Addendum to Robust extenders

We present a simplified version of the methods needed for a proof of [RE] § 1 Theorem 1, which says that Steel's main realizability lemma for arrays with "background certification" also holds for arrays in which the extenders satisfy the weaker condition of robustness. As in [RE] we prove the theorem in ZFC, whereas Steel worked in \( \text{H}_\kappa \) for an inaccessible \( \kappa \). Our new proof is technically simpler. In particular, we dispense with Steel's function \( c(\alpha, \beta) \).

§1 Basic definitions and results

We recall the definition of robustness,

**Def.** Let \( \mathcal{N} = \langle J^\mathcal{N}, F \rangle \) be an active premouse, \( F \) in robust wtt. \( \mathcal{N} \) iff whenever \( UC \lambda = \lambda \), and \( W \subseteq P(\lambda) \cap W (\kappa = \kappa_\nu) \) is countable,

then there is \( g: U \rightarrow \kappa \) s.t.

\[ \forall x \in X \quad \exists \bar{z} \in F(x) \] \[ \forall a \in g(\bar{z}) \quad \exists \bar{y} \in F(a) \] \[ \exists \bar{z} \in F(\lambda) \]

whenever \( a_1, \ldots, a_m \in U \) and \( X \in W \).

(Here \( \bar{z} \) is a Gödel-tuple function on ordinals.)

(b) Let \( U_1, \ldots, U_m \subseteq U \), \( T = \text{lub} U \), \( E = \text{lub} g " U \).

Let \( \varphi \) be a \( \Sigma_1 \) formula. Then

\[ C^E_{\zeta, \kappa} \models \varphi[\bar{g} " U_1, \ldots, " U_m] \quad \forall \bar{z} \in \mathcal{N} \]

(Here \( C^E_{\zeta, \kappa} \) is the Chang hierarchy up to \( \alpha \) over \( \langle L^E_{\zeta}, \mathcal{E}, E \cap L^E_{\zeta} \rangle \). The precise definition is given in [RE] § 1.)

\* [RE] = "Robust Extenders"
(We recall the condensation principle; let $C \subseteq C^E$ r.t. $\gamma \leq C$ and $[C] \subseteq C$.
Then $C$ is isomorphic to a $C^{E^F}$ for any $\gamma \leq \kappa$. From this it follows that if $\lambda^{<\kappa} \leq \kappa$ for all $\alpha < \kappa$, then $C^{E^{\lambda, \kappa}} \subseteq C^{E_{\lambda, \kappa}}$ for all $\gamma \leq \kappa$. This holds in particular for $\kappa = (2^\beta)^+$.)

As in [RE]§2 we also define:

**Def.** Let $N = \langle V, \in \rangle$ be as above, let $\kappa = \kappa$, $\lambda = \kappa$ be as above. Let $\kappa \leq \lambda, \kappa \leq \lambda$.

* $F$ is **robust** up to $\kappa$ in $N$ iff for every pair of countable sets $U \subseteq \lambda, W \subseteq \kappa \cap \lambda$ there is a $g : U \rightarrow \kappa$ r.t. (a) holds as above and

  (b') Let $c = \text{lub } \{g^{\infty}(u) : u \in U\}$.

Then for all $u_1, u_2, \ldots, u_m \in U$ and all $\Sigma_1$

* $F$ is **robust up to $\kappa$** in $N$ iff whenever

$N[\gamma] = \langle V, \in \rangle$ is active and $\delta \subseteq (\kappa, \delta)$$

is a cardinal in $N$, then $F$ is robust up to $\delta$ in $N[\gamma]$.

As we remarked in [RE], if $\langle N_i : 1 \leq i \leq \theta \rangle$

$(\theta \leq \omega)$ is an array in which all extenders added are robust, then each $N_i$ is a robust premouse.
We again prove only a special case of the theorem, which reduces to:

**Main Claim** Let $N$ be a monotone, robust premouse satisfying $\text{ZFC}^-$. Let $\gamma = \langle \langle p_i \rangle, \langle \nu_i \rangle, \langle \pi_i \rangle, T \rangle$ be a countable normal putative iteration of a countable $p_0$ without truncation, where $\sigma : p_0 \prec N$.

Then one of the following holds:

(a) $\text{lh}(\gamma) = h+1$ and there is $\sigma' : p_0 \prec N$ such that $\sigma' \pi_0 h = \sigma$.

(b) $\gamma$ has a maximal branch $b$, which is of limit length, and there is $\sigma'' : p_b \prec N$ such that $\sigma'' \pi_b = \sigma$.

Before beginning the proof, we reintroduce the notions of "world" and "enlargement". For technical reasons, some definitions will differ in essential ways from the earlier ones. The normal iteration $\gamma$ is fixed from now on. Let $\text{ZFC}^*$ be the theory $\text{ZFC}^- + \forall \alpha \ [\alpha]^\omega \in V$.

As before, we define:

**Def** A *world* is a transitive structure $\mathcal{W}$ such that $\mathcal{W} \models \text{ZFC}^*$ and $[\tau]^\omega \cap \mathcal{W} = [\tau]^\omega \cap V$ for $\tau = \text{On} \cap \mathcal{W}$.

*"putative" means that if $\text{lh}(\gamma) = h+1$, then $P_h$ does not need to be well-founded.*
(Note Since we shall often find our worlds in generic extension of $V$, the clause
$[\tau]^{\omega} \forall W < V \text{ not vacuous.}

Fix $A < \text{ On } \omega$ s.t. whenever $\beta > \omega, \beta = \overline{\beta}$, then
$V^*_\beta = L^*_\beta [A]$ and $L^*_\beta [A] = (\beta \in \text{ the largest cardinal})$
Clearly $L^*_\beta [A]$ is a world whenever $\beta = \overline{\beta}$
and $\text{cf}(\beta) > \omega$.

Def By a standard world we mean a
$W = \langle L^*_\beta [A], \varepsilon, A, \text{An}_{\beta}, N, P_1, \ldots, P_m \rangle$ s.t.
$P_1, \ldots, P_m \in W$ and $N \in L^*_\beta [A]$ is
a robust morelike premouse,
and $\text{cf}(\beta) > \omega$.
(Hence $(C^e_{\tau, \infty}) = C^e_{\tau, \beta^+} \lessdot C^e_{\tau, \infty}$ for
$e, \tau \in W$.)

Clearly $Y \subseteq H_\omega \subseteq W$ for any world $W$.
For standard worlds we shall assume, in addition, that $Y \in W$ - definable
(e.g. $Y = P_1$).

[Note We generally write:
$W = \langle L^*_\beta [A], N, P \rangle$ or $W = \langle W_1, N, P \rangle$.
At $W = \langle L^*_\beta [A], N, P \rangle$ in a standard
world, we define for each $\mu < \infty$
the $\mu$-th enhancement:
\[ W(\mu) = \langle L_{\mu}^+ [\vec{A}], N, \bar{p}, \beta > \], \text{ where} \\
\beta_i = \text{the } i^{\text{-th}} \beta \equiv \beta \text{ s.t. } \beta = \bar{\beta} \text{ and} \\
\text{cf}(\beta) > \omega. \\
(\text{Hence } W^0 = \langle W, \beta >.) \\
\text{Note that } \langle \beta_i | i \leq \mu \rangle \text{ is uniformly} \\
W(\mu) \text{ - definable.}

We then define a \underline{good world} to be an arbitrary world which has the salient features of a standard world:

\text{Def: } \tilde{W} = \langle L_\delta [\tilde{A}], \varepsilon, \tilde{A}, \tilde{N}, \tilde{p}, \ldots, \tilde{p}_n > \text{ is a } \\
good \text{ world : if } \tilde{W} \text{ is a world in which the following hold:} \\
(a) \text{ If } \beta = \bar{\beta}, \text{ then } \bar{\beta} = L_\beta [\tilde{A}]. \text{ Moreover, if} \\
\beta^+ \text{ exists, then } \beta \text{ is the largest cardinal in } L_{\beta^+} [\tilde{A}]. \\
(b) \text{ There is a largest cardinal } \beta. \text{ Moreover,} \\
\beta = \bar{\beta} \text{ and } \text{cf}(\beta) > \omega. \\
(c) \tilde{N} \in \bar{\beta} \text{ is a robust, im once like premouse.}

