{\theta} = \text{On}(\bar{V})$. 
Let $\sigma(E) = E'$, $\sigma(\delta) = \delta'$. Then $\delta'$ is Woodin in $L^E'$, hence in $L^E$, since $E'$ is inaccessible. Hence by A1, $T_\theta$ is closed under $\#$.
Hence $\overline{T}$ is closed under $\#$. Now let $ht(Q_b) = \bar{3}$ and let $\bar{\delta} > \bar{3}$ be admissible in $\overline{y\bar{a}}$. Let $\overline{L} = L_{\bar{\delta}, \bar{3}, \overline{y\bar{a}}}$ be as in Case 1.2. Then $\overline{L}$ is consistent. $T_\bar{\delta}$ is the first indiscernible for $L[\overline{y\bar{a}}]$ given by $(y\bar{a})^\#$, then $L[\overline{y\bar{a}}] \models L[\overline{y\bar{a}}]$. Hence there are $\bar{\delta}, \bar{\delta} < \bar{\delta}$, $\overline{L} = L_{\bar{\delta}, \bar{\delta}, \overline{y\bar{a}}}$ is consistent.
Let $\overline{M}$ be a good model of $\overline{L}$ and let $\overline{b} = b^{\overline{M}}$. Then $\overline{b}$ is a cofinal well founded branch through $\overline{y\bar{a}}$ and $ht(Q_{\overline{b}}) = \bar{3} < \bar{\delta}$.
Contradiction!

QED (Theorem 7)
We recall that $M$ was called a weak mouse iff whenever $\Vdash \exists^+ \mathcal{A} \subseteq M$ and $\mathcal{A}$ is countable, then $\mathcal{A}$ is countably iterable. Call $M$ a very weak mouse iff every such $\mathcal{A}$ is countably normally iterable.

Then:

Corollary 7.1 Every very weak mouse is a weak mouse in $V_\alpha$.

Hence:

Corollary 7.2 Every very weak MS-mouse is a weak mouse.

We believe that these facts should be provable without A1-A3, but don’t know how.
The condition A3 was imposed only to ensure enough inaccessible cardinals to verify the background conditions for the construction of $K^c$. As mentioned in §2 of this addendum, we can get by with a weaker background condition if we are willing to work with $MS$-mice. Thus it would seem that we can prove Thm 5 for the $K^c$ of $MS$-mice without use of A3. However, we also made use of A3 in proving Lemma 5.1. It would be natural to prove Lemma 5.1 replacing (i') by (i'') $d = \overline{v}_d$ and cf$(d) > \omega$. The rest of the proof would then be essentially as before. However A don't see how to prove even this version of Lemma 5.1 without adopting some new assumption in place of A3. The following will do: A4. A2 holds for $M \subseteq V_\theta$ if $\theta$ is a $1$-small premouse.

(As particular, this will hold for $M = K^c$.)
The details are left to the reader. We then get:

**Thm 8** Assume A1, A2, A4. Let Kc be the Kc of MS-mice. Then Kc is universal wrt. 1-small premouse in Vθ.

We would expect this to yield a version of Corollary 6.1 for MS-mice. We must, however, be careful in formulating. At Q ∈ Vθ is a normally iterable 1-small premouse in Vθ; then Thm 8 shows that Q is normally iterates up to a normal iterate Q′ of a sound MS-mouse ∃Q. Let π : Q → Q′ be the iteration map. Q′ itself may not be MS-iterable and hence the existence of π does not entitle us to assert that Q in MS-iterable. At can be shown, however, that core (Q′) is an MS-mouse. Suppose now that Q′ is a solid core mouse. Then π(ρ) = ρ′, and π induces a map
\[ \Pi : Q \to \text{core}(Q') \]. Hence \( Q \) is MS-iterable. Thus we get:

**Cor 8.1** Assume \( A_1, A_2, A_4 \). Let \( Q \in V_\theta \) be a normally iterable 1-small solid core mouse in \( V_\theta \). Then \( Q \) is an MS-mouse in \( V_\theta \).

Similarly, modifying the proof of Thm 7

**Thm 9** Assume \( A_1, A_2, A_4 \). Let \( Q \) be a countable 1-small solid core mouse which is countably normally iterable. Then \( Q \) is an MS-mouse in \( V_\theta \).

Hence:

**Thm 10** Assume \( A_1, A_2, A_4 \). Let \( Q \) be a very weak solid core mouse. Then \( Q \) is a weak MS-mouse.