§3. The fine structure of the constructible hierarchy

Let $u$ be a transitive set which is closed under the formation of finite sets (thus, in particular, $u$ is closed under $n$-tuples and $\text{Emb}_u \subseteq u$). In this section, we attempt to determine the p.i. closed levels $\alpha \in L_\alpha[u]$ of the constructible hierarchy over $u$ at which interesting things happen, for instance when is $L_\alpha[u]$ admissible? It turns out that this is the case iff for no $\beta < \alpha$ there is a map $\Delta_\beta$ map of $L_\beta[u]$ onto $L_\alpha[u]$ exists. As a corollary we get:

$\lambda \in L_\lambda[u] \text{ is admissible} \iff \lambda \rightarrow \omega_1^\omega$.

(Corollary: $\lambda \not\rightarrow \omega_1$ is a singular cardinal and $\rho \in \lambda$, then $L_\rho[u]$ is admissible.)
Throughout this section, \( u \) will be a fixed transitive set which is closed under finite subsets. \( L_d[u] \) will always be p.r. closed. 'p.r.' will always mean 'p.r. in parameters from \( uu \cup \{u\} \)'. We begin with an observation on p.r. functions:

**Lemma.** There is a p.r. function \( \gamma(x,y) \) which maps \( uu \times uu \) onto \( L_d \).

**Proof.**

There is a p.r. map \( \exists! : On^2 \rightarrow On \)

s.t. \( \beta, \delta \leq \langle \beta, \delta \rangle \) (hence the inverses \( l, r \), defined by \( d = \langle l(d), r(d) \rangle \) are p.r.). To see this, we order \( On^2 \) by:

\[ \langle d, \beta \rangle \mathrel{R} \langle \xi, \delta \rangle \iff \max(d, \beta) < \max(\xi, \delta) \]

\[ v. \max = \max \land d < \xi. v. \]

\[ v. \max = \max \land d = \xi \land \beta < \delta. \]
Let \( \beta : \mathbb{R} \rightarrow \mathbb{E} \). To see that \( \langle \rangle \) is p.r., we first define the function 
\( \langle \langle 0, \beta \rangle \mid \beta \in \mathbb{E} \rangle \) by:

\[ \langle 0, \beta \rangle = \sup \{ \langle 0, \nu \rangle + \nu \cdot 2 \mid \nu < \beta \} \]

and then set:

\[ \langle \nu, \beta \rangle = \langle 0, \beta \rangle + \nu \text{ if } \nu < \beta \]

\[ \langle 0, \nu \rangle = \langle 0, \beta \rangle + \beta + \nu \text{ if } \nu \leq \beta \]

We can represent n-tuples of ordinals by:

\[ \langle \beta_1, \ldots, \beta_m \rangle = \langle \beta_1, \langle \beta_2, \ldots, \beta_m \rangle \rangle. \]

Define a function \( h(\varepsilon, y) \) by:

\[ h(\langle \varphi, m, i \rangle, x) = \varphi \left( \frac{\langle \bar{v}_i, \ldots, \bar{v}_m \rangle}{\bar{L}_{\xi_1[u]}, \ldots, \bar{L}_{\xi_m[u]}} \right) \]

if \( \varphi \in \text{Fml}_{u}, m < \omega, i < \omega, \)

\[ i = \langle i_1, \ldots, i_m \rangle, x = \langle \xi_1, \ldots, \xi_m \rangle. \]

\[ h(\varepsilon, y) = 0 \text{ if otherwise}. \]
Then \( h \) is a p.r. function which maps \( \mathbb{N} \times \mathbb{N} \) onto the set of \( \Sigma_0 \) formulae \( \varphi \) containing only the constants \( \alpha (x \in \mathbb{N}) \), \( \beta x [u] \) (\( \forall \mathbb{N} \)).

We have seen that every \( y \in \mathbb{L}_d [u] \) has the form:

\[
y = \{ x \in \mathbb{L}_\beta [u] \mid \models \varphi (\gamma[\gamma/x]) \}
\]

where \( \beta \leq d \) and \( \varphi \) is such a formula. Hence, we may define the desired function \( \gamma \) by:

\[
\gamma (x, \beta, \varphi) = \{ z \in \mathbb{L}_\beta [u] \mid \models h (x, \gamma (\gamma[z])) \}
\]

\( \Box \) E D.
Def: The function $r(\bar{x})$ uniformly
the relation $Ry \bar{x}$ iff $\text{dom}(r) = \text{dom}(R)$
and $\forall \bar{x} \ (\forall y \ Ry \bar{x} \iff Rr(\bar{x}) \bar{x})$.

A structure $M = \langle \text{IM1}, e, A_1, \ldots, A_n \rangle$ is
called $\Sigma_m$-uniformizable $(m \geq 1)$
iff each $\Sigma_m$ relation $R$ s.t.
$\text{rng}(R) \subseteq \text{On}$ is uniformizable by
a $\Sigma_m$ function.

(This motion should really be
called ‘ordinal uniformizability’).
We use it in preference to the stronger
motion because the latter may fail
for lack of a nice well-ordering of
$\text{IM1}$.)
Thm. 1.

Let \( \Sigma \) uniformizable.

**Proof.** Let \( R(\nu, \vec{x}) \) be a \( \Sigma \) relation s.t. any \( (R) \subset \mathbb{N} \). Let

\[
R \nu \vec{x} \iff \forall y \exists y \in R(\nu, y) \vec{x},
\]

where \( \nu \in \Sigma_0 \). Set:

\[
Q(\nu \exists \vec{x}) \iff \forall y \exists y \in R(\nu, y) \vec{x}.
\]

Then \( Q \) is p.r. (in the parameters entering the \( \Sigma_1 \) definition of \( R \) and

\[
R \nu \vec{x} \iff \forall \nu \exists \vec{x}.
\]

Set:

\[
Q(\vec{x}) \equiv \mu \exists \vec{x} \ Q \nu(\vec{x}) \nu(\vec{x}) \vec{x}
\]

(where \( \nu(\vec{x}), \nu(\vec{x}) \vec{x} = \vec{x} \)). Then

\[
Q \in \Sigma_1 \quad \text{since:}
\]

\[
R = Q(\vec{r}) \iff Q \nu(r(\vec{x})) \vec{x}
\]

\[
\land \exists \vec{d} > \vec{r} \ Q \nu(r(\vec{x})) \vec{x}.
\]

Hence, \( r(\vec{x}) = \nu Q(\vec{x}) \in \Sigma_1 \) function which uniformizes \( R \): \( \Box \).
(Note: By the proof of Thm 1, we may take \( r \) as being \( \Sigma_1 \) in the same parameters which enter the \( \Sigma_1 \) definition of \( P_\lambda \)).

**Def.** \( \exists \Sigma_m \subseteq L_d[u] \) ("\( \exists \Sigma_m \)-elementary submodel of \( L_d[u] \)) \( \iff \)
\( \iff \) \( u \cup \{ u \} \subseteq X \) and for every \( \varphi \in \operatorname{Fml}_{\Sigma_m}^X : \)
\( \models \varphi \iff \models \varphi \quad \langle X, \varepsilon \rangle \quad \langle L_d[u], \varepsilon \rangle \)

In other words, \( \exists \Sigma_m \subseteq L_d[u] \) iff \( u \cup \{ u \} \subseteq X \) and for each \( R \in L_d[u]^{m+1} \) which is \( \Sigma_1 \) in parameters from \( X \):

\( \forall y \exists y \models R \quad \iff \quad \forall y \in X \quad \exists y \models R \quad \text{for } \overline{x} \in X. \)
In particular, if $X \subseteq \Sigma_{a_1}$, then $X$ is closed under $\Sigma_1$ functions definable with parameters from $X$. Since $p.r.$ functions are $\Sigma_1$ in the parameter $\omega$, $X$ is $p.r.$ closed.