\text{If } \tilde{W} \text{ is a good world, we denote its largest cardinal by } \beta_{\tilde{W}}. \\
(\text{Note: if we were doing the full proof for arrays, then an array would take the place of } N.)
Let \( \tilde{w} = \langle L, [A \setminus \tilde{A}], \tilde{N}, \tilde{P}, \beta \rangle \) be a good world. By an enhancement of \( \tilde{w} \) we mean a good world
\[
W' = \langle L', [A'], \tilde{N}, \tilde{P}, \beta \rangle
\]
\( \beta = \beta_{\tilde{w}}, \quad \tilde{A} = A \setminus \tilde{A}, \quad d = \beta + w \) and
\( \nu_{\beta, w} = \nu_{\beta, w}' \). At \( W' \) is an enhancement of \( \tilde{w} \). Then in \( W' \) we can define a sequence \( \langle \beta_i \rangle_{i \leq \mu} \) by:
\[
\beta_i = \text{the least } \beta_i \geq \beta \text{ s.t. } \beta_i > \beta_n \text{ for } h < i, \beta_i = \nu_{\beta_i} \text{, and cf}(\beta_i) > \omega.
\]
Then \( \beta_{\mu, w'} = \beta_{\mu, w} \). By \( \tilde{w}(i) \) \( (i \leq \rho) \) we denote \( \langle L, \beta_i^+, [A'], \tilde{N}, \tilde{P}, \beta \rangle \), where \( \beta_i^+ \) denotes \( (\beta_i^+)_w \).
We denote \( \beta_i \) by \( \beta_{i, w} \) and \( \tilde{A} \) by \( \tilde{A}_w \).
Clearly each \( \tilde{w}(i) \) is a good world.

We now define the notion of enlargement. We shall demand less of enlargement than we did in [RE],
Def Let \( \gamma \leq \mathfrak{h}(\mathcal{U}) \). \( \mathcal{W} = \{\langle \mathcal{W}_i, \sigma_i \rangle \mid i < \gamma \} \) is an enlargement of \( \mathcal{U} \) if:

(a) \( \mathcal{W}_i = \langle L_{\alpha_i}, [A_i], N_i, \mathcal{P}_i \rangle \) is a good world

(b) \( \sigma_i \in \mathcal{V} \) s.t. \( \sigma_i \prec \mathcal{P}_i < N_i \)

(Hence \( \sigma \in \mathcal{W}_i \), since \( \sigma \in \bigcap \mathcal{W}_i \) is countable.)

(c) Set: \( \delta_i = \) the largest \( \delta \leq \lambda_i \) which is a cardinal in \( \mathcal{P}_i \) (hence \( \delta_i \leq \lambda_i \leq \delta_i \) for \( i < j < \gamma \)).

Set: \( c_i = \sup \sigma_i \delta_i; \quad E_i = E_N \kappa_i \).

Then

\[
\sigma_i \cap \delta_i = \sigma_i \delta_i, \quad \bigcup_{c_i} = \bigcup_{\mathcal{E}_i} \quad \text{for } h < i.
\]

(d) Set: \( \delta_i = \sigma_i \delta_i \).

Then \( \delta = \{\langle \delta_i \rangle \mid i < \gamma \} \in \mathcal{V} \).

(Hence \( \langle \delta_i \rangle \mid h < i \rangle \in \mathcal{W}_i \).)

(e) Set \( t_i = \text{th}(\langle \mathcal{W}_i, \delta_i, \langle \mathcal{E}_i \rangle \mid h < i \rangle) \)

(where \( \text{th}(\mathcal{W}) \) is the complete theory of \( \mathcal{W} \)).

Then \( t = \{\langle t_i \rangle \mid i < \gamma \} \in \mathcal{V} \).

(Hence \( t \in \mathcal{W}_i \), since \( t \in H_\omega \) is countable.)

Note The same definition gives the notion "enlargement of a phalanx", since we have not used the fact that \( \mathcal{U} \) is an iteration but merely that \( \mathcal{P}_i \succ \lambda_h = \mathcal{P}_h \succ \lambda_h \) and \( \lambda_h \) is a cardinal in \( \mathcal{P}_i \) for \( i < h \).

Def \( \langle t, \delta \rangle \) is called the trace of \( \mathcal{W} \).

We refer to \( t \) as the first trace and \( \delta \) as the second trace.

We write \( \langle t, \delta \rangle = \text{tr} (\mathcal{W}) \).
Def. \( z = \langle t, S \rangle \) is a trace if there is a \( z \in \text{coll}(w, d) \) such that \( \forall E \), \( z = \text{tr}(E) \).

Note that the trace of an enlargement \( E \) is always in \( V \), even if \( E \) is not.

Def. Let \( N = \langle JE, F \rangle \) be a premove.

\( z = \langle t, S \rangle \) is an \( N \)-conforming trace for \( y \) if and only if there is a \( z \in \text{coll}(w, d) \) such that \( \forall E \), \( z = \text{tr}(E) \cap \bigcap_{i=0}^{y} c_i = JE \) for \( i < y \).

Def. Let \( N \) be an above and let \( \sigma : P \leq N \). \( z = \langle t, S \rangle \) is an \( N, \sigma \)-conforming trace if and only if \( z \) is an \( N \)-conforming trace and \( S_i = \sigma^i \delta_i \) for \( i < y \).

(Note. This definition makes sense for an arbitrary \( \sigma : \sup \delta_i \rightarrow 0 \).

(\( i < y \))

A very basic lemma in the following:
Lemma 1. There is a $\Sigma_1$ formula $\varphi$ such that whenever $N = \langle \omega^N, F \rangle$ is a premouse, then $\sigma$ is in an $N$-conforming trace for $V|N$. If the following hold:

(a) $\sigma = \langle t, \delta \rangle$ where $\delta = \langle \delta_i \mid i < \gamma \rangle$ and $\delta_i : \gamma_i \rightarrow 0$ on $N$ for $i < \gamma$.

(b) $C_E \in \varphi[\omega, \gamma|\gamma]$ where $C = \sup_{i < \gamma} C_i$, $C_i = \sup_{i < \gamma} \delta_i$.

Proof:
(a) is obviously a necessary condition. Now let (a) hold. For appropriate $\varphi$ we show:

Claim: $\sigma$ is an $N$-conforming trace $\iff$ (b).

The condition $\varphi$ says:

There exist $\beta, \delta \in N$.

(a1) $\gamma < \beta < \delta$, $\sigma = \langle t, \delta \rangle \in C_{e, \beta}$, $C_{e, \beta}$ is admissible.

(b1) $T$ in consistent, where $T = T_{\beta, \delta}(\sigma)$ is the theory in $L_{C_{e, \beta}}$ with:

Predicate: $\in$

Constants: $x \in (x \in C_{e, \beta})$, $\bar{E}, \bar{\beta}, \bar{A}, \bar{N}, \bar{s}, \bar{w}, \bar{p}$

Axioms:

(1) $\exists \sigma (x \in x \leftrightarrow \forall y \bar{y} \in x)$ for $x \in C_{e, \beta}$

(2) $\bar{E} = \langle \langle \bar{w}, \bar{s} \rangle \mid i < \gamma \rangle$ where

The following hold:
(i) Each of \( W, \beta, \dot{A}, \ddot{o}, \dot{p}, \dot{N} \) maps \( \gamma \) into \( C_{c, \beta}^E \).

