V A Remark on $\Box$ in $L^E$

Consider a model $K = L^E$ with the property that each $K/\mu^*$ is a weak mouse in the sense of $I$, Schimmerling and Zeman have shown that $\Box_\lambda$ holds in $K$ for many cardinals $\lambda$.

Let $S_\lambda = \{ \xi \in (\lambda, \lambda^+) \mid \xi \in K \}$.

$E_\xi$ is a supercompact extender (i.e., $E_\xi \neq \emptyset$ and $\lambda = \text{lh}(E_\xi)$). By the methods of Schimmerling and Zeman, $\Box_\lambda$ holds if $S_\lambda$ is not stationary in $\lambda^+$ (in $K$). We now show that this result is best possible:

Thus let $S_\lambda$ be stationary in $\lambda^+$.

Then $\Box_\lambda$ fails.

Proof. Suppose not.

Set $H = L^E_{\lambda^{++}}$. Set $D = \{ \delta \in (\lambda, \lambda^+) \mid \delta = \lambda^+ \cap h_1(\delta) \}$. 


Then $D$ is cub in $\lambda^+$. Let $\delta \in \mathcal{D} \cap \mathcal{S}_\lambda$.
Set $X = \Theta^\delta (x)$ and let $\overline{H} = \mathcal{L}^\delta$, $\sigma$. be defined by $\sigma : \overline{H} \rightarrow \mathcal{H} | X$. Then $\sigma : \overline{H} \rightarrow \mathcal{L}^\delta$, $\overline{H}$ is a premouse which is sound above $\delta$ and $\omega \beta^\delta = \sigma$. We apply Lemma 4' in IV (which generalizes Lemma 4 of §8 [NF5]). Clearly $\overline{H} \not\models \varphi_2 (t_1)$ since $\rho^\delta = \delta \leq \lambda^{++} = \rho^\delta$, hence (a) in Lemma 4' fails. Moreover $\overline{H}$ is not an initial segment of $\mathcal{H}$, since $E_\delta^\overline{H} \neq \emptyset$, $E_\overline{H} = \emptyset$. Hence (b) fails. (c) also fails, since $\delta = \alpha$ is the largest cardinal in $\overline{H}$, where $\alpha = \omega \beta (\sigma)$. Thus (d) must hold, and $\overline{H}$ is a segment of $\mathcal{L}^\delta$, where $\tau : \mathcal{L}^\delta \rightarrow \mathcal{L}^\delta'$. 
We are assuming that $L^\kappa$ has a $\square^\kappa$ sequence. Let

$$C = \langle C_\gamma \mid \chi < \gamma < \chi^+ \land \text{fin}(\gamma) \rangle$$

be the $<_{L^\kappa}$-least such. Since $\sigma \in \Sigma_1$ preserving, there is $\bar{C} \in H^\kappa$ and $\sigma(\bar{C}) = C$. Hence $H$ thinks that $\bar{C}$ is a $\square^\kappa$-sequence. Since $d = \chi^+ + L^\kappa$, and $H$ is an initial segment of $L^\kappa$, it follows that $\bar{C}$ is the $<_{L^\kappa}$-least $\square^\kappa$ sequence in $L^\kappa$.

Since $\pi(\bar{C}) = \chi$, where $\kappa = \text{crit}(E_{\chi^+})$ and $\tau = \kappa^+$ in $L^\kappa$, it follows that there is $C' = \pi^{-1}(\bar{C})$ which is a $\square^\chi$ sequence in $L^\kappa$.

$$\therefore \pi = \pi \upharpoonright L^\kappa_{\tau}$$

Then

1. $\pi : \langle L^\kappa_{\tau}, C' \rangle < \langle L^\kappa_{\kappa^+}, \bar{C} \rangle$
2. $\langle L^\kappa_{\kappa^+}, \bar{C} \rangle < \langle L^\kappa_{\chi^+}, C \rangle_{\Sigma_1}$

Hence:
(3) $\overline{\pi} : \langle L^E, C' \rangle \rightarrow \langle L^E, C \rangle$,

where $C'$ is a $\Sigma_1$ sequence.

This is known to yield a contradiction. Consider $C_d$.

Since $d = \sup \overline{\pi}^\ast \omega$ and $\omega$ is regular, the set $C^*_d \cap \text{rng}(\overline{\pi})$ is unbounded in $\omega$. Suppose $\gamma \in C^*_d \cap \text{rng}(\overline{\pi})$, $\overline{\pi}(\gamma') = \gamma$.

Then $\overline{\pi}(C_d') = C_\gamma = \gamma \land C_d$.

Since $C_\gamma$ is a proper segment of $C_d$, we have $\text{otp}(C_\gamma) < \omega$. Hence $\text{otp}(C_d') < \omega$. Since $\overline{\pi}(C_\gamma) = C^*_d \cap \text{rng}(\overline{\pi})$, $\text{otp}(C_\gamma) < \omega$, since $\overline{\pi}^\ast \omega = \text{id}$.

Thus $\text{otp}(C_d) = \sup \text{otp}(C_d') \leq \omega$,

$\forall \gamma \in C^*_d \cap \text{rng}(\overline{\pi})$

Contr. since $cf(\omega) = 2 > \omega$ in $L^E$.

QED