Clearly, $\varepsilon I X$ satisfies the axiom of extensionality, since, if $x, y \in X$ and $x \neq y$, then $\forall z (z \in x \iff z \in y)$, hence $\forall z \in X (z \in x \iff z \in y)$. Therefore there exists a map $\pi$ of $X$ onto a transitive set $\omega$ s.t.

$$\pi : \langle x, \varepsilon \rangle \leftrightarrow \langle u, \varepsilon \rangle,$$

$\vdash \varphi$ in a $\Sigma_1$ formula (without constants), then

$$(+) \quad \vdash \varphi(x) \iff \vdash \varphi(\pi(x)) \quad \text{for } x \in X.$$

$\vdash$
We may conclude:

(++) \( \pi f(\overline{x}) = f(\overline{\pi(x)}) \) for all p.n. \( f \), since, by the stability lemma, each p.n. \( f \) has a \( \Sigma_1 \) definition which is absolute with respect to p.n. closed domains, i.e. there is a \( \Sigma_0 \) formula \( \varphi_f \) (with constants \( \kappa \) (\( \kappa \in \nu \cup u \cup u \cup \omega \)) s.t.

\[
y = f(x) \iff \forall z \in \varphi_f(\overline{z}, y, \overline{x}) \text{ for all } y, x
\]

\[
y = f(x) \iff \forall z \in L_d[u] \models \varphi_f(\overline{z}, y, \overline{x})
\]

for \( x \in L_d[u] \).

Hence, if \( \overline{x} \in X \), we have:

\[
y = f(x) \iff \forall z \in L_d[u] \models \varphi_f(\overline{z}, y, \overline{x}) \leq_{L_d[u]}
\]

\[
\iff \forall z \in u \models \varphi_f(\overline{z}, \overline{\pi(y)}, \overline{\pi(x)})
\]

\[
\rightarrow \overline{\pi(y)} = \pi f(\overline{\pi(x)}). \quad \text{QED (++)}
\]

(Note (++) implies that \( u \in \text{p.n. closed} \))
By (++) we get:

Lemma 2: \( X \leftrightarrow L_d[u], \pi : \langle x, \varepsilon \rangle \leftrightarrow \langle \nu, \varepsilon \rangle \)

and \( \omega \cup \nu \subseteq \nu \), then \( V_\beta \leq \alpha \quad \omega = L_\beta \).

Proof:
Since \( \langle L_\nu[u] \mid \nu \in \text{On} \rangle \) is p.r., we have

by (++) \( : \pi L_\nu[u] = L_{\pi(\nu)}[u] \).

For all \( x \in X \), we have:

\( V_\nu \in L_d[u] \quad x \in L_\nu[u] \), hence

\( V_\nu \in X \quad x \in L_\nu[u] \), hence

\( V_\nu \in X \quad \pi(\nu) \in L_{\pi(\nu)}[u] \).

Let \( \beta \) be the least ordinal not in \( \nu \).

Then \( \beta = \pi \text{ "on } \cap X \). Hence:

\( \nu = \pi \text{ "on } X = \bigcap_{\nu \in X} L_{\pi(\nu)}[u] = L_\beta[u] \quad \omega \in \Omega \)
Using the fact that $L_\beta[u] \subset L_d[u]$, we can strengthen (++) to:

**Lemma 3** Let $X, \pi$ be as in Lemma 2 and let $f$ be a function which is $\Sigma_1$ without parameters (or at most parameters $x \in X$ s.t. $\pi(x) = x$). Then, whenever $\overrightarrow{x} \in X$ and $f(\overrightarrow{x})$ is defined, so is $f(\pi(\overrightarrow{x}))$ and

$$\pi f(\overrightarrow{x}) = f(\pi(\overrightarrow{x})) .$$

**Proof.**
Let $\varphi$ be a $\Sigma_1$ formula defining $f$ in $L_d[u]$ (containing at most constants $x$ s.t. $\pi(x) = x$). Then

$$y = f(\overrightarrow{x}) \iff \models_{L_d[u]} \varphi(y, \overrightarrow{x})$$

$$\iff \models_{L_\beta[u]} \varphi(\pi(y), \overrightarrow{x})$$

$$\iff \models_{L_d[u]} \varphi(\pi(y), \overrightarrow{x})$$

$$\iff \pi y = f(\pi(\overrightarrow{x})) .$$

Q.E.D.
Lemma 4. There is a $\Sigma_1$ function $h$ s.t. $\text{dom}(h) \subseteq u \times L_d[u]$ and
\[ \forall x \in L_d[u]. \forall y \in L_u. \forall z \in L_d[u]. h((x, y, z)) \leq L_d[u]. \]

Proof.
Define $r(z, x)$ by:
\[ r(\langle \psi, i, j \rangle, x) = \varphi(\langle \psi, i, j \rangle/u, x) \]
if $\varphi \in \text{Fml}_u^{\Sigma_1}$, $i, j < u$, $i \neq 1$.
\[ r(z, x) = 0 \text{ otherwise}. \]
Then $r$ is p.r. and maps $u$ onto $\text{Fml}_u^{\Sigma_1}$.

Note that, since $\models u$ is p.r., $\varphi \in L_d[u], \varphi \in \Sigma_1$. Set:
\[ R \varphi \iff \models u \varphi(\langle \psi, i \rangle/u). \]
and let $r$ uniformize $R$. Let $\Sigma$ be $\Sigma_1$ in the parameter $p$ and set:
\[ h(\langle z, w \rangle, x) = \varphi(z, x, r(z, w, \langle z, x, p \rangle)) \]
i.e. $\langle z, w \rangle \in u$; (otherwise undefined).
Clearly, \( h \in \Sigma_1 \).
Let \( x \in L_d[u] \). Set \( X = h^{\circ}(u \times \{x\}) \).

**Claim** \( X \subseteq L_d[u] \)

Let \( A \subseteq L_d[u] \) be \( \Sigma_1 \) in parameters from \( X \). We must show:

\[ V_y \mathcal{A} y \leftrightarrow V_y \mathcal{E} x \mathcal{A} y. \]

Let \( \bar{z} \subseteq x \) be the parameters of \( A \);
since \( \bar{z}_i = h(w_i, x) \) (\( w_i \in u \)), \( A \) is \( \Sigma_1 \) in parameters from \( uu \{u, x, p\} \).
Assume \( A \mathcal{A} y \). Then \( y = \gamma(z, \nu) \) for some \( z \in u, \nu < d \). The set \( A' = \{ \nu \mid A \gamma(z, \nu) \} \) is \( \Sigma_1 \) in parameters from \( uu \{u, x, p\} \). Hence there is a \( \varphi \) with constants from \( uu \{u, x, p\} \) s.t.

\[ A' \nu \leftrightarrow \Sigma^1 \varphi(x). \]

Set \( y' = \gamma(z, r(\varphi)) \). Then \( A \mathcal{A} y' \).
But \( y \in X \), since, letting \( \varphi = \Sigma(w, \langle x, p \rangle) \),
\[
y = \gamma(z, \varphi(\Sigma(w, \langle x, p \rangle))) = h(k, v, x)
\]

**Thm 2** The following conditions are equivalent:

(i) There is a \( \Sigma_1 \) a \( u \) s.t. \( a \notin \Lambda_2[u] \)

(ii) There is a \( \Sigma_1 \) map from a subset of \( u \) onto \( \Lambda_2[u] \).