(ii) \( \beta_i < \beta \), \( \dot{A}_i < \beta \), \( \ddot{o}_i : \dot{p}_i < N_i \),

\[ \dot{W}_i = \langle L_{\dot{A}_i}, C, \dot{A}_i, i \dot{N}_i, \dot{p}_i \rangle \quad (i < \gamma) \]

(iii) \( \dot{W}_i \cap [\beta_i] = [\beta_i] \cap \dot{W}_i \) \( \cap [\beta_i] \) \( \cap [\beta_i] \)

(Note: \( [\beta] \cap [\beta] = \{ x \mid (x < \beta \wedge x = \omega) \} \).

hence \( \langle [\beta] \cap \beta < \gamma \rangle \) \( \cap [\beta] \) \( \cap [\beta] \)

(iv) \( \dot{W}_i \cap (ZFC^c \cap (a) \cap (b) \cap (c)) \), where \( a ), (b), (c) \) are as in the definition of the "good world", \( (i < \gamma) \)

(v) \( t_i = \langle \dot{W}_i, \dot{o}_i, t \dot{N}_i \rangle \) \( \cap [\beta_i] \)

(vii) \( \dot{C}_i : \dot{N}_i = \delta_{\dot{C}_i} : \dot{N}_i = \dot{C}_i : \dot{N}_i \) \( \cap [\beta_i] \) \( \cap [\beta_i] \)

(\( h \leq i < \gamma \)).

Proof of Claim:

\( \text{N-conformally} \)

(\( \rightarrow \) Let \( \Xi = \langle \Xi_i, i < \gamma \rangle \) be an enlargement of \( \gamma \) with trace \( \gamma \), where \( \Xi = V[G] \) and \( G \in \text{coll}(\omega, \delta) \) - generic for some \( \delta \). Let \( \Xi \subseteq V[\beta] \cap G \) and let \( \alpha > \beta \) s.t. \( C_{c, \alpha} \) is admissible. Let \( \delta \geq \mu \) s.t. \( V[\delta] \models \text{ZFC}^- \). Then \( \langle V[G], \Xi \rangle \) models \( T_{\beta, \alpha} \). Hence \( T_{\beta, \alpha} \) is consistent.)
Let \( M \) model \( T_{\beta, \xi} \) (in \( \mathcal{U}[G] \)), where \( G \) is \( \text{coll}(\omega, d) \)-generic. We can assume \( \mathcal{U} \) s. o. y. that \( M \) is \( \text{not} \text{d} \text{i} \text{d} \text{d} \) (i.e., \( \text{core}(\mathcal{U}) \) is transitive and \( \mathcal{U} \subseteq \text{core}(\mathcal{U}) \)).

Then \( x^\mathcal{U} = x \) for \( x \in C_{c, \xi} \). It follows easily that \( E = E^\mathcal{U} \) is an enlargement with traces \( E_D \). Q.E.D. (Lemma 1)

An easy modification of the proof yields:

**Cor 1.1** "\( z \) is a trace" is uniformly \( \Sigma_1(C_{c, \infty}) \)-definable.

The proof of Lemma 1 can be carried out in any good world. Hence:

**Cor 1.2** Let \( W = \langle L_\delta[A], N, p \rangle \) be a good world. Let \( z \in W \). Then \( W \models "z \) is an \( N \)-conforming trace" if and only if (a) holds and (b) \( (C_{c, \infty}^E)_W \models \psi \),

where \( c = \langle t, \delta \rangle \) and \( 0 < \delta \). Hence:

**Cor 1.3** "\( z \) is an \( N \)-conforming trace" is absolute in standard worlds \( W \), since \( (C_{c, \infty}^E)_W \subseteq C_{c, \infty}^E \) for \( c \in W \).
As a corollary of the proof we also get:

**Cor 1.4.** Let \( N^* = \langle J, E^*, F^* \rangle \) be a pronose. Let \( E \) be an \( N^* \)-conforming enlargement of \( Y | \gamma \). Let \( W = \langle W_i, N_i, \rho \rangle \) be a good world. Let \( J^c E^c = J E^* \), where \( c = \sup c_i \), \( i \leq \gamma \). Let \( \delta^c = \sup \delta_i \), Suppose moreover that \( \text{rn}(E) \leq \text{om} W \). Let \( z = \text{tr}(E) \). Then \( W \models z \) in an \( N^* \)-conforming trace.

**Proof.**
Let \( \text{rn}(E) = \beta \). Let \( z > \beta \) be least set \( C^c_i \) in \( C^c \) in admissible. Then \( z \in W_i \). But then the theory \( T_{z, \beta} \) in the proof of Lemma has a model. Hence \( T_{z, \beta} \) is consistent.

QED (1.4)

We shall confine ourselves largely to traces having the following property:

**Def.** Let \( E \) be an enlargement of \( Y | \gamma \), \( E \) not perfect (or self-justifying) in \( H^c \), let \( z = \text{tr}(E) \), we have

\[
W_i \models z \quad \text{in an } N_i \gamma^c \text{-conforming trace}
\]
for \( i < \gamma \). (Here \( W_i \) denotes \( \langle W_i, N_i, \rho_i \rangle \), where \( z = \langle t, \delta \rangle \).

(Note: At follow that
\[
W_i \models z \quad \text{in an } N_i \gamma^c \text{-conforming trace}.
\]
Def a neat trace in the trace of a neat enlargement.

Note that \( t = \langle t, s \rangle = \text{tr}(E) \). Then \( E \) is neat iff \( t \) satisfies a syntactical condition of the form: \( t_i \in \mathbb{E}_i \) for \( i < p \).

Hence:

Lemma 1.5 Let \( t \) be a trace for \( \mathbb{E}_p \).

(a) \( t \) is neat iff it satisfies the above syntactical condition.

(b) If \( t \) is neat and \( E \) is any enlargement with traces \( s \), then \( E \) is neat.

We call a good world reflective if it countenances the existence of smaller imitations of itself in the following sense:
Def. Let \( W \) be a good world. \( W \) is reflective iff whenever \( e, \tau \in W \) and \( t = \text{th}(\langle W, e \rangle) \), then \( t \in W \) and the following holds in \( W \):

For sufficiently large \( \delta \) it is forced by \( \text{coll}(\omega_1, \delta) \) that there is a good world \( \overline{W} \) s.t. \( \tau \in \overline{W} \) and for some \( \overline{e} \in \overline{W} \):

\[
  t = \text{th}(\langle \overline{W}, \overline{e} \rangle) \quad \text{and} \quad \tau \overset{\overline{e}}{=} J \overset{e}{\tau}.
\]

The method of proof used in Lemma 1 yields:

Lemma 2. There is a \( \Sigma_1 \) formula \( \psi \) s.t. for all good worlds \( W \):

\( W \) is reflective iff whenever \( e, \tau \in W \) and \( t = \text{th}(\langle W, e \rangle) \), then \( t \in W \) and:

\[
(C^e_{\tau, \infty})_W \models \psi[t].
\]

Proof.

\( \psi \) says that there exist \( \beta, \lambda \) s.t. \( \tau < \beta < \lambda \), \( C^e_{\tau, \lambda} \) is admissible, and \( T^* = T_{e, \beta, \lambda}(t) \) is consistent, where \( T^* \) is the following theory in the infinitary language of \( C^e_{\tau, \lambda} \).
Predicate: $e$

Constants: $x \in (x \in C_{\varepsilon, \delta})$; $W, e, \Delta, \psi, \rho$

Axioms:

1. $\text{ZFC}^-$, $\forall u \in x (u \in x \iff \exists v \in x \forall z \in z \in x)$ for $x \in C_{\varepsilon, \delta}$.

2. $\hat{W} = \langle L_\beta [A], \in, A, \Delta, \psi, \rho \rangle$, where the following hold:

   (i) $\hat{A} \subseteq \beta$ and $W \models (\text{ZFC}^* + (a) \land (b) \land (c))$, where $(a), (b), (c)$ are in the definition of "good world".