**Proof.**

(iii) \( \rightarrow \) (i) is trivial, since \( a = \{ x \mid x \notin f(x) \} \) is \( \Sigma_1 \) but not an element of \( \Lambda_2[u] \),

for if not, we should have:

\[
x \in a \iff x \notin f(x)
\]

for some \( z \); hence:

\[
x \notin a \iff z \notin a.
\]
(i) → (ii). Let \( a \in u \in \Sigma_1 \), \( a \notin L_2[u] \).

Let \( a \in \Sigma_1 \) in \( x \). Set \( x = h^{"}(u \times \{x\}) \).

Let \( \pi : \langle x, e \rangle \leftrightarrow \langle L_\beta[u], e \rangle \). Then, if \( \varphi(x, z) \in \Sigma_1 \), the definition of \( a \), we have:

\[
\begin{align*}
\varphi(x, z) & \iff L_2[u] \\
& \iff L_\beta[u] \\
& \iff \varphi(x, \pi(x)).
\end{align*}
\]

Hence \( a \in \Sigma_1 \in L_\beta[u] \). But this means that \( \beta = 2 \), since otherwise \( a \in L_{\beta+1}[u] \subset L_2[u] \). Let \( h \) be \( \Sigma_1 \) in the parameter \( p \); in particular let:

\[
y = h(x, z) \leftrightarrow H(p, y, z, x),
\]

where \( H \in \Sigma_1 \) without parameters.

Set \( h'(z, x) = y \leftrightarrow h(x, \pi^{-1}(z)) \beta_{y+1}(+1), \)

(\( y \leftarrow h(z, \pi^{-1}(x)) \)) \( \iff (p(x), \pi^{(y)}, \pi(x)) \).

(\( y \leftarrow h(z, \pi^{-1}(x)) \)) \( \iff (p(x), \pi(y), \pi(x)) \).
Set: \( h'(x, x) \approx_{\pi} h(x, \pi^{-1}(x)) \). By (1):
\[ \pi(y) = h'(x, \pi(x)) \iff H(\pi(p), \pi(y), x, \pi(x)) \]
Thus, \( h' \in \Sigma_1 \) in \( \pi(p) \), and
\[ h' \cup \{ \pi(x) \} = \pi' \quad X = L_d[u] \]
Set \( f(x) = h'(x, \pi(x)) \). Then \( \text{dom}(f) \subseteq u \), \( f \in \Sigma_1 \) and \( f'' u = L_d[u] \). QED

As a corollary of Thm 2, we obtain:

**Thm 3** The following conditions are equivalent:

(a) There is a \( \Sigma_1 \) set \( a \subseteq u \) s.t. \( a \notin L_d[u] \)

(b) There is a \( \Sigma_1 \) map of \( u \) onto \( L_d[u] \)

**Proof:**

\( (b) \Rightarrow (a) \) follows as before

We now prove \( (a) \Rightarrow (b) \)
By Thm 2, there exists a \( \Sigma_1 \) map \( f' \) s.t. \( \text{dom}(f') \subseteq u \) and \( f'^{-1}u = L_d \{u\} \). We must replace \( f' \) by a \( \Sigma_1 \) map which is defined on the whole of \( u \). Since \( A \in \Delta_1 \), we have:

\[ z \epsilon a \iff \forall y \ A_0 y z \]
\[ z \epsilon a \iff \forall y \neg A_1 y z, \]

where \( A_0, A_1 \) are \( \Sigma_0 \). In particular,

\[ \forall z \epsilon u \forall y \ (A_0 y z \lor A_1 y z). \]

Set \( G y z \rightarrow A_0 y z \lor A_1 y z. \)

Set \( G y z \iff \forall y \in L_y \{u\} (A_0 y z \lor A_1 y z). \)

Set \( g \) uniformize \( \ast G \). Then \( g''u \) is unbounded in \( L_d \{u\} \) since if \( g''u \in L_y \{u\}, \ y < x \), we would have \( a \in L_{y+1} \{u\} \subseteq L_d \{u\}. \)
Since $f' \in \Sigma_1$, we have:

$$y = f'(x) \iff \exists z \ F z y x,$$

where $F \in \Sigma_0$. Set:

$$\bar{f}(y, x) = \begin{cases} y & \text{if } \forall z \in L_\Sigma[u] \ F z y x \\
& \text{and } y \in L_\Sigma[u] \\
0 & \text{otherwise} \end{cases}$$

Then $\bar{f}$ is p.r.

Set $f((z, w)) = \bar{f}(y(z), w)$ if $(z, w) \in u$

$f(x) = \emptyset$ otherwise.

Then $f'' u = f' ' u = L_\Sigma[u]$. QED
Non projectible admissible sets

Def Call \( M = \langle \IMI; e, A_1, \ldots, A_m \rangle \) non projectible iff \( M \) is admissible and satisfies the stronger replacement axiom:

\[ \forall u \forall v \forall x \exists u' ( \forall y \varphi \leftrightarrow \forall y v e u \varphi ) \]

where \( \varphi \) is in \( \Sigma_0 \).

One easily establishes the following Lemma: Let \( M \) be admissible; then the following are equivalent:

(a) \( M \) is non projectible

(b) \( x \in M \to x \cap A \in M \) for every \( \Sigma_1 \) set \( A \).

(c) \( x \in M \to \exists^c x \in M \) for every \( \Sigma_1 \) map \( f \).
We wish to characterize the \( \delta \) n.t.
\( L_d[u] \) in mon projectible. Our major
tool in this endeavour will be:

**Lemma 6** Let \( h \) be as in Lemma 4.
Let \( h \) be \( \Sigma_1 \) in the parameter \( x \).
Let \( h \in L_d[u] \) be transitive.
Let \( u \subseteq L_d[u] \) be transitive, closed
under finite sets, and let
\( u \cup \{ x \} \subseteq u \). Then
\[ V \delta \subseteq x \quad h^\infty \cup u = L_\delta [u]. \]

**Proof.** Let \( X = h^\infty \cup u \).
Obviously, \( X \subseteq L_d[u] \). Let
\( \pi : X \rightarrow L_\delta [u] \). Since \( \pi \upharpoonright u = \text{id} \upharpoonright u \),
we have: \( \pi h(x, u) = h(x, u) \) for
\( x \in u \), \( u \subseteq u \). Hence \( \pi \upharpoonright X = \text{id} \upharpoonright X \); \( \pi X = h^\infty \cup X = L_\delta [u] \).

QED
Thm 4. \( L_d[u] \) is mon projectible if there is a normal function \( \langle d, \nu < \lambda \rangle (\text{Lim}(\lambda)) \) s.t. \( d = \nu \cdot p \cdot d \) and \( L_{d, \nu}[u] \subseteq L_d[u] \) for \( \nu < \lambda \).

Proof:

(\( \leftarrow \)) Let \( \varphi \) be a \( \Sigma_0 \) formula. Let \( \nu \in L_d[u] \). Then \( \nu \in L_{d, \nu}[u] \) for some \( \nu \). For all \( x \in u \), we have:

\[
\models_{L_d[u]} \forall y \varphi(y, x) \iff \models_{L_{d, \nu}[u]} \forall y \varphi(y, x).
\]

Hence, for \( w = L_{d, \nu}[u] \):

\[
\models_{L_d[u]} A \times \nu \_{\in u} (\forall y \varphi \iff \forall y \exists w \varphi).
\]

The remaining admissibility axioms hold trivially by the fact that \( d \) is a limit ordinal.
(→) Since the set of \( \beta < \alpha \) n.t. \\
\( L_\beta[u] \leq \subset L_\alpha[u] \) is closed, we need only show that it is unbounded.

Let \( \alpha < \beta \). Claim: There is \( \beta < \alpha \) n.t. \( \gamma < \beta \) and \( L_\beta[u] \leq \subset L_\gamma[u] \).

Let \( \gamma \) be a limit ordinal n.t. \( \gamma \in L_\alpha[u] \), where \( \gamma \in \Sigma_1 \), and let \( \sigma > \gamma \) be a limit ordinal n.t. \( x \in L_\gamma[u] \), where \( h \in \Sigma_1 \) in the parameter \( x \). By Lemma 5:

\[
\forall x (L_\gamma[u]) = L_\beta[u] \leq \subset L_\gamma[u]
\]

for some \( \beta < \alpha \). But, by the non-projectibility of \( L_\alpha[u] \):

\[
\forall (L_\gamma[u]) \in L_\delta[u]
\]

hence \( \beta < \alpha \). QED
Thm 5 \( L_d[u] \) is non-projectible if there is no \( \Sigma_1 \) function which, for some \( \delta < d \), maps a subset of \( L_y[u] \) onto \( L_d[u] \).

**proof.**

\((\rightarrow)\) trivial

\((\leftarrow)\) Let \( L_d[u] \) be projectible. Then there is a \( \Sigma_0 \) relation \( R \) and a \( v \in L_d[u] \) s.t. for each \( \nu < d \) there is an \( x \in v \) with:

\[ \forall y \exists y_x \text{ but } \forall y \in L_y[u] \neg y_x, \]

Let \( r \) uniformize the relation:

\[ \forall y \in L_y[u] \ y_x. \]

Then \( y \upharpoonright \nu \) is unbounded in \( d \).

Let \( h \) be as in Lemma 4. Let \( h, \)

\( x \) be \( \Sigma_1 \) in the parameter \( x \)

and let \( u, x \in L_y[u] \), where
$\gamma$ is a limit ordinal. By Lemma 5:

$h'' \cup \mathcal{L}_\beta [u] = \mathcal{L}_\beta [u] \leq \mathcal{E}_\beta \mathcal{L}_\beta [u]$.

In particular, $\gamma'' \cup \mathcal{L}_\beta [u]$; hence $\beta = \delta$, since $\gamma'' \cup \mathcal{L}_\beta [u]$ is unbounded in $\delta$.

Set $f(\langle x, y \rangle) = h(x, y)$ for $x \in \mathcal{L}_\beta [u]$, $y \in \mathcal{L}_\beta [u]$. Then $f$ is $\mathcal{E}_\beta$; $\text{dom}(f) \subseteq \mathcal{L}_\beta [u]$ and $\text{ran}(f) = \mathcal{L}_\beta [u]$.

QED

We now come to the Thm announced at the outset of this section:

Thm 6 \( \mathcal{L}_\beta [u] \) is admissible if and only if there is no $\Delta_1$ function which, for some $\gamma < \delta$, maps $\mathcal{L}_\gamma [u]$ onto $\mathcal{L}_\beta [u]$.

Proof:

($\Rightarrow$) trivial
(≤) Let \( L_{d}[u] \) not be admissible. Then there is a \( \Sigma_{d} \) relation \( R \) s.t.
\( \forall y \, \forall x \, \forall y \, Ryx \) but for some \( u \in L_{d}[u] \),
there is no \( z < d \) with \( \exists x \in \forall y \, \forall x \in L_{d}[u] \, Ryx \). Let \( q \)
uniformize the relation \( \forall y \in L_{d}[u] \, Ryx \).

Then \( x \) is unbounded in \( d \), \( x \in \Sigma_{d} \) and defined everywhere. By
Thm 5, there is a \( s < d \) and
a \( \Sigma_{d} \) f s.t. dom \( (f) \subset L_{d}[u] \)
and \( s \forall y \), \( (f) = L_{d}[u] \). Let:
\[
\forall y = f(x) \leftrightarrow \forall z \, Fz y x,
\]
where \( F \) is \( \Sigma_{0} \). Set:
\[
\tilde{f}(y, x) = \exists \{ y \mid y \in L_{d}[u] \text{ and } \forall z \in L_{d}[u] \, Fz y x
\}
\]
Then \( \tilde{f} \) is p.l.s.
Take $\bar{f}$ as a limit ordinal $\lambda x$.

Set: $\bar{f}(\langle x, y \rangle) = \bar{f}(\alpha(x), y)$. 

Then $\bar{f}$ is defined everywhere and $\bar{f}'' \cup L_{\bar{\lambda}}[\alpha] = f'' \cup L_{\bar{\lambda}}[\alpha]$. At $\alpha$ is a limit ordinal and $\alpha \in L_{\bar{\lambda}}[\alpha]$, then $\bar{f}$ maps $L_{\bar{\lambda}}[\alpha]$ onto $L_{\bar{\lambda}}[\alpha]$. QED
The projectum

Def \( d^* = \inf \beta \) s.t. there is a \( \Sigma_1^1 \) function mapping a subset of \( \beta \) onto \( L_\beta [u] \). \( d^* \) is called the projectum of \( d \).

By Thm 5, \( L_d [u] \) is non-projectible iff \( d = d^* \).

Thm 4: \( d^* > 0 \), then \( L_{d^*} [u] \) in mon projectible.

Proof: \[ d^* = d, \quad L_{d^*} [u] \]

If \( d^* = d \), the theorem is trivial.

Now let \( d^* < d \). There is no \( f \in L_d [u] \) mapping a \( \epsilon < d^* \) onto \( d^* \), for then: \[ g(\langle x, y \rangle) = \eta (x, f(y)) \quad \text{if} \quad x \in u, \nu < \varepsilon \]
\[ g (z) > 0 \quad \text{if not} \]
would map \( L_\varepsilon [u] \) onto \( L_{d^*} [u] \).

By composition, we would obtain a \( \Sigma_1^1 \) map of \( L_\varepsilon [u] \) onto \( L_d [u] \).
But this means that $d^*$ is p.r. closed, for, as we shall show in an appendix, whenever $\sigma$ is p.r. closed and $\beta$ is the first p.r. closed ordinal after $\sigma$, each $\gamma < \beta$ is 1-1 mappable into $\sigma$ by a map $f \in L_\beta$. If $d^*$ were not p.r. closed, we should have $\sigma < d^* < \beta \leq d$ for such a pair $\sigma, \beta$; $f \in L_\beta$, hence some $f \in L_\beta \lceil \sigma \rceil \subseteq L_{d^* \lceil \sigma \rceil}$ would map $\sigma$ onto $\beta$. But, since $d^*$ is p.r. closed, we may apply Thm 5 to conclude that $L_{d^* \lceil \sigma \rceil}$ is non-projectible, for otherwise there would be $\sigma < d^*$ mappable onto $d^*$ by an $f \in L_{d^* \lceil \sigma \rceil}$. $\square$
$\Sigma_m$-admissibles

**Def** $M = \langle IMI, \varepsilon, A_1, \ldots, A_m \rangle$ is called $\Sigma_m$-admissible ($m \geq 1$) iff $M$ is admissible and satisfies the replacement axiom:

$$\forall x \forall y \varphi \rightarrow \forall u \forall v \forall x \exists u \forall y \exists v \varphi$$

for $\Sigma_{m-1}$-formulas $\varphi$.

(Thus, 'admissible' = '\( \Sigma_1 \)-admissible')

**Def** $M$ is called $\Sigma_m$-non-projectible iff $M$ is admissible and satisfies:

$$\forall u \forall v \forall x \exists u \left( \forall y \varphi \leftrightarrow \forall u \exists v \varphi \right)$$

for $\Sigma_{m-1}$-formulas $\varphi$. 
We can readily establish:

(1) $M \in \Sigma_m$ admissible iff $\langle M, \frac{\Sigma_m}{M} \rangle$ is admissible

(2) $M \in \Sigma_m$ non-projectible iff $\langle M, \frac{\Sigma_m}{M} \rangle$ is non-projectible

(3) If $M \in \Sigma_{\alpha}$ admissible, then $P \in \Sigma_{\alpha}$ iff $P \in \Sigma_{\alpha}$ in $\Sigma_{\alpha}$ relations.