   (ii) $\hat{e} \in \hat{W} \land \varepsilon = \text{th} (\langle W, \hat{e} \rangle)$.

   (iii) $\hat{J}_{\varepsilon} = J_{\varepsilon}$.

   (iv) $\hat{W} \cap [\beta]^{\omega} = [\beta]^{\omega}$

   (where $[\beta]^{\omega} = \{ x | C_\varepsilon^e \models (x \in \beta \land \exists x \leq \omega)^3 \}

   in C_{\varepsilon, \delta}$ - definable.

If $W$ is reflective and $e, \varepsilon \in W$, then clearly there is $\beta$ s.t. letting $\alpha = \text{the smallest} \gamma > \beta$ s.t. $C_\varepsilon^e \models \alpha$ in admissible, then $T^*$ has a model in $W[G]$ for a col(l($\omega, \Delta$)) - generic $G$.

Hence $T^*$ is consistent. Hence $C_\varepsilon^e \models 4^* [\hat{\tau}]$ in $W$. Conversely, if $T^*$ is consistent and $G$ is col(l($\omega, \Delta$)) - generic over $W$, then $T^*$ has
a solid model \( \bar{w} \) in \( W[\sigma] \). Set \( \bar{w} = \bar{w}' \), \( \bar{e} = \bar{e}' \). Then \( \bar{w}, \bar{e} \) have the desired properties. QED (Lemma 2)

**Def.** A good world \( W \) is \underline{enhanceable} iff it has a proper enhancement (i.e. there is a good world \( W' \) of the form \( W^{(\mu)} \)).

**Note.** At \( W = W^{(\mu)} \), then \( W^{(\mu)} \) is enhanceable for all \( \mu \).

As a corollary of the above proof:

**Cor. 2.1** At \( W \) is enhanceable, then it is reflective.

**Proof.**

Let \( W' = W^{(\mu)} \). Let \( \tau \in W \). Let

\[ \beta = \text{omn}\ W \text{ and let } \lambda \text{ be the smallest } \lambda > \beta \]

s.t. \( C^e_{\tau, \lambda} \) is admissible. Then \( \lambda \in W' \),

\( T^* = T_{\mu, \beta, \lambda} \) has the model \( \langle W', W, e' \rangle \), where \( W \) interprets \( W' \) and \( e \) interprets \( e' \). Where \( t = \text{th} (\langle W, e \rangle) \). Then \( (C^e_{\tau, \lambda})_{W'} \models \varphi(t) \).

But

\[ (C^e_{\tau, \lambda})_{W} \models (C^e_{\tau, \lambda})_{W'} \]

Hence

\[ (C^e_{\tau, \lambda})_{W} \models \varphi(t) \). QED (2.1)
Lemma 2.2 Let $\mathcal{E} = \langle \langle W_i, \sigma_i \rangle \rangle_{1 \leq i \leq \eta}$ be a meat enlargement of $\mathcal{M}_\eta$. Let $W_i$ be reflective. Let $s = tr(\mathcal{E})$. Then $W_i \mathcal{E}(sN_i + 1)$ is an $N_i; \sigma_i$-conforming trace.

Proof. Let $s = \langle t, \delta \rangle$. We must show that, if $G$ is coll$(W, \delta)$-generic over $W_i$ for a sufficient $\delta$, then $W_i[G]$ contains an $N_i; \sigma_i$-conforming enlargement of $\mathcal{M}_i + 1$ with trace $sN_i + 1$. Since $\eta$ is meat, we can assume $\delta$ large enough that there is $\mathcal{E} \in W_i[G]$ enlarging $\mathcal{M}_i$ with trace $sN_i$. Writing

$$(a, b, c) = a \times \bar{\delta}_0 \cup b \times \bar{\delta}_1 \cup c \times \bar{\delta}_2,$$

let $t : E = (\sigma_i, E \sigma_i, t \sigma_i)$. Clearly $t_i = \text{th} \langle W_i, \sigma_i, t \sigma_i \rangle$ is recursive in $t^* = \text{th} \langle W_i, \mathcal{E} \rangle$. (In fact, $t_i$ is uniformly recursive in $t^*$ in the sense that $t_i = F(t^*)$ and for any good word $W$ and any $a, b, c \in W$ s.t. $b = N^W$, we have $\text{th} \langle W, a, c \rangle = F(\text{th} \langle W, (a, b, c) \rangle)$.
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At we have chosen $\delta$ large enough, it follows by reflectivity that there is a good world $\overline{\mathcal{W}} \in \mathcal{W}_c [c]$ and a $\overline{\mathcal{E}} \in \overline{\mathcal{W}}$ s.t. $\overline{c} \models \overline{\mathcal{E}}$ and $t^* = \text{th}(\langle \overline{\mathcal{W}}, \overline{\mathcal{E}} \rangle)$. But then, letting $\overline{N} = N \overline{\mathcal{W}}, \overline{E} = E \overline{\mathcal{W}}$, we have: $\overline{E} = (\overline{\sigma}, \overline{E}, \text{tr} i);$, where $\overline{\sigma} : p_i \leq \overline{N}, \overline{J} \overline{E} = J \overline{E} i;$, and $\overline{\sigma} \text{tr} i \overline{\mathcal{E}} = \sigma \text{tr} i \mathcal{E}$. Moreover, $t_i = \text{th}(\langle \overline{\mathcal{W}}, \overline{\sigma}, \text{tr} i \overline{\mathcal{E}} \rangle) = F(t^*)$. But then $\overline{E} \langle \overline{\mathcal{W}}, \overline{\sigma} \rangle \in \mathcal{W} [c]$ is then an enlargement of $\mathcal{Y} [\mathcal{E}]$ with trace $s^i+1$. Q.E.D. (Lemma 2.2)

Before proceeding further we adopt the following linguistic conventions:

We often make a statement of the form:

"There is an enlargement with property A". This means that there is such an enlargement in $V [G ]$ if $G \in \text{coll}(\omega, \delta)$ - i.e. for a sufficiently $\delta$. At we say:

"Let $\mathcal{E}$ be an enlargement with property A. Then there is an enlargement $\mathcal{E}'$ with property B". It is understood that we are working in such a $V [G ]$ and are asserting the existence of an $\mathcal{E}'$ with property B exist in $V [G ][G ]$, where $G \in$
coll(ω, ω) generic over V[ω] for a sufficiently large ω'. (In fact, statements of this sort will generally turn out to be equivalent to a \( \Sigma^1 \) statement in \( C_{\omega}^{\omega} \), as for appropriate \( \varepsilon, \zeta \).)

We recall that the ground model \( V \) is fixed throughout the discussion and was used in defining the notions "world" and "enlargement".

A major tool in proving iterability will be the following "superlemma". We first define:

**Def.** An enlargement \( E \) of \( V \) is powerful iff \( E \) is meet and \( W' \) is reflective (where \( E = \langle \langle W_n, n \rangle \rangle \) where \( 1 \leq n \)).

**Def.** \( W' \) is a segment of \( W \) iff

- \( W \) is a good world satisfying \( \varphi \);
- \( W = \overline{W^{n_1}} \), where \( \overline{W} \) is a good world;
- and \( W' = \overline{W^{n_1}} \) in \( W \) for some \( n_1 \leq n \).
Lemma 3 (Superlemma)
Let $E = \langle \langle w_i, \sigma_i \rangle \mid i \leq \gamma \rangle$ be a powerful enlargement of $J_{\gamma+1}$ with trace $t = \langle t, \delta \rangle$. Let $3 = T(\gamma+1)$ in $\gamma$ and let $W$ be a reflective segment of $W_3$.