Thus, all the theorems of §1 carry over to $\Sigma_{\alpha}$ admissible. Some of the theorems in this section carry over.

In particular, we shall obtain slightly weaker analogues of Thm 4–Thm 6.
Lemma 7. If \( \langle L_2[u], A \rangle \) is admissible, then \( \langle L_2[u], A \rangle \) is \( \Sigma_2 \)-uniformizable.

Proof.
Let \( R \) be \( \Sigma_1 \), any \( (R) \in d \).
Let \( R \supseteq \forall y \exists \exists z. P_y \lor \exists z \),
where \( P \in \Pi_2 \).
Set: \( p(x) \equiv \forall y \exists z. P_x \exists z \).
where \( \langle l(x), r(x) \rangle = \sigma \).
Then \( p \in \Sigma_2 \), since:
\[
\forall y = p(x) \iff \forall y \exists \exists z. P_x \exists z \exists z \ \text{and}
\forall x \in \Pi_2 \\exists \exists z. P_x \exists z \\
\land \forall z \exists x. P_x \exists z \\
\land \forall z \exists x. P_x \exists z \\
\land \forall z \exists x. P_x \exists z \\
\Sigma_2
\]
Set: \( r(x) \equiv l(p(x)) \). Then \( r \) uniformizes \( R \).
QED
(Note: This proof also goes then on the assumption \( \Sigma_4 \in \Sigma_4 \in \Sigma_2 \)).
Since, if \( L_d[u] \) is \( \Sigma_m \)-admissible, \\
\( \langle L_d[u], \Sigma_m^{m-1} \rangle \) is admissible and \\
\( \Sigma_m(\langle L_d[u], \Sigma_m^{m-1} \rangle) = \Sigma_m(L_d[u]) \), \\
we get:

**Corollary 7a** \( \vdash \) \( L_d[u] \) is \( \Sigma_m \) \\
admissible, then \( L_d[u] \) is \( \Sigma_{m+1} \) \\
uniformisible.

**Lemma 8** \( \vdash \) \( L_d[u] \) is \( \Sigma_m \)-uniformisible, then there is a \( \Sigma_m \) \\
function \( h \) s.t. \( \text{dom}(h) \subseteq u \times L_d[u] \) \\
and \\
\[ \forall x ( x \in h^n(u \times \emptyset) \subseteq L_d[u] ). \]

**Lemma 8** is proved exactly like 
**Lemma 4**, which is a special 

case of it.
Lemma 2 obviously holds with $E_m$ in place of $E_1$ \((m \geq 2)\), since $X \leq_{E_m} L_d[u]$ implies $X \leq_{E_1} L_d[u]$.

Lemma 3 does not hold, but we do get the weaker form:

\underline{Lemma 9} \quad \forall X \leq_{E_m} L_d[u] \quad \text{and} \quad \pi : \langle X, e \rangle \overset{\sim}{\leftrightarrow} \langle L_d[u], e \rangle,

Then for every $E_1$-f (which is $E_1$ in parameter $x \in X$ s.t. $\pi(x) = x$):

$$\pi f(\vec{x}) \equiv f(\pi(\vec{x})) \quad \text{for} \quad \vec{x} \in X.$$ 

The proof is obvious.

Using Lemmas 8, 9 in place of Lemmas 4, 3, we get

\underline{Thm 7} \quad \forall L_d[u] \in E_m \quad \text{uniformly},

then the $E_m$ analogues of Thm 2, Thm 3 hold.
The proofs of Thm 2, Thm 3 can be repeated word for word to obtain Thm 7.

By Lemma 7, then, the $\Sigma_{n+1}$ analogues of Thm 2, 3 hold whenever $L_d [u]$ is $\Sigma_n$ admissible.

We shall show later that this result can be greatly strengthened. The hypothesis of Thm 7 is always satisfied. But first we turn to the question of criteria for $\Sigma_n$ admissibility and non-projectivity.

The $\Sigma_n$ analogue of Thm 6 does not hold. For, in, letting $L_{\omega_1} [u]$ admits no function mapping an element onto the entire domain, yet
$L_\omega[u]$ is not admissible, since $\langle \omega \mid m < \omega \rangle$ is $\Sigma_2$ (understanding $\omega$ in the sense of $L[u]$).

The analogues of Thms 4, 5, 6 do hold, however, on the assumption that, for some $\beta < \omega$, $L_\beta[u]$ can be mapped onto each $x \in L_\alpha[u]$ by an $f \in L_\alpha[u]$. Since $L_\alpha[u] = L_\alpha[L_\beta[u]]$, it suffices to prove this for the case $\beta = 0$ ($L_0[u] = u$).

Def: $L_\alpha[u]$ is $u$-dense iff for all $\delta < \alpha$ there is an $f \in L_\alpha[u]$ mapping $u$ onto $\delta$. 
By Lemma 1, \( u \)-density is equivalent to the condition, that \( u \) can be mapped onto each \( x \in L_d[u] \) by an \( f \in L_d[u] \).

**Lemma 40** \( \forall X \subset L_d[u] \) in \( u \)-dense and \( X \subset \bigcup_m L_d[u] \), then

\[ \forall \beta \leq d \quad X = L_\beta[u] \]

**Proof.** By Lemma 2 it suffices to show that \( X \) is transitive.

Let \( x \in X \). We wish to show: \( x \subset X \). The statement:

\[ \forall f : f \circ u \rightarrow x \]

holds in \( \langle L_d[u], \epsilon \rangle \), hence in \( \langle X, \epsilon \rangle \). Thus there is an \( f \in X \) s.t. \( f \circ u \rightarrow x \). But then \( f(z) \in X \) for each \( z \in u \); hence \( x = f^\prime u \subset X \). Q.E.D
Using Lemma 10 in place of Lemma 6, we can repeat the proofs of Thms 4, 5, 6 to obtain:

(*) If \( L_2[u] \) is \( \Sigma_n \) uniformizable, then the \( \Sigma_n \) analogues of Thms 4, 5, 6 hold.

(The proofs can be repeated word for word)

But this enables us to prove the \( \Sigma_n \) analogues of those Thms outright. We use induction on \( m \). For \( m=1 \) the Thms are proven. Now suppose the Thms to hold for \( m \). Then either \( m \) is admissible, or else the Thms hold trivially for all \( m \geq m \). But if \( m \) is admissible, then by Lemma 7 \( L_2[u] \) is \( m+1 \) uniformizable and the Thms hold for \( m+1 \) by (*).
Thus:

**Thm 8**  If \( L_\Delta[u] \) is \( u \)-dense, then the \( \Sigma_n \) analogues of Thms 4, 5, 6 hold for \( m \geq 1 \).

\[ \ldots \ldots \ldots \]

\underline{u-uniformizability}

**Def**  A function \( \Phi(z, \bar{x}) \) is called a \( u \)-uniformization of a relation \( R_{\bar{x}} \) iff \( \text{dom}(r) = u \times \text{dom}(R) \), \( \text{rng}(r) \subseteq \text{rng}(R) \) and \( \forall y \, R_{\bar{x}} \iff \forall z \in u \, R_{\bar{z}, \bar{x}} \).

**Def**  \( M = \langle \text{IM1}, e, A_1, \ldots, A_m \rangle \) (\( u \)-t. \( u \in M \)) is \( \Sigma_n \) \( u \)-uniformizable iff every \( \Sigma_n \) relation is \( u \)-uniformizable by a \( \Sigma_n \) function.
Until now we have worked with the notion of ordinal uniformizability (i.e., uniformizability of relations with ordinal range) rather than $u$-uniformizability. However, ordinal uniformizability implies $u$-uniformizability for $L_d[u]$ (and, indeed, the efficacy of ordinal uniformizability as a tool depends on this fact).