(A) $E^{\gamma+3}$ extends to a powerful enlargement $E^* = \langle \langle \overline{w}_i, \overline{\sigma}_i \rangle \mid i \leq \gamma+1 \rangle$ of $J_{\gamma+2}$ with trace $\overline{t} = \langle \overline{t}, \overline{\delta} \rangle$ s.t.

\begin{itemize}
  \item (i) $\overline{t}_{\gamma+3} = \overline{t}_{\gamma+3}$,
  \item (ii) $\overline{t} = \overline{t} \upharpoonright \gamma+1$,
  \item (iii) $\overline{w}_{\gamma+1} = \overline{w}_{\gamma+1}$,
  \item (iv) $\overline{\sigma}_{\gamma+1} \overline{w}_{\gamma+1} = \overline{\sigma}_{\gamma+1}$
\end{itemize}

(B) $A + W$ in a proper segment of $W_3$, then $E$ extends to a powerful enlargement $E' = \langle \langle w_i, \sigma_i \rangle \mid i \leq \gamma+1 \rangle$ of $J_{\gamma+2}$ with trace $t' = \langle t', \delta' \rangle$ s.t.

\begin{itemize}
  \item (i) $E' = E' \upharpoonright \gamma+1$,
  \item (ii) $O_{w_i} < O_{w_i}$ in $\gamma+1$
\end{itemize}

\begin{itemize}
  \item (iv) $t' = \overline{t} \upharpoonright \gamma+1$, where $\overline{t}, \overline{\sigma} = \langle \overline{t}, \overline{\delta} \rangle$
\end{itemize}

are as in (A).
We first prove (A).

Since $N_\gamma$ is robust in $W_\gamma$, there is a $g : \gamma \to \sigma_\gamma(\kappa_\gamma)$ s.t.

$$\varphi(\gamma, \sigma_\gamma(\kappa_\gamma)) \land \varphi$$

for $\varphi(x_1, \ldots, x_m) \in \mathcal{L}_{\kappa_\gamma}$, $x \in \mathcal{F}(\kappa_\gamma) \cap P_\gamma$.

Then in $W_\gamma$ we have:

$$C_{\sigma_\gamma(\kappa_\gamma)} \models \varphi$$

for all $\Sigma$-formulae $\varphi$.

Then:

1. $g \upharpoonright \kappa_\gamma = \sigma_\kappa(\kappa_\gamma)$

(Apply (*)) with $X = \{ x_1, \ldots, x_m \}$ for $\varphi \models \kappa_\gamma$.

By (*), there is $\sigma : \kappa_{\gamma + 1} \to N_{\kappa_\gamma}$ defined by:

$$\sigma(\kappa_{\gamma + 1}) = \sigma \upharpoonright \kappa_{\gamma + 1}$$

Hence $\sigma \Gamma_{\kappa_{\gamma + 1}} = \sigma \upharpoonright \kappa_{\gamma + 1}$.

Let $x = \langle t, \delta \rangle = \tau \kappa(\kappa_\gamma)$. Since $W_\gamma$ is reflective, we know by Lemma 2.2:

3. $W_\gamma \vdash x \models \kappa_\gamma$, $\sigma_\kappa(\kappa_\gamma)$-conforming trace.

Let $f : W \rightarrow \kappa_{\gamma + 1}$. Set:
\[ t^* = \{ i, k \mid i < \omega_1, k \in t_{f(i, i)} \} \], where 
\[ \omega_1 \] is Gödel's pairing function on ordinals. Then \( t^* \subset \omega_1 \). (We assume \( t_n \subset \omega \) for \( n \leq \gamma \))

We also set:
\[ U = \{ i, \mu \mid i < \omega, \mu < \gamma, (i, \mu) \} . \]

Then \( U \subset \delta_\gamma \).

Moreover \( \delta_\gamma (i, \mu) = \langle i, \sigma_\gamma (\mu) \rangle \) for \( \mu < \gamma \) and \( i \leq \gamma \).

Since \( \delta_\gamma = \sigma_\gamma \wedge i \gamma_i \) for \( i \leq \gamma \), we see that \( s = \langle t, s \rangle \) is straightforwardly coded by \( t^*, \overline{\sigma}_\gamma U \), and \( e \), where \( e \in \omega \) coding \( \gamma \mid \gamma + 1 \) and \( f \).

Clearly \( \overline{\sigma}_\gamma \uparrow \omega = \text{id} \). By Lemma 2 it follows that (3) can be expressed as
\[ \frac{t \in \gamma}{C \in \gamma} \]

(4) \[ c \gamma_\gamma = \Psi [ t^*, \sigma_\gamma U, e ] \]

where \( t \in \Sigma_7 \). Hence by (**):

(5) \[ C \in \gamma \]

But \( \overline{\sigma}_\gamma (\mu_\gamma) \)

But \( t^*, \sigma_\gamma U, e \) code \( \overline{s} = \langle t, \overline{s} \rangle \)

where \( \overline{s}_i = \sigma_\gamma \wedge \overline{s}_i \) for \( i \leq \gamma \).

But \( J C \gamma_\gamma = J C \gamma \), since

\[ c \leq \sigma_\gamma (\mu_\gamma) = \sigma_\gamma (\mu_\gamma) \leq c \gamma_\gamma \]. Thus
(6) \( C^{E_3}_{c, \delta (x)} = \Psi [t, \eta, u, e] \).

But this tells us \( \omega_l \circ \delta (x), \omega \leq \bar{\omega} \).

(7) \( \bar{\omega} \) in an \( N_3, \sigma \) - conforming trace for \( \gamma + 1 \) in \( \bar{\omega} \).

Note that \( \bar{\omega} \mid 3 = 2 \bar{\omega} \), since \( \sigma \circ \delta_h = \delta \circ \delta_h \) for \( h < 3 \). By (7), if \( G \in \text{coll}(\omega, \delta) \) - generic over \( \bar{\omega} \) for a sufficient \( \delta \), there is \( \bar{\omega}'' \in \bar{\omega} \) [6] which is an enlargement of \( \gamma \mid \gamma + 1 \) with trace \( \bar{\omega} \).

Set \( \bar{\omega}' = \bar{\omega} \circ \delta_h \in \bar{\omega} \mid 3, \gamma + 1 \). Then \( \bar{\omega}' \) is also an enlargement of \( \gamma \mid \gamma + 1 \) with trace \( \bar{\omega} \). Finally set:

\( \bar{\omega} = \bar{\omega}' \setminus \langle \bar{\omega}, \delta \rangle \), \( \bar{\omega} \) is easily seen to have the desired properties.

We now prove (B).

Let \( t' = \bar{\omega} \mid \gamma + 1 = \text{th}(\langle \bar{\omega}, \sigma, t \rangle) \). For \( c < \beta < \delta \) with \( C^{E_3}_{c, \delta} \) admissible, consider the theory \( T \alpha \) in the language of \( C_{c, \delta} \) with:

\[
\begin{align*}
\text{Predicate } & \subseteq \\
\text{Constante } & \in \{ x \in C_{c, \delta} \mid \bar{\omega}, \sigma, \delta, \delta, \delta, \delta, \delta \}
\end{align*}
\]
Axioms

(A) $ZF C^-$, $\forall u (u \in x \iff \exists v \in x \forall w (w \in x \iff v \subseteq w))$ for $x \in C_c, \alpha$

(B) $\dot{W} = \langle L_\beta^+ [A^+], \in, A^+, \dot{N}, \dot{p} \rangle$ with:

(i) $\dot{A} \in \beta^+, W, \vdash \text{ZFC}^* + (a) \land (b) \land (c)$, where

(a), (b), (c) are in the def of "good world"

(ii) $\dot{W} \models ^{\dot{V}} \text{Th} \langle \dot{W}, \dot{\sigma}, \dot{\mathcal{E}} \rangle \land \dot{\sigma} \in \dot{W} \land$

\[ \dot{V} \in \dot{W} \land \dot{\mathcal{E}} \subseteq \dot{W} \land \dot{\sigma} \in \dot{W} \]

(iii) $\dot{W} \models [\beta^+]_\omega = [\beta]_\omega$

(Recall that $[\beta]_\omega$ is $C_{c, \alpha}$-definable in $\beta$)

(iv) $\dot{W}$ is reflective.