**Lemma 11** \[ \Rightarrow L_d[u] \in \Sigma_\infty \]

If ordinal uniformizable, then $L_d[u]$ is $u$-uniformizable.

**Proof.**

Let $R \subseteq \infty$ be $\Sigma_\infty$. Let

\[ R \subseteq V \]  

where $R \subseteq \Pi^1_n$.

Set $s \in P \langle s \rangle \iff \langle s \rangle_0 \subseteq \langle s \rangle_1 = y$. Set $s \in x \iff P [P \langle s \rangle x] \forall x \in u$.
proof of Lemma II.

We first show that each $T_{n-1}$ relation is $\Sigma_n$-uniformizable by a $\Sigma_n$ function. Let $R$ be $T_{n-1}$. Set: $G_{r, \bar{x}} \iff \forall z \forall \eta (x, y) \exists \bar{x}$, $G \in \Sigma_n$. Let $q$ uniformize $G$.

We may assume w.l.o.g. that $R \neq \emptyset$, hence that $y \in \text{rng}(R)$.

Set:

$$r(z, \bar{x}) = \left\{ \begin{array}{l}
\eta (z, q(\bar{x})) \vdash R (z, q(\bar{x})) \exists \bar{x} \\
y \vdash \neg R (z, q(\bar{x})) \exists \bar{x}.
\end{array} \right.$$  

Then $r$ uniformizes $R$.

Now let $R$ be $\Sigma_n$. Let:

$$G_{r, \bar{x}} \iff \forall z P z y \bar{x},$$

where $P \in T_{n-1}$. Set:

$$P' (z, y) \bar{x} \iff P z y \bar{x},$$

and let $p$ uniformize $P$. Set:

$$r(w, \bar{x}) = (p(w, \bar{x})).$$
(where \( \langle z, y \rangle \), \( z \in z, (\langle z, y \rangle), y \in y \)).

Then \( \pi \) \( u \)-uniformizes \( R \). QED

All previous theorems in which ordinal uniformizability was mentioned as an assumption hold on the (apparently) weaker assumption of \( u \)-uniformizability. In particular:

**Lemma 12**: \( \forall L_d[U] \in \Sigma m \) \( u \)-uniformizable, then there is a \( \Sigma m \) function \( h \) s.t. \( \text{dom}(h) \subset u \times L_d[U] \) and

\[
\forall x \in L_d[U] \ (x \in h^{-1}(u \times \{x\}) \subseteq L_d[U])
\]

**Proof**: We imitate the proof of Lemma 4.

Letting \( s(z, x) \) s.t. \( s : u \times \{x\} \xrightarrow{\text{onto}} \Sigma m \cup \{u, u \times u, x\} \)

be as before, we set

\[
R \times \varphi \leftrightarrow \Sigma m \varphi(\overline{v_0} / x)
\]

and let \( \pi \) uniformize \( R \). Set:

\[
h(s(x, x), x) \equiv \pi(z, s(u, (x, p)))
\]

and \( \pi \) uniformize \( R \) in the proposition. QED.
Carrying through the earlier proofs, again virtually without change, we get:

**Thm 9** Let \( L_d[u] \) be \( \Sigma_m \) u-uniformizable. Then the following are equivalent:
(a) There is a \( \Sigma_m \) set \( a \subset u \) s.t. \( a \notin L_d[u] \).
(b) There is a \( \Sigma_m \) map \( f \) s.t.
\[ \text{dom}(f) \subset u \text{ and } f``u = L_d[u]. \]

**Thm 10** Let \( L_d[u] \) be \( \Sigma_m \) u-uniformizable. Then the following are equivalent:
(a) There is a \( \Delta_m \) set \( a \subset u \) s.t. \( a \notin L_d[u] \).
(b) There is a \( \Delta_m \) map of \( u \) onto \( L_d[u] \).

We now prove:

**Thm 11** \( L_d \) is \( \Sigma_m \) u-uniformizable \((m \geq 1)\).
The proof of Thm 11 extends over several lemmas. From now on, we shall write "uniformizable" to mean "u-uniformizable".

**Lemma 13** Let $L_d[u]$ be admissible and let $A \subseteq L_d[u]$ be s.t.
\[ x \in L_d[u] \rightarrow A \land x \in L_d[u]. \]
Then $\models_A \Xi^p \models \Delta_4$ in $\langle L_d[u], A \rangle$. Moreover, $R \models \Xi_4$ in $\langle L_d, A \rangle$ iff $R \models \Xi_4$ in $\langle L_d[u], \models_u \Xi^p \rangle$.

**Proof.**
We first show that $\models_A \Xi^p \models \Delta_4$.
Set $T : a(x) = \models_A A \land x$.
$L_d[u]$ is closed under $a$. $a \models \Xi_4$ since
\[ q = a(x) \iff q \land \forall z \in x (z \in y \iff Ax). \]
Thus $\models_A \Xi^p \models \Delta_4$.
\[ \frac{\varphi \vdash \psi}{\frac{\vdash \varphi}{\varphi}}, \quad \langle C(\varphi), a(C(\varphi)) \rangle \]

But, by the same argument,

\[ \frac{\varphi \vdash \psi}{\frac{\vdash \varphi}{\varphi}}, \quad \langle C(\psi), a(C(\psi)) \rangle \]

This establishes the second part of the lemma. \( QED \)

Using Lemma 13, we can repeat the proofs of Thm 1 and to obtain the analogues.
**Lemma 14.** If $L_d[u]$, $A$ are as in Lemma 13, then $\langle L_d[u], A \rangle$ is $\Sigma_1$ uniformizable.

Since the only two facts used in the proof of Lemma 12 were: $\Sigma_1$ uniformizability and the $\Sigma_1$ definability of $E_n$, we may repeat the proof to obtain:

**Lemma 15.** If $L_d[u]$, $A$ are as in Lemma 13 and if $\langle L_d[u], A \rangle$ is $\Sigma_1$ uniformizable, then there is a $\Sigma_1$ Skolem function (i.e. an $h$ s.t. $\text{dom}(h) \subseteq u \times L_d[u]$ and $\forall x (x \in h \iff (u \times L_d[u]) \subseteq \langle L_d[u], A \rangle$). In particular, by Lemma 14, there is a $\Sigma_1$ Skolem function.

**Def.** $\langle L_d[u], A \rangle$ is called feasible iff for every $\Delta_1$ set $B$ we have:

$$x \in L_d[u] \rightarrow \exists n x \in L_d[u]$$

$\langle L_d[u], A \rangle$ is called $\Sigma_1$-feasible iff this holds for every $\Delta_1$ set $B$. 
Lemma 16 Let $\langle L_d[u], A \rangle$ be $\Sigma_n$-feasible but not $\Sigma_n$ admissible. Let $\langle L_d[u], A \rangle$ be $\Sigma_n$ uniformizable. Then a relation $R$ in $\Sigma_1$ in $\Sigma_n$ iff $R$ in $\Sigma_{n+1}$.

Proof:
(\rightarrow) trivial, since each $\Sigma_{n+1}$ relation in $\Sigma_1$ in $\Sigma_n$.
(\leftarrow) Since $\langle L_d[u], A \rangle$ is not $\Sigma_n$ admissible, there is a $\Pi_{n+1}$ relation $R$ and a $\beta < d$ s.t. $\forall x \exists y Ryx$ but for each $\delta < d$: $\forall x \in L_\beta[u] \rightarrow \forall y \in L_\delta[u] Ryx$.
Set: $Gyx \iff \forall y \in L_\delta[u] Ryx$
and let $y$ uniformize $G$. Then $y \in \Delta_n$, $\text{dom}(y) = u \times L_\beta[u]$ and $\forall x \in L_\beta[u] = L_d[u]$. Let $h$ be a $\Sigma_n$ Skolem function for $\langle L_d[u], A \rangle$. ($h$ exists by Lemma 15).
Let:
\[ y = h(x, z) \iff \forall u \in H \land y = x, \]
where \( H \in \mathcal{P}(m-1). \) Set:
\[ h^*(u, z, x) = \begin{cases} y & \text{if } \exists y \in L_y(u) \land \exists u \in L_y(u) \land y = x \land \text{not } \end{cases} \]
Then \( \text{dom}(h^*) = (u \times L^0_y(u)) \times L_d [u] \) and
\[ h^*(u, x, x) = h^*(u, x, x) \]
for all \( x \in L_d [u]. \)

For \( \delta < \alpha \) set:
\[ \delta = (u \times L^0_y[u]) \times L^0_y[u] \]
\[ e(\delta) = \{ \langle x, y \rangle \mid x, y \in \delta \land h^*(x) \in \delta^*(y) \} \]
\[ a(\delta) = \{ x \mid x \in \delta \land A(h^*(x)) \} \]
Since, for each \( \delta < \alpha, \) \( e(\delta), a(\delta) \) are \( \Delta_0 \) subsets of \( \delta, \) we have:
\[ e(\delta), a(\delta) \in L_d [u] \quad \text{for } \delta < \alpha. \]
Let $m(v) = \langle u, e(v), a \rangle$, where $u \in \Sigma$ and for some $\pi$:

$$\pi : \langle \bar{v}, e(x), a(x) \rangle \rightarrow m(v).$$

By the admissibility of $L_\Sigma[u]$ (Thm 3, Thm 6), we may conclude that $m(v)$, $\pi$ are elements of $L_\Sigma[u]$. This follows by the recursion theorem, since the factorisation of $e(v)$ by extensional equivalence is certainly in $L_\Sigma[u]$ and the factorised $e(v)$ is well founded.

Thus, $L_\Sigma[u]$ is closed under the function $m(v)$. We show now that $m$ in $\Delta_m$. Since $e(x)$ in $\Sigma_{m+1}$ we have:

$$\forall y \in e(x) \implies y \in \bar{e}^2 \implies \lambda \exists \Sigma, \nu \in \bar{e} \langle \bar{v}, \bar{w} \rangle \in y \iff \lambda x (z) \in \bar{h}(y).$$

$$\Delta_m \implies \Sigma_m.$$
Similarly, \( a(\varphi) \) is \( \Delta_{n+1} \).

This means that \( m \in \Delta_{n+1} \), since:

\[
y = m \varphi(x) \iff \forall \pi \left( \pi : \langle \varphi, e(x), a(\varphi) \rangle \leq y \right)
\]

\( \pi : x \leq y \) being \( \Delta_1 \).

To establish the lemma, we need only show that relations \( \Sigma_0 \) in \( \Sigma_m \) relations are \( \Sigma_{m+1} \).

Let the formula \( \varphi \) be \( \Sigma_0 \) in \( \Sigma_m \) (i.e., built up from \( \Sigma_m \) formulae by sentential operations and bounded quantifications).

Then

\[
| \langle L_a[u], A \rangle | \varphi \iff \forall x < d \left( \exists m(x) \wedge
\varphi \in L_a[u] \wedge \frac{\varphi}{m(x)} \right)
\]

QED
Note that the assumption $\Sigma_{m+1} = \Sigma_1$ in $\Sigma_m$ can be used alternatively to $\Sigma_m$ admissibility to carry out the proof of Lemma 7; hence:

Lemma 17: If $\langle L_d[u], A \rangle$ is $\Sigma_m$-feasible, then $\langle L_d[u], A \rangle$ and $\Sigma_m$-uniformisable, then $\langle L_d[u], A \rangle$ is $\Sigma_{m+1}$ uniformisable.

We are now ready to prove Thm 11. We proceed by induction on $m$. For $m=1$ the theorem is proven. We now suppose it to hold for $m$ and prove it for $m+1$.

Case I: $L_d[u]$ is $\Sigma_m$-feasible.

The conclusion follows by Lemma 17.
At Case I fails, there is a $\beta < \alpha$ s.t. a $\Delta^m_\alpha a \in L^\beta[u]$ exists with $a \in L^\beta[u]$. Let $\beta$ be the least such. By Thm 10, there is a $\Delta^m_\alpha$ map from $L^\beta[u]$ onto $L^\alpha[u]$. Case II. $\beta = 0$ (hence $L^\beta[u] = u$).

We first show that each $\Delta^m_\alpha$ relation is uniformizable by a $\Sigma^m_\alpha$ function. Let $R \subseteq^x x$. Assume (w.l.o.g.) $y \in \text{rng}(R)$.

Set $\pi(x, \overline{x}) = \begin{cases} f(x) & R f(x) \overline{x} \\ y & R f(x) \overline{x} \wedge \forall y \neg R y \overline{x} \end{cases}$

Then a uniformizer $P$. At $R \subseteq^x x$, there is $\Pi^m_{\alpha+1} P$ s.t.

$R \overline{x} \iff \forall z P z \overline{y} \overline{x}$.

Set $P'(z, y, \overline{x}) \iff P y \overline{x}$. 

Let $p$ uniformize $P'$ and set:
\[ r(w, x) = (p(w, x'))_1 \]
Q.E.D. Case II

Case III $\beta > 0$.
Then $L_\beta[u]$ will be admissible by the same argument which demonstrated that the projectum of $d$ is admissible.

**Lemma 18** At $A$ in $\Delta_n (L_\delta[u])$ and $A \subseteq L_\beta[u]$, then each $T \subseteq L_\beta[u]$ which is $\Sigma_m (\langle L_\beta[u], A \rangle)$ is $\Sigma_{m+1} (L_\delta[u])$.

**Proof.**
At suffices to show: At $R$ in $\Sigma_0 (\langle L_\beta, A \rangle)$, then $R \in \Sigma_{m+1} (L_\delta)$.
Let $\varphi$ be a $\Sigma_0$ formula of $\langle L_\beta, A \rangle$. 
Then

\[ \frac{\varphi \iff \varphi}{A} \]

\[ \langle C(\varphi), A \land C(\varphi) \rangle \]

But \( a(u) = A \cup u \) is a \( \Sigma_2 (L_{\alpha}[u]) \) function which is defined on all \( L_{\beta}[u] \). Hence \( \frac{\varphi \in \Sigma_2 (L_{\alpha}[u])}{\varphi \in \Sigma_2 (L_{\alpha}[u])} \) \( \omega E D \)

Letting \( f : L_{\beta} \to L_{\alpha} \) be \( A \circ (L_{\alpha}) \), pick \( A \in L_{\beta} \) in such a way that:

\[ \{ \langle x, y \rangle \mid f(x) \in f(y) \} \subseteq f^{-1}[L_{\beta}[u]], \quad f^{-1}[L_{\beta}[u]], ~ \]

\[ \{ \varphi \in \text{Form} \mid \frac{\varphi}{L_{\beta}} \}\]

\[ \frac{\varphi}{L_{\alpha}} \]

are \( \Sigma_0 \) in \( \Delta \langle L_{\beta}, A \rangle \). (Setting

\[ \overline{f}(\varphi(x)) = \varphi(f(x)) \]

Then every \( \Sigma_0 (L_{\alpha}[u]) \) relation

\[ R \subseteq L_{\beta}[u]^m \in \Sigma_0 (\langle L_{\beta}, A \rangle), \]
Using an obvious abbreviation, we have then:

\[ \Sigma_m(L_d) \subset \Sigma_1(\langle L_\beta, A \rangle) \subset \Sigma_{m+1}(L_d). \]

On which side, if any, of this chain of conclusions does the identity lie?