Then $T = T_{\beta, \alpha} (\dot{V})$ is consistent, since

$\langle W_{\dot{v}+1}, \dot{W}, \dot{\sigma}, \dot{\mathcal{A}}, \dot{W}, \dot{p} \rangle \models \text{a model}$

The statement: "There exist $\beta, \alpha$ s.t., $c \in \beta \in \alpha$, $C_{c, \alpha}$ is admissible, and $T_{\beta, \alpha} (\dot{V})$ is consistent" has the form:

(7) $C_{c, \infty} \models X[\dot{A}, \dot{\mathcal{E}}']$

Hence it has the form:

(8) $C_{c, \infty} \models X[\dot{t}^x, q''u, e, t']$, where

$X \in \Sigma_1$.

Since (7) holds in $W_3$ and

$(C_{c, \infty}^E)_W \models \Sigma_1 (C_{c, \infty}^E)_{W_3}$, we
conclude that (7) holds in $\overline{W}$. But it follows just as in [RE] by the mouse likeness and reobtaintness of $N^\xi$ in $\overline{W}$ that:

$$ (9) \quad C_{c, \mu \gamma}^E \prec (C_{c, \mu \gamma}^E)_{\overline{W}} \prec_\xi \Xi_1 $$

Hence:

$$ (10) \quad C_{c, \mu \gamma}^E \models \chi[t^*, q^* "u_1 e, t'] $$

By (**1) we conclude:

$$ (11) \quad (C_{E^\gamma})_{\overline{W}^\eta} \models \chi[t^*, q^* "u_1 e, t'] $$

in other words:

$$ (12) \quad (C_{E^\gamma_{\zeta}})_{\overline{W}^\eta} \models \chi'[s', t'] $$

(Where $s = tr(W)$)

But this means that there are $\beta, \delta \in W^\eta$ s.t. $\xi_\gamma < \beta < \delta$ and $C_{E^\gamma_{\zeta}}$ is admissible, and $T_{\beta, \delta}$ (2) is consistent. Now let $G$ be coll($\omega_1, 2$)-generic over $W^\eta$. Then $T_{\beta, \delta}$ (2.1) has a solid model $M^G$ in $W^\eta[G]$. Set: $W' = W^{M^G}, \sigma' = \sigma^{M^G}$. Then:

$$ (13) \quad W'$ is a reflective good world;

$$ (14) \quad s \in W', \sigma' \in W'$ and $\sigma': \rho < N = N' \text{ with } \sigma' = \sigma' \rho^{M^G}$.
Moreover, (15) \( t' = th(\langle W', \sigma',z \rangle) \), hence:

(16) \( W' = z \) in an \( N', \sigma' \) - conforming trace.

If we set \( E' = E \setminus \langle W', \sigma' \rangle \), then

\( E' \) has the desired properties.

\textit{QED (Superlemma)
§ 2 Aterability

We now give the proof of the Main Claim in §1, N is a mouse-like, robust premouse satisfying ZFC. We can place it inside a standard world \( W = \langle L_\alpha[A], V_1, p \rangle \), where \( Y \) is \( W \)-definable. Hence there is a \( W \)-definable bijection \( m^*: lh(Y) \to \omega \).

Following Steel we define:

\[ m(i) = \min \left\{ m^*(j) \mid i \leq_T j \text{ in } Y \land (i < lh(Y)) \right\} \]

**Def.** \( i \) survives at \( j \) \( (i \text{ surv}_j) \iff i \leq j \wedge m(i) = m(j) \wedge m(l) \geq m(l') \text{ for all } l \leq (i, j) \).

Steel establishes:

**Lemma**
(a) \( i \text{ surv}_j \to i \leq_T j \)
(b) \( h \text{ surv}_i \to h \text{ surv}_j \)
(c) \( h \text{ surv}_j \land h <_T i \leq_T j \to h \text{ surv}_i \text{ surv}_j \)
(d) Let \( b \) be a branch of limit length in \( Y \), \( b \) is maximal in \( Y \) iff
\[
\forall i \in b \forall j \in b (i < j \land T(i) \land \forall i \in b (i < j \to i \text{ surv}_j))
\]

(Hence if \( b = \{ h \mid h <_T 3, \text{ surv}_3 \} \), then
\[
\forall i \in b \forall j \in b (i < j \to i \text{ surv}_j)
\]

We now define:

\[ i \prec_* \iff (i \leq_T j \land i \text{ surv}_j) \]

**Def.** \( i \) dies at \( j \) \( (i \text{ die}_i) \iff i < lh(Y) \land \exists i < j \land \text{ whenever } h \geq i, T(h) = i, \text{ then } T(h) <_* h \).
Finally:

Def. \( i \) is a break point at \( s \leq \text{lh}(y) \) \iff 
\[ i \leq s \text{ and whenever } i < h < s \text{ and } T(h) \leq i, \text{ then } T(h) < *h. \] (Another words:
An \( \forall \) \( \forall \) \( \forall \) every \( l \leq i \) dies at \( i+1. \)

We now turn to the proof of the main claim. We are given \( \sigma : P \leq N \), where \( P \)

in a putative iteration of \( P \), and wish to show that one of the following holds:

(a) \( \text{lh}(y) = h+1 \) and there is \( \sigma' : P, h < N \)

\( \lambda i. \sigma{\pi_0^i}, h = \sigma \).

(b) There is a maximal branch \( b \) of \( \text{lim} \) length in \( Y \) and a \( \sigma' : P < N \) \( \lambda h. \sigma') \text{ is } \sigma. \)

We shall assume (b) to be false and prove (a). We begin by reformulating (b):

Def. \( R = R_N \) is defined by:

\[ R = R \upharpoonright N = \{ \sigma \mid V_i \sigma. \sigma : P < N \} \]

\[ R = \{ \langle \sigma', \sigma \rangle \mid V_i \lambda i < * \lambda \sigma'. \sigma : P < N \} \]

\[ \text{and } \sigma{\pi_i^c} = \sigma^{-1} \]

\[ A \forall \sigma \in R \text{ we also set:} \]

\[ R \sigma = R \sigma = \{ \langle \sigma', \sigma'' \rangle \mid \sigma \sigma' \land (\sigma' \pi \nu \sigma'' = \sigma) \}. \]
Then (b) is equivalent to:
(b) $R^\sigma$ is ill-founded for our given $\sigma : \rho \subseteq N$. Thus we are assuming $R^\sigma$ to be well-founded.

In the following we shall always deal with good worlds $W' = \langle W', N', \rho' \rangle$ in which $Y$, $m'$ are $W'$-definable by the same definition as in our standard world $W$. Hence there is a relation $R_{W'} = R_N'$ definable in $W'$ as $R$ was defined in $W$. With this convention we define:

Def: An enlargement $E = \langle \langle W_h, \sigma_h \rangle \mid h \leq i \rangle$ of $Y|_{i+1}$ is semi-prond iff

(a) $E$ in meet
(b) $R_h = R^\sigma_h \upharpoonright W_h$ is well-founded for $h \leq i$
(c) $W_h$ has the form $\overline{W_h}(\mu_h)$, where $\mu_h \leq$ the rank of $\sigma_h$ in $R_h$ ($h \leq i$)
(d) $W_i$ is reflexive
(e) If $h < i$ does not die at $i+1$, then $W_h$ is reflexive.

Def: $E$ is semi-prond iff (a)-(c) hold.
Def: $E$ is a (semi) prond trace iff $E$ is the trace of a (semi) prond enlargement.
Note Semi-prudence is equivalent to a syntactic condition of the form:
\[ \forall h \leq i \text{.} \] Hence any enlargement of a semi-prond trace is semi-prond.