We consider two cases:

Case 1: There is a \( \gamma < \beta \) and a \( \Sigma_m(L_d) \) function \( q \) such that \( \text{dom}(q) = L_\gamma \cdot [\alpha] \) and \( q \upharpoonright L_\gamma \cdot [\alpha] \) is unbounded in \( d \).

In this case, we prove that, for an appropriate choice of \( A \):

\[ \Sigma_1(\langle L_\beta, A \rangle) = \Sigma_{m+1}(L_d). \]

But, by Lemma 14, \( \langle L_\beta, A \rangle \) is \( \Sigma_1 \) uniformizable.
Case 2: Case 1 fails.

In this case we show that
\[ \Sigma_m(L_d) = \Sigma_1(\langle L_\beta, A \rangle). \]

But then \( \langle L_\beta, A \rangle \) is feasible and, by Lemmas 14, 17, \( \langle L_\beta, A \rangle \) is \( \Sigma_2 \) uniformizable, whereby:
\[ \Sigma_{m+1}(L_d) = \Sigma_2(\langle L_\beta, A \rangle). \]

In either case, we may conclude that, if \( R \subseteq L_\beta[u]^{m+1} \) in \( \Sigma_{m+1}(L_d) \), then \( R \) is uniformizable by a \( \Sigma_{m+1}(L_d) \) function. Now let \( R \subseteq L_d[u]^{m+1} \).

Set:
\[ R' \Delta \leftarrow R f(y) f(x). \]

Let \( \rho' \) uniformize \( R' \). Let \( f' \)

uniformize \( f(x) = y \) and set:
\[ \rho(\langle \varepsilon, \overline{w} \rangle, \overline{x}) \sim f' \varepsilon, f'(\overline{w}, \overline{x}), \ldots, f'(\overline{w}_m, \overline{x}_m). \]

Then \( \rho \) uniformizes \( R \).
Thus, it remains only to prove the assertions made in Case 1, 2.

**Lemma 19.** Let \( \alpha < \beta \) and let there be a \( \Delta_m(\mathbb{L}_d) \) function \( \varphi \) which maps \( \mathbb{L}_\alpha[u] \) onto \( \mathbb{L}_\beta[u] \). Then \( A \) can be so chosen that every \( R \in \mathbb{L}_\beta[u]^m \) which is \( \Sigma_{m+1}(\mathbb{L}_d) \) in \( \Sigma_1(\langle \mathbb{L}_\beta, A \rangle) \).

**Proof.** It suffices to show: If \( R \in \mathbb{L}_\beta[u]^m \) in \( \Sigma_m(\mathbb{L}_d) \), then \( R \) is \( \Delta_m(\langle \mathbb{L}_\beta, A \rangle) \). For this, it suffices that

\[
\{ \varphi \in \text{Fml}[\mathbb{L}_\beta[u]] : R_{\beta[u]} \varphi \}
\]

is \( \Delta_1(\langle \mathbb{L}_\beta, A \rangle) \). Let \( h \) be a \( \Sigma_m \) Sholem function for \( \mathbb{L}_\alpha[u] \).

Let \( \varphi \in \text{Fml}[\mathbb{L}_\beta[u]] \).
Let $y = h(z, x) \rightarrow V u H y z x$, where $H$ is $T n-1 (L_d[u])$. Define:

$$h^* (\langle w, z \rangle, x) = \begin{cases} y & \text{if } y \in L_q(w)[u] \text{ and } V u \in L_q(w) H y z x \\ 0 & \text{if not} \end{cases}$$

$h^*(z, x) = 0$ in all other cases.

Then $\text{dom}(h^*) = L_\kappa[u] \times L_d[u]$ and

$$\forall x \in L_\kappa[u] \times \{x\} \exists y \in L_\kappa[u] \times \{x\} \Rightarrow h^*(x) = h^*(y).$$

Choose $A$ in such a way that $R$ in $\Sigma_0 \forall (\langle L_\beta, A \rangle)$, where:

$$R x y \leftrightarrow x, y \in L_\kappa \times L_\beta \land h^*(x) \in h^*(y).$$

$R \in \Delta_m (L_\beta)$.

Set: $\bar{\delta} = L_\kappa \times L_\delta$, 

$$e(\delta) = \bar{\delta}^2 \cap R$$

Then $\forall \delta < \beta e(\delta) \in L_\beta[u]$, since $e(\delta)$ is a $\Delta_m (L_\beta)$ subset of $\bar{\delta}^2$. 


The function \( e(\bar{s}) \in \Sigma_1(L_\beta, A) \), since:

\[
e = e(\bar{s}) \iff \bar{e} \in \bar{s}^2 \land \forall x \in \bar{s} \quad (x, y) \in e \iff R_{xy},
\]

Set: \( m(\bar{s}) = \text{that } \langle v, e(v) \rangle \text{ s.t. } uv < v \text{ and } \langle v, e \rangle \iff \langle \bar{s}, e(\bar{s}) \rangle \iff \langle k^* \bar{s}, e \rangle \).

Imitating the methods of the proof of Lemma 16, we get: The function \( m \in \Sigma_1(L_\beta, A) \) and is defined everywhere. But this means that \( \{ \varphi \in \text{Fm}(L_\beta) \mid \models \varphi \} \in \Sigma_1(L_\beta, A) \),

\[
\Delta_1(L_\beta, A), \text{ since } \bar{s},
\]

nothing \( s(\varphi) = \mu s(\varphi) \cap \varphi \in \text{Fm}(L_\beta) \),

we have:

\[
\models^\Sigma \varphi \iff \models^m \varphi
\]

for \( L_\beta \models \varphi \iff \text{Fm}(\varphi) \). \( \Box \)
Lemma 20: If the hypothesis of Lemma 19 fails, then every $\Sigma_1(\langle L_\beta, A \rangle)$ relation is $\Sigma_\mu(L_d)$ (hence $\langle L_\beta, A \rangle$ is feasible).

Proof:
It suffices to show: If $R \in \Sigma_0(\langle L_\beta, A \rangle)$, then $R \in \Delta_\mu(L_d)$. We show this by induction on the defining formula of $R$, using

(*) If $Ry \leftrightarrow y \in L_\beta[u] \land \exists z_\psi Rz \rightarrow x$, then

so we:

$Ry \leftrightarrow \exists y \in L_\beta[u] \land \exists z_\psi Rz \rightarrow x$, proof of (*):

Let $Ry \leftrightarrow \forall u Puy \rightarrow x$, where $P \in \text{TM}_\mu$. For $y \in L_\beta[u]$, we

have:

$\exists z_\psi \forall y \in Puy \rightarrow x \rightarrow \forall z_\psi \exists y \in Puy \rightarrow x$. \hfill \square
\[ \lambda z \epsilon y \ P_{\epsilon z} x \rightarrow \forall z \alpha \lambda z \epsilon y \ V_{\epsilon z} y \ P_{\epsilon z} x. \]

since otherwise, letting \( p(w, z) \)
uniformize the relation:

\[ \neg \forall z \epsilon \ L \ [ ( P_{\epsilon z} x ) ] \]

\( p \) would map \( u \) \( xy \) unboundedly
into \( a. \) (Contradiction!)

Hence:

\[ \lambda z \epsilon y \ P_{\epsilon z} x \leftrightarrow V w \lambda z \epsilon y \ V_{\epsilon w} y \ P_{\epsilon w} x. \]

We apply the same reduction
to \( V_{\epsilon w} y \ P_{\epsilon w} x \) etc. until we
are left with a \( \Sigma \) \( \in \) formula. Q.E.D.