We prove:

**Main Lemma** Let \( j+1 \leq h(Y) \). Let \( i < j \) and let \( \mathcal{E} \) be a proud enlargement of \( Y[i+1] \).

(a) If \( i \) is a breakpoint at \( j+1 \), then \( \mathcal{E} \) extends to a proud enlargement \( \mathcal{E}' \) of \( Y[j+1] \) s.t. \( \text{On}_{\mathcal{E}'} < \text{On}_{\mathcal{E}} \) for \( i < l \leq j \).

(b) If \( i \) survives at \( j \), then \( \mathcal{E} \mathcal{I} \) extends to a proud enlargement \( \mathcal{I}' \) of \( Y[j+1] \) s.t. \( \text{On}_{\mathcal{I}'} < \text{On}_{\mathcal{I}} \) for \( i < l \leq j \) and \( W_{i}^{\mathcal{I}} = W_{i}^{\mathcal{I}'} \), \( \sigma_{i}^{\mathcal{I}'} = \sigma_{i}^{\mathcal{I}} \).

Before proving this, we show that it implies the main claim. Note that if \( \mu = \text{the rank of} \sigma \) in \( \mathcal{E} \), then \( \langle W(\mu), \sigma \rangle \) is a proud enlargement of \( Y[i] \). Suppose, first, that \( h(Y) = j+1 \). W.l.o.g., we may suppose \( m^*(j) = 0 \).
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Then 0 survives at \( \vdash \). Hence if \( \exists 0 = \emptyset \) extends to a profile enlargement \( I_1 \) of \( Y \) with \( \Psi_1^{I_1} = \Psi \) and \( \exists_1^{I_1} \leq 0 \).

Now suppose that \( Y \) has limit length. We derive a contradiction. We suppose w.l.o.g. that \( m^*(0) = 0 \).

Hence 0 is a breakpoint at \( \ell h(Y) \).

Define \( \exists_i^0 \) (\( i < \omega \)) by: \( \exists_0^0 = 0 \); \( \exists_{i+1} = \{ \exists_i^0 \} \) is a set, \( m^*(\exists) = \min \{ m^*(h) \mid h > \exists_i \} \).

An easy induction shows that each \( \exists_i^0 \) is a breakpoint at \( \ell h(Y) \), hence at \( \exists_i+1 \). By successive applications of (a) we get enlargements \( E_i \) of \( Y \mid \exists_i+1 \) such that \( W_\exists \leq W_i \) for \( i < \omega \). Contr!
We now prove the main lemma by induction on \( j \). Let it hold below \( j \).

We first prove (a), assuming (b) to hold:

**Case 1.** There is an \( h < j \) which survives at \( j' \). Then \( h > i' \), since \( i \) is a break point at \( j' + 1 \). By the induction hypothesis \( \mathcal{E} \) extends to a proud enlargement \( \mathcal{E}' \) of \( \mathcal{E} \).

\[ \text{On} W_i < \text{On} W_i', \text{ for } i < l \leq h, \text{ we then apply (b) to } \mathcal{E}', \text{ getting a proud } \mathcal{E} \text{ extending } \mathcal{E}' \text{ s.t. } \text{On} W_i < \text{On} W_i'. \]

\[ \text{On} W_i < \text{On} W_i', \text{ for } h \leq l < i', \text{ and } W_i' \mathcal{E} = W_i \mathcal{E}' \text{ (hence } \text{On} W_i' < \text{On} W_i \mathcal{E}' \text{ ).}

Then \( \mathcal{E} \) extends \( \mathcal{E} \) and has the right properties.

**Case 2.** Case 1 fails. Then \( j' \) is a successor ordinal \( h+1 \) and \( T(j') < \alpha(j) \). By the induction hypothesis \( \mathcal{E} \) extends to a proud \( \mathcal{E}' \) enlarging \( \mathcal{E} \) s.t. \( \text{On} W_i < \text{On} W_i', \text{ for } i < l \leq j', \text{ we then apply the Superlemma. By Superlemma (A) there: } \exists \sigma \in W_3 \cdot (\exists T(j') \text{ s.t. } \sigma ; \pi' \leq \mathcal{N}_3 \text{ and } \ldots) \)
\[ \mathbf{\overline{1} \overline{1}, i+1} = \mathbf{\overline{1} \overline{1}}. \text{ Hence } \mathbf{\overline{1} \overline{1}} \text{ and hence } \mathbf{\overline{1} \overline{1}} < m^2, \text{ where } \mathbf{\overline{1} \overline{1}} \text{ is the rank of } \mathbf{\overline{1} \overline{1}} \text{ in } \mathbf{\overline{1} \overline{1}}. \text{ We then apply Superlemma (B) to } \mathbf{\overline{1} \overline{1}}, \text{ getting an enlargement } \mathbf{\overline{1} \overline{1}} \text{ of } \mathbf{\overline{1} \overline{1}} \text{ extending } \mathbf{\overline{1} \overline{1}} \text{.}

On } \mathbf{\overline{1} \overline{1}} \text{, } \mathbf{\overline{1} \overline{1}} \text{, and } \mathbf{\overline{1} \overline{1}} \text{ is reflective. Moreover } \mathbf{\overline{1} \overline{1}} \text{ is neat, } \mathbf{\overline{1} \overline{1}} \text{ is well founded and } \mathbf{\overline{1} \overline{1}} = \mathbf{\overline{1} \overline{1}} \text{ for } \mathbf{\overline{1} \overline{1}} = \text{ the rank of } \mathbf{\overline{1} \overline{1}} \text{ in } \mathbf{\overline{1} \overline{1}}; \text{ since }

\[ \langle \mathbf{\overline{1} \overline{1}}, \mathbf{\overline{1} \overline{1}} \rangle \equiv \langle \mathbf{\overline{1} \overline{1}}, \mathbf{\overline{1} \overline{1}} \rangle. \text{ Hence } \mathbf{\overline{1} \overline{1}} \text{ is proper. } \text{ QED } (\text{b}) \rightarrow (\text{a}) \]

We now prove (b). We first note:

**Fact**: If } \mathbf{\overline{1} \overline{1}} \text{ survives at } \mathbf{\overline{1} \overline{1}}, \text{ then every } \mathbf{\overline{1} \overline{1}} \text{ dies at } \mathbf{\overline{1} \overline{1}}.

**Proof**: Suppose not.

Let } \gamma > \mathbf{\overline{1} \overline{1}}, \mathbf{\overline{1} \overline{1}} = \Gamma(\gamma) \text{, and } \mathbf{\overline{1} \overline{1}} \text{ survives at } \gamma \text{ and } \mathbf{\overline{1} \overline{1}} \in \mathbf{\overline{1} \overline{1}}. \text{ Then } m(\gamma) = m(\mathbf{\overline{1} \overline{1}}) = m(\mathbf{\overline{1} \overline{1}}), \text{ since } \mathbf{\overline{1} \overline{1}} \in \mathbf{\overline{1} \overline{1}}. \text{ But } m(\gamma) \neq m(\mathbf{\overline{1} \overline{1}}), \text{ since } \gamma \neq \mathbf{\overline{1} \overline{1}} \text{ and } \mathbf{\overline{1} \overline{1}} \neq \gamma. \text{ Hence } m(\gamma) > m(\mathbf{\overline{1} \overline{1}}). \text{ Hence } \mathbf{\overline{1} \overline{1}} \text{ does not survive at } \gamma \text{, since } m(\gamma) > m(\mathbf{\overline{1} \overline{1}}), \mathbf{\overline{1} \overline{1}} \in \mathbf{\overline{1} \overline{1}}. \text{ Contr!}
Case 1: \( \lim (\mathbf{i}^m) \)

Let \( \langle \mathbf{i}^m : m < \omega \rangle \) be monotone r.t. \( i_0 = i, \sup \mathbf{i}^m = \mathbf{i} \). (Hence \( \mathbf{i}^m \) sur i.m. for \( m < \omega \)). We first apply the induction hypothesis to get successive enlargements \( E_n (m < \omega) \) r.t., \( E_0 = \mathcal{E} \) and \( E_{n+1} \) extends \( E_n \) i.m. with:

- \( E_{n+1} \) is proud r.t. \( W_{n+1} = W_n E_n = W_i \),
- \( \mathfrak{E}_{n+1}, \mathfrak{i}_{n+1}^{m+1} = \mathfrak{E}_n \) for \( m < \omega \).

Let \( \mathbf{i}^m = \langle t_m, s_m \rangle = t_\omega (E_m) \). Then \( \mathbf{i}^m \) is a semi proud trace in \( W_i = W_n E_n \), since \( E_m \) is semi proud (hence meet) and \( W_i \) is reflective. Hence \( \langle (\mathbf{i}^m, \mathfrak{E}_m) \rangle (m < \omega) \) forms a descending chain in the following relations \( S \), which is defined in \( W_i \):

- **Def**: \( D = \) the set of \( \langle \mathbf{r}, \sigma \rangle \) r.t. \( \sigma : P_i \leq N_r \), \( r \) in an \( N_r \), \( \sigma \) - conforming, semi proud trace for \( \mathbf{r} \) i.m.\( +1 \), and \( z_0 = \mathbf{z}_0 \cap N_r \). (\( z_0 = t_\omega (\mathcal{E}) \))

\( S = \) the set of \( \langle \langle \mathbf{r}', \sigma' \rangle, \langle \mathbf{r}, \sigma \rangle \rangle \in \mathcal{E} \times \mathcal{D}^2 \) r.t. for some \( m < \omega \):

- \( \sigma : P_i \leq N_r \), \( \sigma' : P_i \leq N_r \), \( \sigma' \mathfrak{E}_m i_m = \sigma \), \( \mathfrak{i}_m = \mathfrak{i}_m \).
Thus $S$ is ill-founded. Let $\langle s_m, \sigma_m \rangle \in W_i$ be a chain through $S$. Define $\sigma : P_i < N_i$ by $\sigma \cap \Gamma_i = \sigma_m$. Set $S = \bigcup_{m < w} S_m \cap \Gamma_i$.

**Claim:** $S$ is an $N_i, \sigma$-conforming trace in $W_i$.

**Proof:**
$S$ is obviously $\sigma$-conforming, since $\sigma \cap \Gamma_i = \sigma_m \cap \Gamma_i = \sigma_m$ for $m < w$. Now let $\mu_m$ be least $\not \exists t_i$.

$coll (\omega, \mu_m) \uparrow$ (There is an $N_i$-conforming enlargement $IF$ of $Y_i \mid \Gamma_i$ s.t. $\sigma_{\mu_m} = t_i (IF)$)

Then $\langle \mu_m, \Gamma_i \rangle \in W_i$ - definable from $\langle s_m, \Gamma_i \rangle$. Set $\mu = \text{lub} \mu_m$. Let $G$ be $\text{coll} (\omega, \mu)$ - generic over $W_i$. Then in $W_i [G]$ we find $\langle IF, \mu \rangle \inW_i$ s.t. $IF_m$ is an $N_i$-conforming enlargement of $Y_i \mid \Gamma_i$.

Define $IF$ by:

$IF [\Gamma_i] = IF_0$,

$IF [\Gamma_i \cap \Gamma_{i+1}] = IF_{\mu_{i+1}} \cap [\Gamma_i \cap \Gamma_{i+1}]$. Then $IF$ is an $N_i$-conforming enlargement of $Y_i \mid \Gamma_i$.

QED (Claim)

Now let $IF$ be an $N_i$-conforming enlargement of $Y_i \mid \Gamma_i$. Define
by $E^i = E$, $E^i \{i, j\} = E^i \{i, j\}$. Then $E^i$ is an $N_i$-conforming enlargement of $Y|_{Y^i}$. Note that $i$ is a neat extension of $Y|_{Y^i}$. Since each $s_m$ is neat, set $E^* = E^i \langle W_i, \sigma \rangle$. Then $E^*$ is a neat enlargement of $Y|_{Y^i+1}$ by the above claim. Moreover $E^* \{i' = E^i \{i', \sigma \}$ and $s_{\pi_i} = s_i$. We note that $E^*$ is semi-proud since each $s_m$ is semi-proud (hence $W_h \vdash \sigma_i^h$ is well-founded and $W_h = \sigma_i^h$ for a $\mu_h \geq \text{rank of } \sigma_i^h$ in $R^h$), and $W_i = \sigma_i$ is well-founded and $W_i = \sigma_i$ for a $\mu_i \geq \text{rank of } \sigma_i$ in $R_i$). Thus $\mu_i \geq \text{rank of } \sigma_i$ in $R_i$, since whenever $\sigma_i \in \sigma_i^\prime$ (i.e., $\sigma_i^\prime : P < N_i$ for an $l > j$ and $\sigma_i^\prime \pi_j = \sigma_i^\prime$), then $\sigma_i \in \sigma_i^\prime$, since $\sigma_i^\prime \pi_j = \sigma_i^\prime$. Thus $E^*$ is semi-proud. But $W_i = W_i$ is semi-proud and $W_h = W_h$ is reflective for all $h < i$ which does not die at $j$ (since it then does not die at $i+1$). By the above fact it then follows.
That $E$ is proud. \( \varnothing \text{ED (Case 1)} \)

Case 2 \( j = h+1 \).

Let $i = T(j)$. Then $i$ is a break point at $j$, since $i \neq k \in (i, j)$ and $T(k) \leq i$, then $T(k) < k$. (Otherwise $m(T(k)) = m(k) \geq m(i), \text{ since } k \in (i, j)$. But $m(k) \neq m(i) = m(i)$, since $k \neq i$ and $j \neq k$. Hence $m(k) > m(i)$, where $i \in T(k), k$. Hence $T(k) \neq m i k$.)

Clearly $i \geq i$ and $i$ survives at $i$. By the inductive hypothesis $E \cap i$ extends to a proud $E'$, enlarging $y_{i+1}$ at $i$.

On $W_{i+1}^{E'}$, for $i \leq l < i$, $W_{i+1}^{E'} = W_{i+1}^{E}$, and $\sigma_{i+1}^{E'} = \sigma_{i+1}^{E}$. Hence we can assume w.l.o.g. that $i = i$. By the inductive hypothesis, we can apply (a) to $E'$, getting a proud $E'$ extending $E$ and enlarging $y_{i+1}$ at $i$. On $W_{i+1}^{E'}$, for $i < l < j$, applying Superlemma (A) to $E'$, then gives the desired result. We obtain $E$ extending $E \cap i$ at $i$, letting

$\bar{t} = \langle t, \bar{\sigma} \rangle = t \cap (\bar{E})$, $\bar{\lambda} = \langle t, \bar{\delta} \rangle = t \cap (\bar{E})$,

we have $t = t \cap \bar{E}$, $\bar{\sigma}_{i+1}^{\bar{E}} = \sigma_{i+1}^{\bar{E}}$, ($\bar{\sigma}_{h} = \sigma_{h}^{\bar{E}}$), and $W_{i+1} = W_{i+1}^{E'}$ ($\bar{W}_{h} = W_{h}^{\bar{E}}$).
Then $\bar{E}$ is semi-prond, since $s$ is a semi-prond trace and (as in Case 1) $W_i = \{ \bar{P}_i, \bar{I}_i \in \text{well founded and } \gamma = \bar{W}^{\langle a \rangle} \}
\text{for } a \geq \text{rank of } \bar{I}_i \text{ in } \bar{P}_i \).

But $W_i$ is reflective and $\bar{W}_h = W_h$ is reflective for all $h \prec i$ which do not die at $i+1$ (hence for $h \prec i$ which do not die at $j+1$). By the above Fact it follows that $\bar{E}$ is prond.

QED (Case 2).

This proves the Theorem.